PAPERS COMMUNICATED

19. *A Generalization of Tauber's Theorem.*

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1. It was proved by Prof. Tauber[1] that:

If $na_n \to 0$ *as* $n \to \infty$, *and if*

$$\lim_{x\to 1-0} \sum_{n=1}^{\infty} a_n x^n = A ,$$

then
$$\sum a_n = A .$$

The condition $na_n \to 0$ was replaced by the broader condition $a_n = O\left(\frac{1}{n}\right)$ by Prof. Littlewood[2], and Professors Hardy and Littlewood[3] replaced it again by $na_n > -K$. Finally Dr. R. Schmidt[4] proved that it is sufficient to assume

$$\underline{\lim}_{m,\, n\to\infty} (s_m - s_n) \geqq 0 ,$$

when $m > n$ and $m/n \to 1$.

On the other hand Prof. Littlewood[5] proved that:

Suppose that

$$0 < \lambda_{n-1} < \lambda_n , \qquad \lambda_n \to \infty ,$$

(1)
$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \to 0 ;$$

and further that

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right);$$

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1) Tauber: Monatshefte für Math. u. Physik, **8** (1897).
2) Littlewood: Proc. London Math. Soc. (2) **9** (1910).
3) Hardy-Littlewood: ibid. (2) **13** (1913).
4) R. Schmidt: Math. Zeits. **22** (1925). The direct proof of this theorem was given by Dr. Vijayaraghavan (Journ. London Math. Soc **1** (1916)).
5) Littlewood: loc. cit.

then the existence of the limit

$$\lim_{s\to+0} \sum_1^\infty a_n e^{-\lambda_n s} = A$$

implies the convergence of the series $\sum a_n$ *to the sum* A.

Dr. Ananda-Rau[1] remarked next that (1) is superfluous. and got the quite analogous theorem to the Littlewood's theorem concerning power series.

The author of the present note has obtained the following theorem:

Theorem. *Suppose that*

$$\varliminf_{p,\, q\to 0} (s_q - s_p) \geqq 0\,,$$

where $s_n = a_1 + \ldots . + a_n$; $q > p$, $\lambda_q/\lambda_p \to 1$.

Then the existence of the limit

$$\lim_{s\to+0} f(s) = \lim_{s\to+0} \sum_1^\infty a_n e^{-\lambda_n s} = A \tag{2}$$

implies the convergence of the series $\sum a_n$ *to the sum* A.

This corresponds to Schmidt's theorem concerning power series, and it contains the analogue to the Hardy-Littlewood's theorem concerning power series:

Suppose that

$$a_n > -K \frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\,,$$

then the existence of the limit (2) *implies the convergence of* $\sum a_n$ *to* A.

From our theorem we can derive the following special case:

When λ*'s satisfy the condition*

$$\varliminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} > 1\,,$$

then the existence of the limit (2) *implies the convergence of* $\sum a_n$.

This is nothing but Hardy-Littlewood's theorem[2].

2. To prove this we use the

Lemma. *If for the sequence* (s_n)

$$\varliminf_{p,\, q\to\infty} (s_q - s_p) \geqq 0\,,$$

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1) Ananda-Rau: Journ. London Math. Soc. **3** (1928).

2) Hardy-Littlewood: Proc. London Math. Soc. **25** (1925).

where $q>p$, $\lambda_q/\lambda_p \to 1$, *then to every constant* c *there corresponds a constant* $K=K(c)$, *such that*

$$s_q - s_p > -\left(K\log\frac{\lambda_q}{\lambda_p} + c\right)$$

for all values of p *and* q.

First we prove that the sequence (s_n) does not infinitely oscillate. If it does, (2) tends to $+\infty$ or $-\infty$, which contradicts the hypothesis.

Next we prove that the sequence does not oscillate even finitely. At this stage we use the Littlewood's lemma, in addition to our Lemma, which asserts that, *if* $f(s) \to A$ *as* $s \to 0$,

$$I^r = s^{r+1}\int_0^\infty s(t)\, t^r e^{-st}\, at \to A \cdot r\,!$$

for all positive integers r, *where*

$$\begin{aligned} s(t) &= 0 \quad \text{for} \quad 0 \leqq t < \lambda\,, \\ &= s_n \quad \text{for} \quad \lambda_n \leqq t < \lambda_{n+1}\,. \end{aligned}$$