177. A Generalization of Almost Periodic Functions.

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Introduction. The Bohr's theory[1] of almost periodic functions has its origin in the problem: What function $f(x)$ can be decomposed, in the interval $-\infty < x < \infty$, into pure oscillations, that is, into oscillations of the form $e^{i\lambda x}$? The simplest functions of this kind are the sum of a finite number of oscillations:

$$s(x) = \sum_{\nu=1}^{N} a_\nu e^{i\lambda_\nu x}.$$

Prof. Bohr adjoined the limit functions to the class (F) of such functions; we understand by a limit function $f(x)$ of the class (F), if there exists a sequence $s_1(x)$, $s_2(x)$, $s_3(x)$, ... of functions in (F), such that

$$(1) \qquad f(x) = \lim_{n=\infty} s_n(x)$$

uniformly for every x, that is

$$(1) \qquad \underset{-\infty<x<\infty}{\text{Upper Boundary}} \; |f(x)-s_n(x)| \to 0, \quad \text{as} \quad n \to \infty.$$

Any function belonging to the adjoined class (C) is called almost periodic. In (1) we can take $s_n(x)$ as an almost periodic functions, without affecting the class (C).

The theory of almost periodic functions was extended by many authors in replacing the limiting equation (1) by more general ones. It seems to me, however, that a natural extension of (1) is the mean convergence. By

$$(2) \qquad \lim_{n=\infty} s_n(x) = f(x) \qquad (C, k),$$

where $k \geqq 0$, we mean that

$$(3) \quad \text{Upper Boundary} \int_{\alpha-T}^{\alpha+T} \left(1-\frac{|x|}{T}\right)^k d\,|f(x)-s_n(x)| \to 0, \text{ as } n \to \infty$$

1) Bohr, zur Theorie der fastperiodischen Funktionen, I-III, Acta Math. **45-47** (1925-26).

for every x and α. When we replace (1) by (2), we get a more general class $C^{(k)}$ than (C). Putting $k=1$ and remembering

$$\lim_{T=\infty}\int_{\alpha-T}^{\alpha+T}\left(1-\frac{|x|}{T}\right)d|f(x)-s_n(x)|=\lim_{T=\infty}\frac{1}{T}\int_{\alpha}^{\alpha+T}|f(x)-s_n(x)|\,dx,$$

we see the class $C^{(1)}$ is equivalent to the Stepanoff's generalization[1] of (C). The class $C^{(k)}$ contains $C^{(h)}$, for $k\geqq h$, so Stepanoff's class is contained in $C^{(k)}$ $(k>1)$.

We will now give two fundamental theorems concerning $C^{(k)}$-class, as in the Bohr's theory.

1. We can prove the following elementary properties, from the definition and the properties of the almost periodic functions.[2]

1°. Every function in $C^{(k)}$-class is bounded in mean in $-\infty<x<\infty$, that is, the integral

$$I^{(k)}(f(x))=\int_{\alpha-T}^{\alpha+T}\left(1-\frac{|x|}{T}\right)^k d\,|f(x)|$$

is bounded.

2°. Every function in $C^{(k)}$-class is uniformly mean continuous in $-\infty<x<\infty$, that is

$$I^{(k)}(f(x+h)-f(x))<\varepsilon \quad \text{for} \quad |h|<\delta,\ -\infty<x<\infty.$$

3°. The sum of the functions belonging to $C^{(k)}$ also belongs to $C^{(k)}$.

4°. $f(x)e^{i\lambda x}$ belongs to the $C^{(k)}$-class, when $f(x)$ is so.

5°. Every function in $C^{(k)}$-class has a mean value:

$$\lim_{T=\infty}\int_{-T}^{T}\left(1-\frac{|x|}{T}\right)^{\mu}df(x)=M^{(\mu)}(f(x)) \qquad (\mu\geqq k).$$

6°. For every function in $C^{(k)}$-class, the value

$$M^{(\mu)}\{f(x)e^{-i\lambda x}\}=a(\lambda)$$

differs from zero for at most an enumerable set of values of real λ.

We can denote these λ's by $\Lambda_1, \Lambda_2, \Lambda_3, \ldots.$ and the corresponding $a(\lambda)$ by $A_1, A_2, A_3, \ldots.$ We express this symbolically by

1) Stepanoff, Ueber einige Verallgemeinerungen der fastperiodischen Funktionen, Math. Annalen, **95** (1926).

2) Cf. Besicovitch and Bohr: Generalisations of almost periodic functions, Det Kgl. Danske Vid. Selskab. VIII, **5** 1928).

$$f(x) \sim \sum A_\nu e^{i\Lambda_\nu x},$$

and we call the series on the right-hand side the Fourier series of the function $f(x)$.

2. We now pass to the characteristic property of the $C^{(k)}$-class. As in the Bohr's and Stepanoff's theory, we introduce a translation number, defined as follows: A number τ is said to be a translation number of the function $f(x)$, belonging to ε, if

$$\mathrm{I}^{(k)}(f(x+h)-f(x)) < \varepsilon.$$

We call $f(x)$ a generalized almost periodic function (g. a. p. function), if, for every positive ε, there exists a relatively dense set of translation numbers $\tau(\varepsilon)$ of $f(x)$; the set of $\tau(\varepsilon)$ being relatively dense in the sense that any interval of a certain length $l = l(\varepsilon)$ contains at least one such number $\tau(\varepsilon)$.

Every function of $C^{(k)}$-class is g. a. p. function, as easily to be seen. To prove the converse we introduce almost periodic functions:

$$\varphi_\delta(x) = \int_x^{x+\delta} \left(1-\frac{|x|}{\delta}\right)^k df(\xi).$$

Then we can prove that

$$(4) \qquad \lim_{\delta=0} \varphi_\delta(x) = f(x) \qquad (C, k')$$

for $k' \geqq k$. Thus we get the

Theorem. *The $C^{(k)}$-class is identical with the class of g. a. p. functions.*

This gives the generalization of the Bohr's theorem, which asserts (4) for $k'=1$, $k=0$.

3. The next problem is to find a simple algorithm, which gives a sequence of finite sums $s_n(x)$ tending to $f(x)$ in the mean, that is

$$\mathrm{I}^{(k)}(f(x)-s_n(x)) \to 0, \quad \text{uniformly as} \quad n \to \infty.$$

We gives the representation of Fourier exponents Λ_ν by the help of the base $a_1, a_2, \ldots.$, such that every Λ_ν can be represented as a finite linear form of $a_1, a_2, \ldots.$ with rational coefficients, and write

$$s(x) = \sum_{\substack{-n_1 \leqq \nu_1 \leqq n_1 \\ \cdots\cdots\cdots \\ -n_p \leqq \nu_p \leqq n_p}} \left(1-\frac{|\nu_1|}{T}\right)^k \cdots\cdots \left(1-\frac{|\nu_p|}{T}\right)^k A_\nu e^{i\Lambda_\nu x},$$

where

$$\wedge_\nu = \frac{\nu_1}{N_1!} a_1 + \ldots . + \frac{\nu_p}{N_p!} a_p ,$$

and A_ν denotes zero when $\frac{\nu_1}{N_1!} a_1 + \ldots . + \frac{\nu_p}{N_p!} a_p$ is not one of the Fourier exponents of $f(x)$. Then we can prove that

$$\mathit{lim}\ s(x) = f(x) \qquad (C, k),$$

provided p; N_1, N_2, ... ; n_1, n_2, ... tend to ∞ and $\frac{n_1}{N_1!}, \frac{n_2}{N_2!}, \ldots$ tend to ∞.

If we replace the condition (3) by

$$\text{Upper Boundary} \int_{\alpha-T}^{\alpha+T} \left(1 - \frac{|x|}{T}\right)^k d|f(x) - s_n(x)|^p \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

$(p > 0)$, we get more general class. In this case we can also derive the quite similar results.