56. Normed Rings and Spectral Theorems, V.

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1. *Introduction.* Recently, M. Krein[1] published a generalisation of the Plancherel's theorem to the case of locally compact (=bicompact) abelian group. The result is much important, since it reveals the hitherto hidden algebraic character of the classical Fourier analysis. However, Krein's proof of the positivity of the functional J is somewhat complicated and moreover it seems that his paper lacks the proof of 3° which is the key to the proof of the positivity of J. The purpose of the present note is to show that a complete proof may be obtained by making use of the preceding note[2]. It is also to be remarked that the theorem below constitutes an extension of 3° and that, by virtue of this extension, Krein's arguments may be much simplified.

2. *A theorem of positivity.* Let G be a locally compact, separable abelian group and let X be the group (without topology for the moment) of continuous characters $\chi(g)$ of G. Then, by Haar's invariant measure dg, we may define the linear space $L_p(G)$ $(\infty > p \geqq 1)$ of complex-valued measurable functions $x(g)$ such that $|x(g)|^p$ is summable over G:

$$\|x\|_p = \sqrt[p]{\int |x(g)|^p dg} < \infty . \tag{1}$$

A multiplication $x * y$ is introduced in $L_1(G)$ by the convolution:

$$x * y(g) = \int x(g-h)y(h)dh . \tag{2}$$

By adjoining formally[3] a unit e to $L_1(G)$ we obtain a normed ring $R(G)$ by the norm $\|z\|$ and the multiplication $*$:

$$\begin{cases} z = \lambda e + x(g), \quad \|z\| = |\lambda| + \|x\| \quad (\lambda = \text{complex number}), \\ z_1 = \lambda_1 e + x_1(g), \quad z_2 = \lambda_2 e + x_2(g), \\ z_1 * z_2 = \lambda_1\lambda_2 e + \lambda_1 x_2(g) + \lambda_2 x_1(g) + x_1 * x_2(g). \end{cases} \tag{3}$$

Such ring is considered by *I.* Gelfand and *D.* Raikov[4].

We next introduce a new normed ring to be denoted as $\overline{R}_{op}(G)$. For any $x \in L_1(G)$ and for any $y \in L_2(G)$ we have

$$x * y(g) \in L_2(G), \qquad \|x * y\|_2 \leqq \|x\|_1 \cdot \|y\|_2 , \tag{4}$$

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1) C. R. URSS, **30** (1941), No. 6.

2) Proc. **19** (1943), p. 356. This note will be referred to as (I).

3) The trivial case of the discrete group G is excluded in the following lines.

4) C. R. URSS, **28** (1940), No. 3.

since, by the invariance of Haar's measure,

$$\int |x*y(g)|^2 dg \leqq \int \left| \int |x(g\text{-}h)|^{\frac{1}{2}} |x(g\text{-}h)|^{\frac{1}{2}} |y(h)| dh \right|^2 dg$$

$$\leqq \int \left\{ \int |x(g\text{-}h)| dh. \quad \int |x(g\text{-}h)| |y(h)|^2 dh \right\} dg \leqq \|x\|_1^2 \cdot \|y\|_2^2 \cdot$$

Thus $x \in L_1(G)$ induces a linear operator T_x of the Hilbert space $L_2(G)$:

$$(5) \qquad T_x \cdot y = x * y, \qquad \|T_x\| = \sup_{|y|_2 \leq 1} \|T_x \cdot y\|_2 \leqq \|x\|_1 \cdot$$

The set $\{T_x; x \in L_1(G)\}$ together with the identity operator E constitute a ring $R_{op}(G)$ isomorphic to the ring $R(G)$ by the correspondence $x \leftrightarrow T_x$:

$$(6) \qquad \begin{cases} e \leftrightarrow E, \quad \alpha x \leftrightarrow \alpha T_x, \quad x+y \leftrightarrow T_x + T_y, \quad x*y \leftrightarrow T_x T_y, \\ x^* \leftrightarrow T_x^*, \quad \text{where } x^*(g) = \overline{x(-g)} \text{ and } T^* = \text{adjoint of } T. \end{cases}$$

The closure $\overline{R}_{op}(G)$ by the topology defined by the norm $\|T\|$ of the ring $R_{op}(G)$ surely constitutes a normed ring by the norm $\|T\|$.

It is proved in (I) that $\overline{R}_{op}(G)$ is representable isomorphically as a function ring $R(\overline{\mathfrak{M}})$ of complex-valued continuous functions $T(\overline{M})$ on the compact Hausdorff space $\overline{\mathfrak{M}}$ of all the maximal ideals $\overline{M}$ of $\overline{R}_{op}(G)$:

$$(7) \qquad T \leftrightarrow T(\overline{M}), \qquad E \leftrightarrow E(\overline{M}) \equiv 1,$$

such that

$$(8) \qquad \begin{cases} \|T\| = \sup_{\overline{M}} |T(\overline{M})|, \\ T(\overline{M}) \text{ is real-valued if and only if } T \text{ is symmetric } (T = T^*), \\ T(\overline{M}) \text{ is non-negative-valued if and only if } T \text{ is symmetric} \\ \quad \text{and positive definite (as operator).} \end{cases}$$

Hence

$$(9) \qquad \begin{cases} T_z(\overline{M}) \text{ is real-valued if and only if } z = z^* \text{ viz.} \\ \lambda = \overline{\lambda} \text{ and } x(g) = x^*(g) = \overline{x(-g)}, \\ T_z(\overline{M}) \text{ is non-negative-valued if and only if } z = z^* \text{ and} \\ \quad (T_z \cdot y, y) = z * y * y^*(0) \geqq 0 \text{ for all } y \in L_2(G). \end{cases}$$

Lemma 1. Let $\overline{M}$ be a maximal ideal of $\overline{R}_{op}(G)$, then the image $M \leqq R(G)$ of $\overline{M} \frown R_{op}(G)$ by the correspondence $z \leftrightarrow T_z = \lambda E + T_x$ is a maximal ideal of $R(G)$.

Proof. Since $T_z \equiv \big(\lambda + T_x(\overline{M})\big)E$ (mod $\overline{M}$), we have $z \equiv \big(\lambda + T_x(\overline{M})\big)e$ (mod M). Q. E. D.

Now, by Gelfand-Raikov's results[1], we have

$$(10)\quad \begin{cases} \lambda+T_x(\overline{M}_\infty)=\lambda & \text{if} \quad M_\infty=L_1(G), \\ \lambda+T_x(\overline{M})=\lambda+\int x(g)\chi_M(g)dg, \quad \chi_M\in X, & \text{if} \quad M\neq L_1(G). \end{cases}$$

Therefore, by (9), (10) and the proof of the Lemma 1, we obtain the

Theorem. Let the Fourier transform $(P\cdot z)(\chi)=\varphi_z(\chi)$ of $z=\lambda e+x\in R(G)$ be defined by

$$(11)\quad (P\cdot z)(\chi)=\varphi_z(\chi)=\lambda+\int x(g)\chi(g)dg \qquad (\chi\in X).$$

If $\varphi_z(\chi)$ is real-valued on X, then $z=z^*$. If $\varphi_z(\chi)$ is non-negative-valued on X, then $z=z^*$ and $z*y*y^*(0)\geqq 0$ for all $y\in L_2(G)$.

Corollary. Let $x(g)\in L_1(G)$ be continuous on G, viz. let $x\in L_1(G)$ $C(G)$ and let $\varphi_x(\chi)=\int x(g)\chi(g)dg\geqq 0$ on X, then $x(g)$ is positive definite in Bochner's sense:

$$x*y*y^*(0)\geqq 0 \quad \text{for all} \quad y\in L_1(G).$$

Thus a complete proof of 3° is obtained.

2. *Plancherel's theorem.* The Fourier transform P is defined by Krein only for $x\in L_1(G)$, $L_2(G)$. Our extension of P for all $z\in R(G)$ together with the theorem (instead of the corollary) will much simplifies Krein's proof of the Plancherel's theorem.

Following Krein, we introduce the additive homogeneous functional $J(\varphi_z(\chi))$:

$$(12)\quad J(\varphi_z(\chi))=\lambda+x(0) \quad \text{for} \quad z=\lambda e+x, \quad x(g)\in L_1(G)\frown C(G).$$

Lemma 2. J is positive viz. $J(\varphi_z(\chi))\geqq 0$ if $\varphi_z(\chi)\geqq 0$ on X.

Proof. Since, by the theorem,

$$z*y*y^*(0)=\lambda\iint y(s)\overline{y(t)}dsdt+\iint x(s-t)y(s)\overline{y(t)}dsdt\geqq 0$$
$$\text{for all} \quad y\in L_2(G),$$

we have $\lambda+x(0)\geqq 0$. Q. E. D.

Therefore J is a linear (=continuous additive) functional on the space of the continuous functions $\varphi_z(\chi)$, where $z=\lambda e+x$, $x\in L_1(G)$ $\frown C(G)$. Here the topology on X is defined by the weak topology induced by the neighbourhood:

$$(13)\qquad U(\chi_0)=\{\chi\,;\,|\varphi_{z_i}(\chi)-\varphi_{z_i}(\chi_0)|<\varepsilon, \qquad i=1,2,\dots,n\}.$$

Moreover we have

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1) See 4).

$$(14)\qquad J\big(\varphi_z(\chi+\chi_1)\big)=J\big(\varphi_z(\chi)\big) \quad \text{for any} \quad \chi_1 \in X,$$

$$(15)\qquad x(g)=J\big(\varphi_z(\chi)\cdot\chi(-g)\big), \qquad x \in L_1(G)\frown C(G),$$

since

$$\varphi_z(\chi+\chi_1)=\lambda+\int x(g)(\chi\chi_1)(g)dg=\lambda+\int\{x(g)\chi_1(g)\}\chi(g)dg,$$

$$\int x(g+h)\chi(h)dh=\int x(g+h)\chi(g+h)\chi(-g)dh=\chi(-g)\int x(k)\chi(k)dk$$

(by the invariance of the Haar's measure).

Now define a formal character $\chi_\infty(g)$ by

$$(16)\qquad \varphi_z(\chi_\infty)=\lambda, \qquad z=\lambda e+x,$$

then the set $X\smile\chi_\infty$ is compact separable by the weak topology defined by the neighbourhood (13), since $X\smile\chi_\infty$ corresponds to the totality of the maximal ideals M of $R(G)$ in one to one manner[1]:

$$\chi_M \leftrightarrow M, \qquad z\equiv\varphi_z(\chi_M)e \qquad (\text{mod } M).$$

Thus X is a locally compact separable abelian group.

Since $x\in L_1(G)\frown C(G)$ is dense in $L_1(G)$, $\{\varphi_z(\chi);\ z=\lambda e+x,\ x\in L_1(G)\frown C(G)\}$ is dense[2] (by the topology of the uniform convergence on $X\smile\chi_\infty$) in the space of complex-valued continuous functions on $X\smile\chi_\infty$.

Thus the positive linear functional J is of the form

$$(17)\qquad J\big(\varphi_z(\chi)\big)=\int_{X\smile\chi_\infty}\varphi_z(\chi)d\chi,$$

with a measure $d\chi$ countably additive on Borel sets $\leqq X\smile\chi_\infty$. By (16), we have $\varphi_x(\chi_\infty)=0$ and hence

$$(18)\qquad x(0)=J\big(\varphi_x(\chi)\big)=\int_X\varphi_x(\chi)d\chi \quad \text{for} \quad x\in L_1(G)\frown C(G).$$

Because of the invariance (14), the measure $d\chi$ in (18) must be the Haar measure on X. Thus we obtain, by (11) and (15), the duality relation for $x\in L_1(G)\frown C(G)$:

$$(19)\qquad \begin{cases}\varphi_x(\chi)=\int_G x(g)\chi(g)dg, \qquad x(g)=\int_X\varphi_x(\chi)\chi(-g)d\chi,\\ \text{where } dg,\ d\chi \text{ being respectively Haar measures on } G, X.\end{cases}$$

Let $y\in L_1(G)\frown L_2(G)$, then, since $x=y*y^*\in L_1(G)\frown C(G)$, we have by (18)

$$x(0)=\int|y(g)|^2dg=\int\varphi_x(\chi)d\chi=\int\varphi_y(\chi)\varphi_{y^*}(\chi)d\chi=\int|\varphi_y(\chi)|^2d\chi$$

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1) See 4). The separability follows from the separability of the normed ring R(G): Cf. I. Gelfand's paper in Rec. Math., **9** (1941), No. 1.

2) See 4).

and thus $(P\cdot y)(\chi)\in L_2(X)$ and

$$(20)\quad \begin{cases} \| P\cdot y\|_2=\int_X |\varphi_y(\chi)|^2 d\chi=\| y\|_2=\int_G |y(g)|^2 dg, \\ Q(P\cdot y)=y, \quad \text{where} \quad (Q\cdot\varphi)(g)=\int_X \varphi(\chi)\chi(-g)d\chi, \end{cases}$$

by (19). Since $L_1(G)\frown L_2(G)$ is dense in L_2(G), (20) is valid also for all $y\in L_2(G)$.

Thus we arrived at the generalised Plancherel's theorem.