148. *On Some Properties of Meromorphic Functions.*

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Consider a class C of meromorphic functions

$$f(z)=\sum_{n=0}^{\infty} b_n z^n \Big/ \sum_{n=0}^{\infty} c_n z^n, \tag{1}$$

where $\sum_{n=0}^{\infty} b_n z^n$ and $\sum_{n=0}^{\infty} c_n z^n$ are integral functions with the following properties:

1) $|b_0|>\varepsilon>0,\ |c_0|>\varepsilon'>0,$ and $|b_0-c_0|>\varepsilon''>0,$ (2)

2) $|b_n|<L_n$ and $|c_n|<L'_n$ for $n=0, 1, 2,$ (3)

where L_n and L'_n are positive numbers such that $\sum_{n=0}^{\infty} L_n z^n$ and $\sum_{n=0}^{\infty} L'_n z^n$ are also integral functions,

3) of the two sets of inequalities

$$\left.\begin{array}{ll} \text{i)} & 0<l_n<|b_n| \text{ for } n=n_1, n_2, \cdots\cdots\cdots \\ \text{ii)} & 0<l'_{n'}<|c_{n'}| \text{ for } n'=n'_1, n'_2, \cdots\cdots \end{array}\right\} \tag{4}$$

where l_n and $l'_{n'}$ are any positive constants for a given sequence of suffixes $n=n_1, n_2,\cdots\cdots\cdots$ and $n'=n'_1, n'_2\cdots\cdots\cdots$ respectively, at least one is satisfied.

Then we have the following

Theorem: There exists an infinite number of concentric ring-regions $|z|<R_1$ *and* $R_i<|z|<R_{i+1}$ $(i=1, 2,\cdots\cdots)$, R_i *depending only on the class* C, *in which all the functions* (1) *take at least p times the value* 1, *or q times zero, or have r poles.*

Proof.[1] From (2) and (3) it follows that

$$|f^{(n)}(0)|<L''_n \text{ for } n=0, 1, 2,\cdots\cdots \tag{5}$$

where L''_n are finite quantities depending on L_n, L'_n and ε'. First, there

1) A more detailed proof and allied theorems will appear in Proc. Phy-Math. Soc. Japan, Ser. (3), 8 (1926).

exists a circle $|z| < R_1$ in which all the functions (1) assume at least p times the value 1, or q times 0, or have r poles. For, if not, the functions (1) must form by Montel's theorem[1] a quasi-normal family, so that there must exist a limiting function $\varphi(z)$, to which a sequence of functions suitably chosen from (1) coverges uniformly in a circle $|z| < R$ except at a finite number of points. R being arbitrary, $\varphi(z)$ is meromorphic in the whole z-plane, and neither reduces to a rational function nor to the constant ∞ by (4) and (5), But this contradicts Picard's theorem.

We now show the existence of a number R_2, such that all the functions (1) take at least p times the value 1, or q times 0, or have r poles in the ring $R_1 < |z| < R_2$. For, if not, the functions (1) must form a quasi-normal family outside the circle $|z| = R_1$, and they have a limited number of zeros, 1-points and poles in the circle of radius $R < R_1$, these being respectively the zeros of $f_1(z) = \sum_{n=0}^{\infty} b_n z^n$, $f_2(z) = \sum_{n=0}^{\infty} (b_n - c_n) z^n$ and $f_3(z) = \sum_{n=0}^{\infty} c_n z^n$ in $|z| < R$, which, since $f_1(z)$, $f_2(z)$ and $f_3(z)$ are " dominated " by $\sum_{n=0}^{\infty} (L_n + L'_n) z^n$, by Jensen's formula[2] must ultimately exceed in absolute value

$$R \sqrt[n+1]{|f(0)|} \Big/ \sqrt[n+1]{\underset{|z|=R}{\mathrm{Max}} \left| \sum_{n=0}^{\infty} (L_n + L'_n) z^n \right|} .$$

There being thus only a limited number of the zeros, 1-points and poles of the functions (1) in $|z| < R-\delta$, where δ is arbitrarily small, they form a quasi-normal family in $|z| < R-\delta$, and since R is arbitrary, we are led to the same contradiction as before, and our theorem is proved.

Remark: For integral functions $f(z) = \sum_{n=0}^{\infty} b_n z^n$, where

$$b_0 = \frac{1}{2}, \quad \left(\frac{e\rho'}{n}\right)^{\frac{1}{\rho'}} < |b_n| < \left(\frac{e\rho}{n}\right)^{\frac{1}{\rho}}, \quad \rho \geqq \rho' > 0,$$

we can find an expression of the radii of the rings in which $f(z)$ takes at least once the value 1 or the value 0 as in the above theorem. For this purpose we must adopt a different method. After Landau we have for the first $n+1$ η-points $z_0, z_1, \cdots\cdots z_n$ of $f(z)$ for $|z| < R$ the inequality:

$$|z_0 z_1 \cdots\cdots z_n| \geqq \frac{M(R)|f(0) - \eta|}{|M(R)^2 - \bar{\eta} f(0)|} R^{n+1}, \tag{6}$$

where $M(R)$ denotes the maximum of $|f(z)|$ for $|z| \leqq R$.

1) Bull. Soc. Math. France, 52, (1924) 85.

2) *Cf.* Bieberbach, Enzyklopädie d. Math. Wissenschaften, Band II 3, 506.

Putting $|z_0| \leqq |z_1| \leqq \cdots\cdots \leqq |z_n|$ in (6), we have for both 1-points and 0-points

$$|z_n| \geqq \sqrt[n+1]{\frac{1}{2M(R)}}\,R.$$

Now $M(R) < e^{R^{\rho}}$ by (3) for $R \geqq R_0$, R_0 being a fixed constant, so that we have

$$|z_n| \geqq \left(\frac{n}{e\rho}\right)^{\frac{1}{\rho}} \sqrt[n+1]{\frac{1}{2}}.$$

Hence in the circle of radius

$$R'_n = \left(\frac{n}{e\rho}\right)^{\frac{1}{\rho}} \sqrt[n+1]{\frac{1}{2}} - \delta,$$

where δ denotes a positive quantity, there exist at most n zeros and n 1-points of $f(z)$.

On the other hand, by Bieberbach's theorem[1], that there exists a circle $|z| < R''_n$ in which all the functions $f(z) = \sum_{n=0}^{\infty} b_n z^n$ with the conditions

$$|b_i| < \left(\frac{i}{e\rho}\right)^{\frac{1}{\rho}} (i=0,\,1,\,2,\cdots\cdots,\,n-1) \text{ and } |b_n| > \left(\frac{n}{e\rho'}\right)^{\frac{1}{\rho'}}$$

have at least n zeros or n 1-points, we find after somewhat long calculation for all $n > n_0$, such that $R''_{n_0} > \overline{R}_0$, ($\overline{R}_0$ being a certain fixed constant of which the exact value can be determined,)

$$R''_n = \frac{2^{mn}(n+1)^{2m-1}\left((e\rho)^{\frac{1}{\rho}} + \frac{1}{2}\right)^m}{2\pi\left\{1 - \frac{1}{4(2n+1)}\right\}^{mn+1}\left(\frac{e\rho'}{mn+1}\right)^{\frac{1}{\rho'}}} + \delta, \tag{8}$$

where $m = 5^{4\pi^{4(2n+1)-1}}$ and δ is positive.

By a wellknown theorem of Landau[2] we can find a circle $|z| < R$ in which $f(z)$ takes at least once the value 1 or 0. Then the radius R of the ring $R_1 < |z| < R_2$ can be found in the following way.

Determine R'_{x_1} in (7) so that

$$R'_{x_1} \geqq R_1,\ R'_{x_1} > R_0,\ R'_{x_1} > \overline{R}_0, \tag{9}$$

where R_0 and $\overline{R}_0$ are the quantities given above and x_1 a positive integer.

1) Math. Ann. 85, 141.

2) Götting. Nachr. 1910, 303.

x_1 being the least integer, satisfying the inequalities (9), we can take R''_{x_1+1} in (8) as R_2. Similarly R''_{x_2+1} in (8) can be taken as the radius R_3 of the ring $R_2 < |z| < R_3$, R'_{x_2} in (7) satisfying $R'_{x_2} \geqq R_2$, and so on.

Thus we have for sufficiently large values of p,

$$R_p = \left[\frac{2^{st+1}(s+1)^{2t-1}\left((e\rho)^{\frac{1}{\rho}} + \frac{1}{2}\right)^t}{2\pi\left(1 - \frac{1}{4(2t+1)}\right)^{st+1}\left(\frac{e\rho'}{st+1}\right)^{\frac{1}{\rho'}}} \right] + 1,$$

where $t = 5^{4\pi^{4(2s+1)-1}}$,

$$S = e\rho\left[\frac{2^{pq+1}(p+1)^{2q-1}\left((e\rho)^{\frac{1}{\rho}} + \frac{1}{2}\right)^q}{2\text{-}\left\{1 - \frac{1}{4(2p+1)}\right\}^{pq+1}\left(\frac{e\rho'}{pq+1}\right)^{\frac{1}{\rho'}}} \right], \quad q = 5^{4\pi^{4(2p+1)-1}}.$$