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1. Approximation of an Irrational Number by Rational Numbers.

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Let ω be any positive irrational number, whose expansion into simple continued fraction is represented by

$$\omega = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots\cdots \frac{1}{+\,a_n} + \cdots\cdots = [a_0 a_1 a_2 \cdots\cdots a_n \cdots].$$

If $\dfrac{P_n}{Q_n} = [a_0 a_1 a_2 \cdots\cdots a_n]$ be the n-th convergent, and

$$S_n = Q_n^2 \left| \omega - \frac{P_n}{Q_n} \right|,$$

then the classical theorem due to HURWITZ and BOREL can be expressed by

$$\text{Mini}\,(S_{n-1}, S_n, S_{n+1}) < \frac{1}{\sqrt{5}},$$

and

$$\text{Mini}\,(S_{n-1}, S_n, S_{n+1}) < \frac{1}{\sqrt{8}}, \text{ if } a_{n+1} = 2.$$

I have extended[1] this theorem in the form

$$\text{Mini}\,(S_{n-1}, S_n, \cdots\cdots, S_{n+3}) < \frac{5}{\sqrt{221}},$$

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1) Bemerkung zur Theorie der Approximation der irrationalen Zahl durch rationale Zahlen, Science Reports of the Tôhoku Imperial University, Ser. I, **14** (1924). See also the Japanese Journal of Mathematics, **1** (1924).

if $a_{n+1}=2, a_{n+2}=1$ and $(a_{n+1}a_{n+2}\cdots\cdots a_{n+8}) \neq (21122112)$. Recently[1] FUKASAWA proved these theorems and their extensions by means of KLEIN's geometrical interpretation of continued fraction and completed my theorem into the more precise form :

$$\text{Mini}\,(S_{n-1}, S_n, S_{n+3}) < \frac{5}{\sqrt{221}}, \text{ if } a_{n+1}=2, a_{n+2}=1.$$

Suggested by these results obtained by FUKASAWA I have returned again to our problem and found that my former method is also capable of giving extended theorems of similar kind.

If $m-n$ be odd, then, since $\omega-\frac{P_n}{Q_n}$, $\omega-\frac{P_m}{Q_m}$ are of different signs, we have

$$\frac{Q_m}{Q_n}S_n+\frac{Q_n}{Q_m}S_m = |P_mQ_n-P_nQ_m| = Q_{n,m}, \tag{1}$$

where

$$\frac{P_{n,m}}{Q_{n,m}} = [a_{n+1}a_{n+2}\cdots\cdots a_m].$$

Hence we get

$$\frac{Q_m}{Q_n} = Q_{nm}\frac{1+\sqrt{1-4\,S_nS_m/Q_{nm}{}^2}}{2\,S_n} \tag{2}$$

$$\frac{Q_n}{Q_m} = Q_{nm}\frac{1-\sqrt{1-4\,S_nS_m/Q_{nm}{}^2}}{2\,S_m} \tag{3}$$

Associating with (2)

$$\frac{Q_{n-1}}{Q_n} = \frac{1-\sqrt{1-4\,S_{n-1}S_n}}{2\,S_n},$$

we have

$$\frac{\sqrt{1-4\,S_{n-1}S_n}+\sqrt{1-4\,S_nS_m/Q_{nm}^2}}{2\,S_n} = \frac{Q_m}{Q_nQ_{nm}}-\frac{Q_{n-1}}{Q_n} = \frac{P_{nm}}{Q_{nm}}. \tag{4}$$

More generally, from (3) and

$$\frac{Q_l}{Q_m} = Q_{ml}\frac{1+\sqrt{1-4\,S_mS_l/Q_{ml}^2}}{2\,S_m}$$

where $l-m$ is odd, we can deduce the relation

$$\frac{\sqrt{1-4\,S_nS_m/Q_{nm}^2}+\sqrt{1-4\,S_mS_l/Q_{ml}^2}}{2\,S_m} = \frac{Q_l}{Q_mQ_{ml}}-\frac{Q_n}{Q_mQ_{nm}} = \frac{Q_{nl}}{Q_{nm}Q_{ml}}. \tag{5}$$

If we put Mini $(S_n, S_m, S_l) = S$, then it follows

$$\frac{Q_{nl}}{Q_{nm}Q_{ml}} \leqq \frac{\sqrt{1-4\,S^2/Q_{nm}^2}+\sqrt{1-4\,S^2/Q_{ml}^2}}{2\,S},$$

where the equality holds good only when $S_n=S_m=S_l$. In this case it

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1) FUKASAWA's Paper will appear in the Japanese Journal of Mathematics, 2 (1925)

results from (1) that S_n must be rational, contradictory to the supposition that ω is irrational. Therefore we have finally, solving for S

$$S < \left\{\left(\frac{Q_{nm}^2 + Q_{nl}^2 + Q_{ml}^2}{Q_{nm}Q_{nl}Q_{ml}}\right)^2 - \frac{4}{Q_{nl}^2}\right\}^{-\frac{1}{2}}. \tag{6}$$

Similarly we have from (4)

$$\text{Mini}\,(S_{n-1}, S_n, S_m) < \left\{\left(\frac{1+P_{nm}^2+Q_{nm}^2}{P_{nm}Q_{nm}}\right)^2 - \frac{4}{P_{nm}^2}\right\}^{-\frac{1}{2}}, \tag{7}$$

a result obtained by FUKASAWA by a simple geometrical consideration.

Let especially

$$\frac{P}{Q} = \Big[2\,\overbrace{11\ldots\ldots11}^{2k}\Big].,$$

then it can be proved by mathematical induction that

$$1+P^2+Q^2 = 3PQ,$$

whence we have

$$\text{Mini}\,(S_{n-1}, S_n, S_{n+2k+1}) < \left(9 - \frac{4}{P^2}\right)^{-\frac{1}{2}},$$

for example : $P = 2$ for $k = 0$, $P = 5$ for $k = 1$, $P = 13$ for $k = 2$.

If we put

$$\frac{P_{nl}}{Q_{nl}} = \Big[2\,\overbrace{11\ldots\ldots11}^{2k}\,22\,\overbrace{11\ldots\ldots11}^{2k}\,2\Big],$$

$$\frac{P_{nm}}{Q_{nm}} = \Big[2\,\overbrace{11\ldots\ldots11}^{2k}\Big],$$

$$\frac{P_{ml}}{Q_{ml}} = \Big[22\,\overbrace{11\ldots\ldots11}^{2k}\,2\Big],$$

then it is not difficult to show that

$$Q_{nm}^2 + Q_{nl}^2 + Q_{ml}^2 = 3\,Q_{nm}Q_{nl}Q_{ml}, \tag{8}$$

whence we have

$$\text{Mini}\,(S_n, S_m, S_l) < \left(9 - \frac{4}{Q_{nl}^2}\right)^{-\frac{1}{2}}, \quad l = n+4k+6,\ m = n+2k+3,$$

for example :

$$k = 0,\ Q_{nl} = 29,\ Q_{nm} = 2,\ Q_{ml} = 5;$$
$$k = 1,\ Q_{nl} = 194,\ Q_{nm} = 5,\ Q_{ml} = 13.$$

As regards the so-called MARKOFF's equation (8), we refer to the papers: MARKOFF, Math. Ann., 17 (1880); FROBENIUS, Berliner Sitzungsber. 1914.