72. *Normed Rings and Spectral Theorems.*

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1. *Introduction.* The purpose of the present note is to show that the simultaneous resolution of the identity for commutative ring of normal operators may easily be deduced from the theory of normed ring[1] by making use of two elementary lemmas in operator theory and of Baire's category theorem. As in the preceding notes[2], our treatment is rather algebraical and integration-free.

2. *Preliminaries.* Let $\boldsymbol{B}$ denote the totality of bounded linear operators in a general euclid space $\mathfrak{H}$. Let a subset $\boldsymbol{A}$ of $\boldsymbol{B}$ satisfy

(1) the commutativity: $TS=ST$ for $T, S \in \boldsymbol{A}$

and

(2) the conjugated condition: if $T \in \boldsymbol{A}$, then its adjoint T^* also $\in \boldsymbol{A}$.

Let $\boldsymbol{A}'$ denote the totality of operators $\in \boldsymbol{B}$ that commute with every operator of $\boldsymbol{A}$, then it is easy to see that $\boldsymbol{R}=\boldsymbol{A}''=(\boldsymbol{A}')'$ is a commutative, conjugated ring with unit and with complex multipliers. $\boldsymbol{R}$ is a normed ring[3] by the norm $\|T\| = \sup_{|f| \leq 1} \|T \cdot f\|$. Moreover it is easy to see that $\boldsymbol{R}$ is closed in the sense of the *strong convergence*.

$$(3) \quad \begin{cases} \text{let a sequence } \{T_n\} \subseteqq \boldsymbol{R} \text{ be such that } \operatorname{strong\,lim}_{n\to\infty} T_n \text{ exist}^{4)}, \\ \text{then the operator } T = \operatorname{strong\,lim}_{n\to\infty} T_n \text{ also belongs to } \boldsymbol{R}. \end{cases}$$

Lemma 1[5]. Let $H \in \boldsymbol{B}$ be hermitian, viz. $H=H^*$, then

$$(4) \quad \|H\| = \sup_{|f|\leq 1} \|H \cdot f\| = \sup_{|f|\leq 1} |(H\cdot f, f)|\,.$$

Lemma 2[6]. Let $T \in \boldsymbol{B}$ and let I denote the identity operator, then $(I+TT^*)$ admits the inverse $(I+TT^*)^{-1} \in \boldsymbol{B}$. It is easy to see that $(I+TT^*)^{-1} \in \boldsymbol{R}$ in case $T \in \boldsymbol{R}$.

From lemma 1 we obtain

$$(5) \quad \|T\| = \|T^*\|,\ \|T^2\| = \|T\|^2 \quad \text{for every} \quad T \in \boldsymbol{R}\,.$$

Proof. We have, by (4), since $H=TT^*$ is hermitian $\left(=(TT^*)^*\right)$, $\|T\|^2 = \sup_{|f|\leq 1} (T\cdot f, T\cdot f) = \sup_{|f|\leq 1} |(T^*T\cdot f, f)| = \|T^*T\| = \|TT^*\| = \|H\|$. Thus $\|T\|^2 = \|T^*\|^2$ and $\|T^2\|^2 = \|T^{*2}T^2\| = \|(T^*T)^2\|$ by the commutativity of $\boldsymbol{R}$.

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1) I. Gelfand: Rec. Math., **9** (1941).
2) K. Yosida and T. Nakayama: Proc., **18** (1942) and **19** (1943).
3) $\boldsymbol{R}$ is a Banach space by the norm $\|T\|$ and satisfies $\|TS\| \leqq \|T\| \cdot \|S\|$.
4) $\operatorname{strong\,lim}_{n\to\infty} T_n = T$ means that $\lim_{n\to\infty} T_n f = T f$ strongly for every $f \in \mathfrak{H}$.
5) See, for example, F. J. Murray's book: Princeton (1941), 41.
6) Murray s book, 42.

Again, by (4), we have $\|H\|^2=\sup\limits_{|f|\le 1}(H\cdot f, H\cdot f)=\sup\limits_{|f|\le 1}|(H^2\cdot f,f)|=\|H^2\|$. Therefore $\|T^2\|^2=\|(T^*T)^2\|=(\|T\|)^4$ and hence $\|T^2\|=\|T\|^2$. Q. E. D.

From (5) we have

$$(6)\quad \|T\|=\|T^*\|=\lim_{n\to\infty}\sqrt[n]{\|T^n\|}\quad \text{for}\quad T\in \boldsymbol{R}.$$

Proof. At first $\|T\|=\lim\limits_{n\to\infty}\sqrt[2n]{\|T^{2n}\|}$ by (5). And since $\|T^{2n+1}\|\leqq \|T^{2n}\|\cdot\|T\|$, $\|T^{2n+2}\|\leqq\|T^{2n+1}\|\cdot\|T\|$, we must have (6). Q. E. D.

The ring $\boldsymbol{R}$ is, by (6), *semi-simple* viz. $\boldsymbol{R}$ does not contain a *generalised nilpotent element*: $\lim\limits_{n\to\infty}\sqrt[n]{\|T^n\|}\neq 0$ if $T\neq 0$. Hence[1], $\boldsymbol{R}$ is ring-isomorphic (with complex multipliers) to the function ring $\boldsymbol{R}(\mathfrak{M})$ of complex-valued continuous functions $T(M)$ on the bicompact Hausdorff space $\mathfrak{M}$ which consists of the totality of the maximal ideals of $\boldsymbol{R}$:

$$(7)\quad \begin{cases} T\leftrightarrow T(M),\quad I(M)\equiv 1 \quad \text{on}\quad \mathfrak{M},\quad \sup\limits_{M}|T(M)|=\lim\limits_{n\to\infty}\sqrt[n]{\|T^n\|}=\|T\| \\ \text{(by (6)).} \end{cases}$$

We next show that the above correspondence $T\leftrightarrow T(M)$ satisfies the following two conditions.

$$(8)\quad \begin{cases} T(M) \text{ is real-valued on } \mathfrak{M} \text{ if and only if } T \text{ is hermitian.} \\ \quad\text{Thus, in particular, } T^*(M)=\overline{T(M)} \text{ (bar indicates complex} \\ \quad\text{conjugate).} \end{cases}$$

$$(9)\quad \begin{cases} T(M) \text{ is non-negative on } \mathfrak{M} \text{ if and only if } T \text{ is non-negative} \\ \quad\text{definite}^{2)}. \end{cases}$$

Proof of (8). Let $T\in\boldsymbol{R}$ be hermitian and let $T(M_0)=a+ib$, $i=\sqrt{-1}$, with $b\neq 0$. Then the hermitian operator $S=(T-aI)/b\in\boldsymbol{R}$ satisfy $(I+S^2)\ (M_0)=1+i^2=0$. Thus the function $(I+S^2)\ (M)$ and hence, by the isomophism $\boldsymbol{R}\leftrightarrow\boldsymbol{R}(\mathfrak{M})$, the operator $(I+S^2)$ does not have inverse, contrary to lemma 2. Next let $T\in\boldsymbol{R}$ be not hermitian and let it be decomposed into hermitian components: $T=\dfrac{T+T^*}{2}+i\dfrac{T-T^*}{2i}$, $\dfrac{T-T^*}{2i}\neq 0$. Then, by the isomorphism $\boldsymbol{R}\leftrightarrow\boldsymbol{R}(\mathfrak{M})$, $\dfrac{T-T^*}{2i}(M)$ is not identically zero on $\mathfrak{M}$. Q. E. D.

For the proof of (9) we need the

Lemma 3. $\boldsymbol{R}(\mathfrak{M})$ is the totality of the complex-valued continuous functions on $\mathfrak{M}$.

Proof. The function ring $\boldsymbol{R}(\mathfrak{M})$ satisfies i) for any pair M_1, $M_2\in\mathfrak{M}$ there exists $T(\mathfrak{M})\in\boldsymbol{R}(\mathfrak{M})$ such that $T(M_1)\neq T(M_2)$, ii) for any $T(M)\in\boldsymbol{R}(\mathfrak{M})$ there exists complex-conjugated function $T^*(M)=T(M)$ $\in\boldsymbol{R}(\mathfrak{M})$ (by (8)). Thus, by Gelfand-Silov's abstraction of Weierstrass' polynomial approximation theorem[3], every continuous function on $\mathfrak{M}$

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1) by Gelfand's theorem, loc. cit.

2) viz. $(T\cdot f,f)\geqq 0$ for every $f\in\mathfrak{H}$.

3) Rec. Math., **9** (1941). Cf. also H. Nakano: 全國紙上數學談話會, **218** (1941).

may be uniformly approximated by functions $\epsilon\, R(M)$. Since $\boldsymbol{R}$ is complete by the norm $\|T\|$ which satisfies (7) we have the lemma.

Q. E. D.

Proof of (9). Let $T(M)\geqq 0$ on $\mathfrak{M}$, then the function $S(M)=\sqrt{T(M)}$ belongs to $\boldsymbol{R}(\mathfrak{M})$ by the lemma 3. Hence, by the isomorphism $\boldsymbol{R}\leftrightarrow\boldsymbol{R}(\mathfrak{M})$ and (8), we have $T=S^2$, $S=S^*\epsilon\,\boldsymbol{R}$. Thus $(T\cdot f,f)=(S^2\cdot f,f)=(S\cdot f, S\cdot f)\geqq 0$. Conversely let T be hermitian. We have, by (8), lemma 3, the isomorphism $\boldsymbol{R}\leftrightarrow\boldsymbol{B}(\mathfrak{M})$ and the result just above obtained, $T=T_1-T_2$, $T_1\cdot T_2=0$, T_i=non-negative definite, $i=1,2$. For the proof we have only to put $T_1(M)=\sup\big(T(M),0\big)$, $T_2(M)=T_1(M)-T(M)$. Let, moreover, T be non-negative definite, then we have, by the above, $0\leqq(TT_2\ f, T_2\cdot f)=(-T_2^2\cdot f, T_2\cdot f)=(-T_2^3\cdot f,f)$, $0\leqq(T_2T_2\cdot f, T_2\cdot f)=(T_2^3\cdot f,f)$ for $f\,\epsilon\,\mathfrak{H}$. Thus $(T_2^3\cdot f,f)=0$ for $f\,\epsilon\,\mathfrak{H}$ and hence $T_2^3=0$ by (4). Since $\boldsymbol{R}$ is semi-simple, we have $T_2=0$. Therefore $T(M)=T_1(M)\geqq 0$ on $\mathfrak{M}$, as was to be proved. Q. E. D.

Summing up we have the

Theorem. $\boldsymbol{R}$ is ring-isomorphic (with complex multipliers) to the function ring $\boldsymbol{R}(\mathfrak{M})$ of all the complex-valued continuous functions on a bicompact Hausdorff space $\mathfrak{M}$ such that (7), (8) and (9) hold good.

3. *The spectral resolution.* The simultaneous resolution of the identity of $\boldsymbol{R}$ may now be obtained by essentially the same idea as in the preceding notes referred to above. For the sake of completeness we will repeat it.

Lemma 4[1]. Let a sequence $\{T_n\}$ of hermitian operators $\epsilon\,\boldsymbol{R}$ be such that $T_1\leqq T_2\leqq\cdots\leqq T_n\leqq\cdots\leqq S\,\epsilon\,\boldsymbol{B}$, then the $\operatorname{strong}\lim_{n\to\infty} T_n=T$ exists. We have $T\epsilon\,\boldsymbol{R}$ by (3). Here $T_n\leqq S$ means that $(S-T_n)$ is non-negative definite.

From lemma 4, theorem and Baire's category theorem we obtain the

Lemma 5[2]. Let a sequence $\{T_n(M)\}$ of real-valued functions $\epsilon\,\boldsymbol{R}(\mathfrak{M})$ be such that $T_1(M)\leqq T_2(M)\leqq\cdots\leqq T_n(M)\leqq\cdots\leqq$ a constants all over $\mathfrak{M}$. Then, if we put $T=\operatorname{strong}\lim_{n\to\infty} T_n$ (its existence is assured by lemma 4 and theorem), we have $T(M)=\lim_{n\to\infty} T_n(M)$ on $\mathfrak{M}$ except possibly on a set of first category.

Now consider the set $\boldsymbol{R}'(\mathfrak{M})$ of all the complex-valued bounded functions $X'(M)$ on $\mathfrak{M}$ such that $X'(M)$ is different from a continuous function $X(M)$ only on a set of first category. We identify two functions from $\boldsymbol{R}'(\mathfrak{M})$ if they differ on a set of first category, then $\boldsymbol{R}'(\mathfrak{M})$ is divided into classes. Since the complementary to a set of first category is dense on a bicompact Hausdorff space, each class X' contains exactly one continuous function $X(M)$ which corresponds, by the isomorphism $\boldsymbol{R}\leftrightarrow\boldsymbol{R}(\mathfrak{M})$, to an element $X\epsilon\,\boldsymbol{R}$.

For any $T\epsilon\,\boldsymbol{R}$ and for any complex number $z=\lambda+i\mu$, we put E_z=the element $\epsilon\,\boldsymbol{R}$ which corresponds to the class E_z' containing the

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1) Yosida and Nakayama: Proc., **18**, loc. cit., 560.
2) Yosida and Nakayama: Proc., **18**, loc. cit., 559.

character isticfunction $E'_z(M)$ of the set $\underset{M}{\mathfrak{E}}\{\Re T(M) \leqq \lambda, \Im T(M) \leqq \mu\}$[1]. We have then

$$\Big| T(M) - \sum_{j=1}^{n} (\lambda_j + i\mu_j)\Big(E'_{\lambda_j+i\mu_j}(M) + E'_{\lambda_{j-1}+i\mu_{j-1}}(M) - \\ E'_{\lambda_{j-1}+i\mu_j}(M) - E'_{\lambda_j+j\mu_{j-1}}\Big)(M)\Big| \leqq \varepsilon$$

on $\mathfrak{M}$, if $\lambda_2 = -\alpha - \frac{\varepsilon}{\sqrt{2}} \leqq \lambda_2 \leqq \cdots \leqq \lambda_n = \alpha = \sup_M |\Re T(M)|$, $\mu_1 = -\beta - \frac{\varepsilon}{\sqrt{2}} \leqq \mu_2 \leqq \cdots \leqq \mu_n = \beta = \sup_M |\Im T(M)|$, $\left(\sup_j (\lambda_j - \lambda_{j-1})^2 + \sup_j (\mu_j - \mu_{j-1})^2\right)^{\frac{1}{2}} \leqq \varepsilon$. Thus, by the definition of $X'(M)$ and lemma 5, we have

$$\Big| T(M) - \sum_{j=1}^{n} (\lambda_j + i\mu_j)\Big(E_{\lambda_j+i\mu_j}(M) + E_{\lambda_{j-1}+i\mu_{j-1}}(M) - \\ E_{\lambda_{j-1}+i\mu_j}(M) - E_{\lambda_j+i\mu_{j-1}}(M)\Big)\Big| \leqq \varepsilon$$

on $\mathfrak{M}$, since the complementary to a set of first category is dense on a bicompact Hausforff space. Therefore, by theorem, we obtain

$$\Big\| T - \sum_{j=1}^{n} (\lambda_j + i\mu_j)\Big(E_{\lambda_j+i\mu_j} + E_{\lambda_{j-1}+i\mu_{j-1}} - E_{\lambda_{j-1}+i\mu_j} - E_{\lambda_j+i\mu_{j-1}}\Big)\Big\| \leqq \varepsilon .$$

We have thus arrived at the desired resolution of the identity $T = \int z dE_z$.

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1) $\Re a$ and $\Im a$ denote respectively the real and imaginary part of a.