13. On the Cluster Set of a Meromorphic Function.

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1. Let $\varDelta$ be a bounded domain on the z-plane and z_0 be a non-isolated accessible boundary point on the boundary Γ of $\varDelta$. We denote the part of $\varDelta, \Gamma$ in $|z-z_0| \leqq r$ by $\varDelta_r, \Gamma_r$ respectively and the part of $|z-z_0|=r$, which lies in $\varDelta$ by θ_r. Let $w=f(z)$ be one-valued and meromorphic in $\varDelta$ and W_r be the set of values taken by $f(z)$ in $\varDelta_r$ and $\overline{W}_r$ be its closure. Then

$$\lim_{r\to 0} \overline{W}_r = H_\varDelta(z_0) \tag{1}$$

is called the cluster set of $f(z)$ in $\varDelta$ at z_0.

Let $\zeta(\neq z_0) \in \Gamma$ and $H_\varDelta(\zeta)$ be the cluster set of $f(z)$ at ζ and

$$V_r(\Gamma) = \sum H_\varDelta(\zeta), \quad \text{added for all } \zeta(\neq z_0) \text{ on } \Gamma_r \tag{2}$$

and $\overline{V}_r(\Gamma)$ be its closure. Then

$$\lim_{r\to 0} \overline{V}_r(\Gamma) = H_\Gamma(z_0) \tag{3}$$

is called the cluster set of $f(z)$ on Γ at z_0.

It is obvious that $H_\varDelta(z_0) \supset H_\Gamma(z_0)$. Iversen[1] proved that *every boundary point of $H_\varDelta(z_0)$ belongs to $H_\Gamma(z_0)$.*

Let $\zeta \in \Gamma$. If for any $\varepsilon > 0$, there exists a neighbourhood U of ζ, such that $|f(z)| \leqq m+\varepsilon$ in U, then we will write: $|f(\zeta)| \leqq m$. Then as an immediate consequence of the Iversen's theorem, we have[2]: *Let $f(z)$ be regular and bounded in $\varDelta$. If $\overline{\lim_{z\to z_0}} |f(z)| \leqq m$, when z tends to z_0 on Γ, then $\overline{\lim_{z\to z_0}} |f(z)| \leqq m$, when z tends to z_0 in $\varDelta$.*

I will here extend the Iversen's theorem in the following way.

Let E be a closed set of capacity zero on Γ and $z_0 \in E$ and $U(\Gamma-E) \neq 0$ for any neighbourhood U of z_0. We denote the part of E in $|z-z_0| \leqq r$ by E_r. Let

$$V_r(\Gamma-E) = \sum H_\varDelta(\zeta), \quad \text{added for all } \zeta(\neq z_0) \text{ on } \Gamma_r - E_r \tag{4}$$

and $\overline{V}_r(\Gamma-E)$ be its closure. Then

1) F. Iversen: Sur quelques propriétes des fonctions monogènes au voisinage d'un point singulier. Öfv. af Finska Vet-Soc. Förh. **58** (1916).

K. Kunugui: Sur un théorème de MM. Seidel-Beurling. Proc. **15** (1939),—Sur l'allure d'une fonction analytique uniforme au voisinage d'un point frontière de son domaine de définition. Jap. Jour. Math. **18** (1942).

K. Noshiro: On the theory of cluster sets of the analytic functions. Jour. Fac. Sci. Hokkaido Imp. Univ. **6** (1938),—On the singularities of analytic functions. Jap. Jour. Math. **17** (1940).

2) K. Kunugui, l. c.

$$\lim_{r\to 0} \overline{V}_r(\Gamma - E) = H_{\Gamma - E}(z_0) \tag{5}$$

is called the cluster set of $f(z)$ on $\Gamma - E$ at z_0.

We will prove

Theorem I. Every boundary point of $H_\Delta(z_0)$ belongs to $H_{\Gamma-E}(z_0)$.

Corollary. Let $f(z)$ be regular and bounded in a bounded domain Δ, z_0 be a non-isolated accessible boundary point on the boundary Γ of Δ and E be a closed set of capacity zero on Γ and $z_0 \in E$ and $U(\Gamma - E) \neq 0$ for any neighbourhood U of z_0. If $\overline{\lim\limits_{z\to z_0}} |f(z)| \leqq m$, when z tends to z_0 on $\Gamma - E$, then $\overline{\lim\limits_{z\to z_0}} |f(z)| \leqq m$, when z tends to z_0 in Δ.

2. We will prove some lemmas.

Lemma 1. Let $f(z)$ be regular and bounded in a bounded domain Δ and $|f(z)| \leqq m$ on the boundary Γ of Δ, except at points of a closed set E of capacity zero on Γ, then $|f(z)| \leqq m$ in Δ.

Proof. Though this is a well known fact, we will prove it for the sake of completeness. By Evans' theorem[1], there exists a positive harmonic function $v(z)$ in Δ, which is ∞ at every point of E.

Consider the domain Δ_k, which is bounded by Γ and the niveau curves: $v(z) = \text{const.} = k$ and z' be any point of Δ. Then for a large k, z' is contained in Δ_k. Now for any $\varepsilon > 0$, $U(z) = \log|f(z)| - \log(m+\varepsilon) - \varepsilon v(z) < 0$ on the boundary of Δ_k for a sufficiently large k. Since $U(z)$ is subharmonic in Δ_k, $U(z') < 0$. Since ε is arbitrary, we have $|f(z')| \leqq m$, q.e.d.

Lemma 2. Let $f(z)$ be regular and bounded in a bounded domain Δ and z_0 be a non-isolated accessible boundary point on the boundary Γ of Δ, which is regular for the Dirichlet problem for Δ and let E be a closed set of capacity zero on Γ and $z_0 \in E$.

If $\overline{\lim\limits_{z\to z_0}} |f(z)| \leqq m$, when z tends to z_0 on $\Gamma - E$, then $\overline{\lim\limits_{z\to z_0}} |f(z)| \leqq m$, when z tends to z_0 in Δ.

Proof. Let $|f(z)| \leqq M$ in Δ. We may suppose $m < M$. We take ε so small that $m + \varepsilon \leqq M$ and then we choose r so that $|f(z)| \leqq m + \varepsilon$ on $\Gamma_{2r} - E_{2r}$ and $U(z)$ be the solution of the Dirichlet problem for Δ with the continuous boundary values, such that $U(z) = \log(m+\varepsilon)$ on Γ_r, $U(z) = \dfrac{(2r - |z - z_0|)\log(m+\varepsilon)}{r} + \dfrac{(|z - z_0| - r)\log M}{r}$ on $\Gamma_{2r} - \Gamma_r$ and $U(z) = \log M$ on $\Gamma - \Gamma_{2r}$. Let e be the set of irregular points on Γ, then e is of capacity zero and Brelot[2] proved that e is F_σ and that there exists a positive harmonic function $v_1(z)$ in Δ, which is ∞ at every point of e. Consider as before, $\log|f(z)| - U(z) - \varepsilon v(z) - \varepsilon v_1(z)$, where $v(z)$ is the Evans' function with respect to E. Since $U(z)$ takes the given boundary values at the regular points on Γ, we have as before, $\log|f(z)| \leqq U(z)$ in Δ.

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1) Evans: Potentials and positively infinite singularities of harmonic functions. Monathefte f. Math. u. Phys. **43** (1936).

2) Brelot: Problème de Dirichlet et majorantes harmoniques. Bull. Sci. Math. France (1939).

Since z_0 is a regular point, $\lim_{z \to z_0} U(z) = \log(m+\varepsilon)$, when z tends to z_0 in Δ. Since ε is arbitrary, we have $\overline{\lim_{z \to z_0}} |f(z)| \leqq m$, when z tends to z_0 in Δ, q. e. d.

Hence if $\dfrac{1}{f(z)}$ is bounded in Δ and $\underline{\lim_{z \to z_0}} |f(z)| \geqq m$, when z tends to z_0 on $\Gamma - E$, then $\underline{\lim_{z \to z_0}} |f(z)| \geqq m$, when z tends to z_0 in Δ. We use Lemma 2 in this form in § 3.

3. *Proof of Theorem I.* Suppose that there exists a boundary point w_0 of $H_\Delta(z_0)$, which does not belong to $H_{\Gamma-E}(z_0)$. We suppose $w_0 = 0$. We take r so small that $\bar{V}_r(\Gamma - E)$ lies outside $|w| = \rho_1 > 0$ and $|z - z_0| = r$ contains no points of E and zeros of $f(z)$ on it. Then there exists $\rho_2 > 0$, such that $|f(z)| \geqq \rho_2 > 0$ on θ_r. For, otherwise, there would exist points $z'_n \to z'$ on $|z - z_0| = r$, such that $f(z'_n) \to 0$. Since, by the hypothesis, z' does not belong to E and $\Gamma - E$, z' must belong to Δ, so that $f(z)$ is meromorphic at z', hence $f(z') = 0$, which contradicts the hypothesis. Hence there exists $\rho_2 > 0$, such that $|f(z)| \geqq \rho_2 > 0$ on θ_r. Let $\rho = \text{Min.}(\rho_1, \rho_2)$ and consider the image of $|w| < \rho$ in Δ_r, which consists of connected domains $\{\Delta^{(n)}\}$.

By the choice of ρ, $\Delta^{(n)}$ contains no points of $|z - z_0| = r$ and $\Gamma - E$ on its boundary. $\{\Delta^{(n)}\}$ consist of two kind of domains; namely the ones of the first kind, which are bounded by closed curves, which contain no points of E and are mapped on the inside of $|w| < \rho$ and the others of the second kind, which contain points of E on their boundaries.

We will prove that there is one domain Δ_0 among $\{\Delta^{(n)}\}$, which contains z_0 on its boundary. If there is no such a domain, then, since $w = 0$ belongs to $H_\Delta(z_0)$, there are infinitely many $\Delta^{(n)}$, which contain points z_n converging to z_0. We will show that such $\Delta^{(n)}$ converges to z_0. For, otherwise, the boundary of $\Delta^{(n)}$ would certain a point z'_n on a circle: $|z - z_0| = r'$ $(< r)$, which contain no points of E on it and we assume that $\lim_{n \to \infty} z'_n = z'$. Then z' does not belong to E. If $z' \in \Gamma - E$, then $H_{\Gamma-E}(z)$ contains points on $|w| = \rho$, which contradicts the hypothesis. Hence $z' \in \Delta$, so that $f(z)$ is meromorphic at z' and in every neighbourhood of z', there are infinitely many niveau curves: $|f(z)| = \rho$, which is impossible. Hence $\Delta^{(n)}$ converges to z_0. If $\dfrac{1}{f(z)}$ is bounded in $\Delta^{(n)}$, then since $\dfrac{1}{|f(z)|} = \dfrac{1}{\rho}$ on the boundary of $\Delta^{(n)}$, except at points of E, we have by Lemma 1, $\dfrac{1}{|f(z)|} \leqq \dfrac{1}{\rho}$ or $|f(z)| \geqq \rho$ in $\Delta^{(n)}$, which contradicts the definition of $\Delta^{(n)}$. Hence $\dfrac{1}{f(z)}$ is unbounded in $\Delta^{(n)}$. Similarly $\dfrac{1}{f(z) - w_1}$ is unbounded in $\Delta^{(n)}$, where w_1 is any point of $|w| < \rho$. Hence $f(z)$ takes in $\Delta^{(n)}$ values, which are dense[1] in $|w| \leqq \rho$.

1) If $\Delta^{(n)}$ is a domain of the first kind, $f(z)$ takes every value of $|w| < \rho$ in $\Delta^{(n)}$.

Consequently, since $\Delta^{(n)}$ converges to z_0, $|w|\leqq\rho$ belongs to $H_\Delta(z_0)$, which contradicts the hypothesis, that $w=0$ is a boundary point of $H_\Delta(z_0)$. Hence there exists one domain Δ_0 among $\{\Delta^{(n)}\}$, which contains z_0 on its boundary. By the same argument as we have proved that $\Delta^{(n)}$ converges to z_0, we see that z_0 is an accessible boundary point of Δ_0.

Next consider the image of $|w|<\frac{\rho}{2}$ in Δ_0. They consist of connected domains $\{\Delta_1^{(n)}\}$. We will show that there is one domain Δ_1 among $\{\Delta_1^{(n)}\}$, which contains z_0 on its boundary.

Suppose that there is no such a domain. Then, since, as before, there are no infinitely many $\Delta_1^{(n)}$ converging to z_0, there exists a neighbourhood U of z_0 in Δ_0, which contains no points of $\Delta_1^{(n)}$, so that $|f(z)|>\frac{\rho}{2}$ in U. Since $|f(z)|=\rho$ on the boundary of Δ_0, except at points of E, we have by Lemma 2, $\varliminf_{z\to z_0}|f(z)|\geqq\rho$, when z tends to z_0 in Δ_0. Since $|f(z)|<\rho$ in Δ_0, it follows that $\lim_{z\to z_0}|f(z)|=\rho$, when z tends to z_0 in Δ_0, so that $|f(z)|\to\rho$, when z tends to z_0 on a curve C in Δ_0. We take off C from Δ, then z_0 is a regular point for the Dirichlet problem for $\Delta-C$. Let w_1 $\left(|w_1|\leqq\frac{\rho}{3}\right)$ be a point, which does not belong to $H_\Delta(z_0)$, then $\frac{1}{f(z)-w_1}$ is bounded in the neighbourhood of z_0 and $\varliminf_{z\to z_0}|f(z)-w_1|\geqq\rho-|w_1|\geqq\frac{2}{3}\rho$, when z tends to z_0 on C and $\Gamma-E$. Hence by Lemma 2, $\varliminf_{z\to z_0}|f(z)-w_1|\geqq\frac{2}{3}\rho$, when z tends to z_0 in $\Delta-C$ and hence in Δ, so that $|w-w_1|<\frac{2}{3}\rho$ and hence $w=0$ does not belong to $H_\Delta(z_0)$, which contradicts the hypothesis. Hence there is a domain Δ_1 among $\{\Delta_1^{(n)}\}$, which contains z_0 on its boundary.

Similarly considering the images of $|w|<\frac{\rho}{2^n}$ $(n=1, 2, \dots)$ in Δ_0, we see that there exists a curve C ending at z_0, such that $f(z)\to 0$ along C. We take off C from Δ, then z_0 is a regular point for the Dirichlet problem for $\Delta-C$. Then as before, we see that $|w-w_1|<\rho_0$ $(\rho_0=|w_1|)$ does not belong to $H_\Delta(z_0)$. We take a point w_2 in $|w-w_1|<\rho_0$, such that $|w_2|=\rho_0$ and see similarly that $|w-w_2|<\rho_0$ does not belong to $H_\Delta(z_0)$. Continuating similarly we conclude that $|w|=\rho_0$ does not belong to $H_\Delta(z_0)$. But we see easily that $H_\Delta(z_0)$ contains a continuum, which connects $w=0$ and $\Gamma-E$, so that $H_\Delta(z_0)$ contains points on $|w|=\rho_0$, which contradicts the above result. Hence there is no boundary point of $H_\Delta(z_0)$, which does not belong to $H_{\Gamma-E}(z_0)$, q. e. d.

4. By Theorem I, $D=H_\Delta(z_0)-H_{\Gamma-E}(z_0)$ is an open set, which, in general, consists of enumerably infinite number of connected domains $\{D_n\}$.

K. Kunugui[1] proved that $f(z)$ takes any value in D_n, except at most two, infinitely many times in the neighbourhood of z_0, when E consists of only one point z_0. We will prove

Theorem II. *$f(z)$ takes any value in D, except values of capacity zero, infinitely many times in the neighbourhood of z_0.*

Proof. Let $D_1 \subset D$ be a closed domain. Then we take δ so small that $\bar{V}_\delta(\Gamma - E)$ has a positive distance $\geqq \rho_1 > 0$ from D_1. Let $w_0 = 0$ be a point of D_1, which is taken by $f(z)$ finite times in the neighbourhood of z_0.

Then we take $\delta_1 < \delta$ so small that $f(z) \neq 0$ in $\varDelta_{\delta_1}$ and on θ_{δ_1} and $|z - z_0| = \delta_1$ contains no points of E on it. Then as in § 3, $|f(z)| \geqq \rho_2 > 0$ on θ_{δ_1}. Let $\rho = \text{Min.}(\rho_1, \rho_2)$ and $\varDelta_0$ be one of the domains, which are the images of $|w| < \rho$ in $\varDelta_{\delta_1}$. Then $\varDelta_0$ contains no points of $|z - z_0| = \delta_1$ and $\Gamma - E$ on its boundary. Since $f(z) \neq 0$ in $\varDelta_{\delta_1}$, $\varDelta_0$ contains points of E on its boundary. Considering the images of $|w| < \frac{\rho}{2^n}$ $(n = 1, 2, \ldots)$ in $\varDelta_0$, we see that there exists a curve C in $\varDelta_0$ ending at a point ζ_0 on E, such that $f(z) \to 0$ along C. Let $z - \zeta_0 = re^{i\theta}$ and $\varDelta_0(r)$ be the part of $\varDelta_0$, which lies between $|z - \zeta_0| = r_0$ and $|z - \zeta_0| = r$ $(r < r_0)$ and $\theta_0(r)$ be the part of $|z - \zeta_0| = r$ which lies in $\varDelta_0$. Let K be the Riemann sphere of diameter 1, which touches the w-plane at $w = 0$ and $S(r)$, $L(r)$ be the area and the length of the image of $\varDelta_0(r)$, $\theta_0(r)$ on K respectively. Then

$$S(r) = \int_r^{r_0} \int_{\theta_0(r)} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} r dr d\theta, \qquad L(r) = \int_{\theta_0(r)} \frac{|f'(z)|}{1 + |f(z)|^2} r d\theta,$$

so that

$$[L(r)]^2 \leqq 2\pi r^2 \int_{\theta_0(r)} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} d\theta = -2\pi r \frac{dS(r)}{dr}.$$

Since $\theta_0(r)$ meets C and the boundary of $\varDelta_0$, we see $L(r) \geqq d > 0$. Hence $d^2 \log \frac{r_0}{r} \leqq 2\pi S(r)$, so that $\lim\limits_{r \to 0} S(r) = \infty$. Now $f(z)$ is regular in $\varDelta_0$, $|f(z)| < \rho$ in $\varDelta_0$, $|f(z)| = \rho$ on the boundary of $\varDelta_0$, except at points of E and $\lim\limits_{r \to 0} S(r) = \infty$.

Hence by a theorem[2] proved by the author, $f(z)$ takes any value in $|w| < \rho$, except values of capacity zero, infinitely many times in $\varDelta_0$ a fortiori in $\varDelta_\delta$. Let e_1 be the set of values in D_1, which are taken by $f(z)$ finite times in the neighbourhood of z_0. Then by the above result, at every point w_0 of e_1, there exists a neighbourhood $U(w_0)$, such that any value in $U(w_0)$, except values of capacity zero, is taken by $f(z)$ infinitely many times in $\varDelta_\delta$. Since by Lindelöf's covering theorem, we can cover e_1 by an enumerably infinite such neighbourhoods and a sum of enumerably infinite number of sets of capacity zero is also of capacity zero, we see that any value in D_1, except values of capacity zero, is taken by $f(z)$ infinitely many times in $\varDelta_\delta$.

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1) K. Kunugui: Sur un problème de M. A. Beurling. Proc. **16** (1940).

2) M. Tsuji: On the behaviour of a meromorphic function in the neighbourhood of a closed set of capacity zero. Proc. **18** (1942).

We take $\delta_1 > \delta_2 > \cdots > \delta_n \to 0$ for δ and the corresponding exceptional set be $e^{(n)}$, then $e_1 = \sum_{n=1}^{\infty} e^{(n)}$ is of capacity zero and any value in D_1, except e_1, is taken by $f(z)$ infinitely many times in the neighbourhood of z_0. Next we approximate D by closed domains $D_1 \subset D_2 \subset \cdots \subset D_n \to D$ and the corresponding exceptional set be e_n and $e = \sum_{n=1}^{\infty} e_n$. Then e is of capacity zero and any value in D, except e, is taken by $f(z)$ infinitely many times in the neighbourhood of z_0, q. e. d.