37. On Green's Lemma.

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1. We will prove the well known Green's lemma in the following generalized form.

Theorem. Let D be a domain on the $z=x+iy$-plane, bounded by a rectifiable curve Γ and $A(z)=A(x,y)$, $B(z)=B(x,y)$ be continuous and bounded functions of z inside D, which satisfy the following conditions:

(i) *$\lim A(z)$, $\lim B(z)$ exist almost everywhere on Γ, when z tends to Γ non-tangentially.*

(ii) *$A(x, y_0)$ is an absolutely continuous function of x on the part of the line $y=y_0$, which lies in D, for almost all values of y_0 and $B(x_0, y)$ is an absolutely continuous function of y on the part of the line $x=x_0$, which lies in D, for almost all values of x.*

(iii) $\iint_D \left(\left| \frac{\partial A}{\partial x} \right| + \left| \frac{\partial B}{\partial y} \right| \right) dx\,dy$ *is finite.*

Then

$$\iint_D \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} \right) dx\,dy = \int_\Gamma \left(A(z)\frac{dy}{ds} - B(z)\frac{dx}{ds} \right) ds,$$

where ds is the arc element on Γ and the line integral around Γ is taken in the positive sense.

The extension of Green's lemma for a domain D, bounded by a rectifiable curve was first proved by W. Gross[1] under the condition that $A(z)$, $B(z)$ are continuous in the closed domain $D+\Gamma$ and $\frac{\partial A}{\partial x}$, $\frac{\partial B}{\partial y}$ are continuous in D. Recently W. T. Reid[2] proved another extension under the condition that $A(z)$, $B(z)$ are continuous in the closed domain $D+\Gamma$ and the conditions (ii) and (iii) of our theorem.

We remark that since $A(x,y)$ is continuous, the Dini's derivatives:

$$\bar{A}_x^+(x,y) = \overline{\lim_{h\to+0}} \frac{A(x+h,y)-A(x,y)}{h},$$

$$\underline{A}_x^+(x,y) = \underline{\lim_{h\to+0}} \frac{A(x+h,y)-A(x,y)}{h}$$

are B-measurable functions of (x,y)[3], so that the set E in which $\bar{A}_x^+(x,y)=\underline{A}_x^+(x,y)$ is measurable. By the condition (ii), $\frac{\partial A}{\partial x}$ exists al-

1) W. Gross: Das isoperimetrische Problem bei Doppelintegralen. Monathefte f. Math. u. Phys. **27** (1927).

2) W. T. Reid: Green's lemma and related results. Amer. Journ. Math. **17** (1941).

3) Saks: Theory of the integral. p. 170.

most everywhere on the line $y=y_0$, hence from the measurability of E and Fubini's theorem, it follows that $\frac{\partial A}{\partial x}$ exists almost everywhere in D and is a measurable function of (x, y). Similarly for $\frac{\partial B}{\partial y}$.

2. To prove our theorem, we map D conformally on $|w|<1$ by $z=z(w)=f(w)$. Let $|w|\leqq r$, $|w|=r$ $(0<r<1)$ correspond to D_r, Γ_r on the z-plane. Since Γ is rectifiable, by F. Riesz' theorem[1], $f(e^{i\theta})$ is an absolutely continuous function of θ and $\lim_{r\to 1} f'(re^{i\theta})=f'(e^{i\theta})$ exists almost everywhere on $|w|=1$ and

$$\lim_{r\to 1}\int_0^{2\pi}|rf'(re^{i\theta})-f'(e^{i\theta})|\,d\theta=0\,. \tag{1}$$

Since on $|w|=r$, $izf'(z)=\frac{df(re^{i\theta})}{d\theta}$, we have from (1),

$$\lim_{r\to 1}\int_0^{2\pi}\left|\frac{df(re^{i\theta})}{d\theta}-\lim_{r\to 1}\frac{df(re^{i\theta})}{d\theta}\right|d\theta=0\,. \tag{2}$$

Since by Fatou's theorem[2], $\lim_{r\to 1}\frac{df(re^{i\theta})}{d\theta}=\frac{df(e^{i\theta})}{d\theta}$, if $\frac{df(e^{i\theta})}{d\theta}$ exists, which occurs almost everywhere by the absolute continuity of $f(e^{i\theta})$, we have from (2),

$$\lim_{r\to 1}\int_0^{2\pi}\left|\frac{df(re^{i\theta})}{d\theta}-\frac{df(e^{i\theta})}{d\theta}\right|d\theta=0\,. \tag{3}$$

If we put $z(re^{i\theta})=x(re^{i\theta})+iy(re^{i\theta})$, then from (3),

$$\lim_{r\to 1}\int_0^{2\pi}\left|\frac{dx(re^{i\theta})}{d\theta}-\frac{dx(e^{i\theta})}{d\theta}\right|d\theta=0\,,$$

$$\lim_{r\to 1}\int_0^{2\pi}\left|\frac{dy(re^{i\theta})}{d\theta}-\frac{dy(e^{i\theta})}{d\theta}\right|d\theta=0\,. \tag{4}$$

By Fubini's theorem and the condition (ii),

$$\iint_{D_r}\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}\right)dxdy=\int_{\Gamma_r}\left(A(z)\,dy-B(z)\,dx\right)$$

$$=\int_0^{2\pi}\left(A(re^{i\theta})\frac{dy(re^{i\theta})}{d\theta}-B(re^{i\theta})\frac{dx(re^{i\theta})}{d\theta}\right)d\theta\,, \tag{5}$$

where we put $A\big(z(re^{i\theta})\big)=A(re^{i\theta})$, $B\big(z(re^{i\theta})\big)=B(re^{i\theta})$. Since for $r\to 1$, $z(re^{i\theta})$ tends to Γ non-tangentially almost everywhere on $|w|=1$ and by F. and M. Riesz' theorem[3], a null set on Γ corresponds to a null

1) F. Riesz: Über die Randwerte einer analytischen Funktion. Math. Z. **18** (1923).

2) Fatou: Séries trigonométriques et séries de Taylor. Acta Math. **30** (1906).

3) F. und M. Riesz: Über die Randwerte einer analytischen Funktion. Quatrième congres des mathématiciens scandinaves à Stockholm, 1916.

set on $|w|=1$, we have by the condition (i), $\lim_{r\to1} A(re^{i\theta})=A(e^{i\theta})$, $\lim_{r\to1} B(re^{i\theta})=B(e^{i\theta})$ exist almost everywhere on $|w|=1$. Now

$$\left|\int_0^{2\pi}\left(A(re^{i\theta})\frac{dy(re^{i\theta})}{d\theta}-A(e^{i\theta})\frac{dy(e^{i\theta})}{d\theta}\right)d\theta\right|$$

$$\leqq \int_0^{2\pi}|A(re^{i\theta})|\left|\frac{dy(re^{i\theta})}{d\theta}-\frac{dy(e^{i\theta})}{d\theta}\right|d\theta$$

$$+\int_0^{2\pi}|A(re^{i\theta})-A(e^{i\theta})|\left|\frac{dy(e^{i\theta})}{d\theta}\right|d\theta$$

$$\leqq M\int_0^{2\pi}\left|\frac{dy(re^{i\theta})}{d\theta}-\frac{dy(e^{i\theta})}{d\theta}\right|d\theta$$

$$+\int_0^{2\pi}|A(re^{i\theta})-A(e^{i\theta})|\left|\frac{dy(e^{i\theta})}{d\theta}\right|d\theta, \tag{6}$$

where we put $|A(z)|\leqq M$ in D, so that

$$|A(re^{i\theta})-A(e^{i\theta})|\left|\frac{dy(e^{i\theta})}{d\theta}\right|\leqq 2M\left|\frac{dy(e^{i\theta})}{d\theta}\right|,$$

hence by Lebesgue's theorem,

$$\lim_{r\to1}\int_0^{2\pi}|A(re^{i\theta})-A(e^{i\theta})|\left|\frac{dy(e^{i\theta})}{d\theta}\right|d\theta$$

$$=\int_0^{2\pi}\lim_{r\to1}|A(re^{i\theta})-A(e^{i\theta})|\cdot\left|\frac{dy(e^{i\theta})}{d\theta}\right|d\theta=0. \tag{7}$$

By (4), (6), (7), we have

$$\lim_{r\to1}\int_0^{2\pi}A(re^{i\theta})\frac{dy(re^{i\theta})}{d\theta}d\theta=\int_0^{2\pi}A(e^{i\theta})\frac{dy(e^{i\theta})}{d\theta}d\theta.$$

Similary

$$\lim_{r\to1}\int_0^{2\pi}B(re^{i\theta})\frac{dx(re^{i\theta})}{d\theta}d\theta=\int_0^{2\pi}B(e^{i\theta})\frac{dx(e^{i\theta})}{d\theta}d\theta.$$

Hence we have from (5),

$$\iint_D\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}\right)dxdy=\int_0^{2\pi}\left(A(e^{i\theta})\frac{dy(e^{i\theta})}{d\theta}-B(e^{i\theta})\frac{dx(e^{i\theta})}{d\theta}\right)d\theta. \tag{8}$$

Let s be the arc length on Γ measured from a fixed point, then by F. and M. Riesz' theorem, $\theta=\theta(s)$ is an absolutely continuous function of s, so that by changing the variable of integration from θ to s in (8), we have

$$\iint_D\left(\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}\right)dxdy=\int_\Gamma\left(A(z)\frac{dy}{ds}-B(z)\frac{dx}{ds}\right)ds, \qquad \text{q. e. d.}$$