82. A Generalization of Poincaré-space.

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The set of symmetrical matrices A of dimension n satisfying the relation $E-A\bar{A}>0$ is called the space $\mathfrak{A}$ and A its points. $\mathfrak{A}$ is bounded, convex, and the points A satisfying the relation $|E-A\bar{A}|=0$ make the boundary of the space $\mathfrak{A}$.

Let $U=\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ be $2n$-dimensional matrices with the properties (1) $U'JU=J$, (2) $U'S\bar{U}=S$, where $J=\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$, $S=\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$, then we call the transformations of the space $\mathfrak{A}$ into itself $W=(U_1Z+U_2)(U_3Z+U_4)^{-1}$ the displacements of the space $\mathfrak{A}$.

We took as a line element the expression $ds=\sqrt{Sp\, dA(E-\bar{A}A)^{-1}d\bar{A}(E-A\bar{A})^{-1}}$ invariant under displacements and regarded $\mathfrak{A}$ as a Riemannian space.[1] However it seems to me more natural to introduce another metric which we will investigate here.

Let $W_i=(U_1Z_i+U_2)(U_3Z_i+U_4)^{-1}$, then by the property (1), we get

$$(W_1-W_4)^{-1}(W_1-W_3)(W_2-W_3)^{-1}(W_2-W_4)$$
$$=(U_3Z_4+U_4)(Z_1-Z_4)^{-1}(Z_1-Z_3)(Z_2-Z_3)^{-1}(Z_2-Z_4)(U_3Z+U_4)^{-1}.$$

Hence $h(Z_1, Z_2, Z_3, Z_4)=|Z_1-Z_4|^{-1}|Z_1-Z_3||Z_2-Z_3|^{-1}|Z_2-Z_4|$ is invariant under displacements. We call it the "anharmonic ratio of the four ordered points Z_1, Z_2, Z_3, Z_4."

Especially $h(0, A, \lambda A, -\lambda A)=\left(\frac{1+\lambda^{-1}}{1-\lambda^{-1}}\right)^n$.

Let $\lambda_1 A$ and $\lambda_2 A$ be the intersecting points of the euclidean straight line $Z=\lambda A$, passing through 0 and $A\neq 0$, with the boundary of the space $\mathfrak{A}$, where λ varies over real numbers. Then $\lambda_2=-\lambda_1$ and λ_1 is the reciprocal of the positive quadratic root of the greatest proper value of the non-negative hermitian form $A\bar{A}$, $(A\neq 0)$[2]; for $|E-\lambda^2 A\bar{A}|=0$.

Now we define the distance $(0, A)$ between 0 and $A\in\mathfrak{A}$ as the quantity $\frac{1}{2n}\log h(0, A, \lambda_1 A, -\lambda_1 A)=\frac{1}{2}\log\frac{1+\lambda_1^{-1}}{1-\lambda_1^{-1}}$, then the distance $(0, A)>0$, because $0<\lambda_1^{-1}\leqq 1$; and $(0, A)\to 0$ when $A\to 0$.

We define $(0, 0)=0$. $(0, A)$ is invariant under any displacement fixing the point 0, $W=U_1AU_1'$, $U_1'\bar{U}_1=E$, because $W\bar{W}=U_1A\bar{A}\bar{U}_1'$ and $(0, A)$ depends only on a proper value of $A\bar{A}$.

Let B^* be the image of $B\in\mathfrak{A}$ by a displacement transforming $A\in\mathfrak{A}$ to 0. We define the distance (A, B) between two points A and

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1) Masao Sugawara, Über eine allgemeine Theorie der Fuchsschen Gruppen und Theta-Reihen. Ann. Math. **41**.

2) λ_1^{-1} is called the norm of a matrix A.

B as $(0, B^*)=\left(0, C_1(B-A)(E-\bar{A}B)^{-1}\bar{C}_1^{-1}\right)$ where $C_1=(C')^{-1}$, $C'\bar{C}=E-A\bar{A}$.

It is invariant by the choice of the displacement which transforms A to 0; $(A, B)\geqq 0$ and $(A, B)=0$ only when $A=B$. Moreover $(A, B)=(B, A)$, because

$$\begin{aligned}B^*\bar{B}^* &= C_1(B-A)(E-\bar{A}B)^{-1}(\bar{B}-\bar{A})(E-A\bar{B})^{-1}C_1^{-1}=\overline{(B^*\bar{B}^*)}'\\ &=\bar{C}_1'^{-1}(E-B\bar{A})^{-1}(B-A)(E-\bar{B}A)^{-1}(\bar{B}-\bar{A})\bar{C}_1'\\ &=\left((E-B\bar{A})\bar{C}_1'\right)^{-1}(A-B)(E-\bar{B}A)^{-1}(\bar{A}-\bar{B})(E-B\bar{A})^{-1}\left((E-B\bar{A})\bar{C}_1'\right).\end{aligned}$$

Now we proceed to prove the third property of the distance. We begin with the

Lemma: Let λ be a real number, then

$$(A, \lambda A)=\begin{cases}(0, \lambda A)-(0, A), & \lambda>1\\ (0, A)-(0, \lambda A), & 1\geqq\lambda\geqq 0\\ (0, A)+(0, \lambda A), & \lambda<0.\end{cases}$$

Proof. Let α^2 be the greatest proper value of $A\bar{A}$, then the greatest proper value of the matrix $(\lambda A-A)(E-\lambda\bar{A}A)^{-1}(\lambda\bar{A}-\bar{A})(E-\lambda A\bar{A})^{-1}=(\lambda-1)^2A(E-\lambda\bar{A}A)^{-1}\bar{A}(E-\lambda A\bar{A})^{-1}=(\lambda-1)^2A\bar{A}(E-\lambda A\bar{A})^{-2}$ is $\dfrac{(\lambda-1)^2\alpha^2}{(1-\lambda\alpha^2)^2}$; so that

$$(A, \lambda A)=\begin{cases}\log\dfrac{1+\dfrac{1-\lambda}{1-\lambda\alpha^2}\alpha}{1-\dfrac{1-\lambda}{1-\lambda\alpha^2}\alpha}=\log\dfrac{1+\alpha}{1-\alpha}\,\dfrac{1-\lambda\alpha}{1+\lambda\alpha}\\ \qquad=\begin{cases}(0, A)-(0, \lambda A), & 1\geqq\lambda\geqq 0\\ (0, A)+(0, \lambda A), & \lambda<0\end{cases}\\ \log\dfrac{1+\dfrac{\lambda-1}{1-\lambda\alpha^2}\alpha}{1-\dfrac{\lambda-1}{1-\lambda\alpha^2}\alpha}=\log\dfrac{1-\alpha}{1+\alpha}\,\dfrac{1+\lambda\alpha}{1-\lambda\alpha}\\ \qquad=(0, \lambda A)-(0, A), \quad \lambda>1.\end{cases}$$

Cor. $$(A, \lambda A)=\frac{1}{2n}\,|\log h(A, \lambda A, \lambda_1 A, \lambda_1 A)|.$$

Proof. From the following properties

$$h(0, A, \lambda_1 A, -\lambda_1 A)\, h(A, \lambda A, \lambda_1 A, -\lambda_1 A)=h(0, \lambda A, \lambda_1 A, -\lambda_1 A),$$

$$h(A, \lambda A, \lambda_1 A, -\lambda_1 A)\, h(\lambda A, A, \lambda_1 A, -\lambda_1 A)=1,$$

$$\frac{1}{2n}\log h(A, \lambda A, \lambda_1 A, -\lambda_1 A)\gtreqless 0 \qquad \begin{matrix}\lambda_1>\lambda>1\\ \lambda=1,\\ -\lambda_1\leqq\lambda<1\end{matrix}$$

$$\frac{1}{2n}\log h(0, \lambda A, \lambda_1 A, -\lambda_1 A)=-(0, \lambda A), \quad \lambda<0,$$

our cor. follows at once.

When $B=A+dA$ is sufficiently near to A, dA is very small and neglecting infinitesimals of the second order we get as a line-element the expression $ds=\frac{1}{2}\log\frac{1+\rho}{1-\rho}=\rho$, where ρ is the positive quadratic root of the greatest proper value of $C_1 dA(E-\bar{A}A)^{-1}\overline{dA}(E-A\bar{A})^{-1}C_1^{-1}$, i.e. $dA(E-\bar{A}A)^{-1}\overline{dA}(E-A\bar{A})^{-1}$. Hence the obtained space is a Finsler-space.

Theorem: Let A, B, C be three points of the space $\mathfrak{A}$, then

$$(A, B)+(B, C)\geqq(A, C). \tag{4}$$

Proof.[1] Firstly we prove the special case in which $C=0$, $B=A+dA$ is sufficiently near to A, $A\bar{A}$ has the diagonal form $\begin{pmatrix} \alpha_1^2 \cdots\cdots 0 \\ \vdots \quad\quad \vdots \\ 0 \cdots\cdots \alpha_n^2 \end{pmatrix}$ and among the proper values of $A\bar{A}$ only $\alpha_1^2=\alpha_2^2=\cdots=\alpha_p^2$ are the greatest. The k-th diagonal element of the matrix $(A+dA)\overline{(A+dA)}$ is $\sum_{i=1}^{n}|a_{ki}+da_{ki}|^2$ and that of $C_1 dA(E-\bar{A}A)^{-1}\overline{dA}(E-A\bar{A}^{-1})C_1^{-1}$ is $\sum_{i=1}^{n}\frac{|da_{ki}|^2}{(1-\alpha_k^2)(1-\alpha_i^2)}$. Put $\sqrt{\sum_{i=1}^{n}|a_{ki}+da_{ki}|^2}=x$, and $\sqrt{\sum_{i=1}^{n}\frac{|da_{ki}|^2}{(1-\alpha_k^2)(1-\alpha_i^2)}}=y$, then neglecting infinitesimals of the second order, we get

$$|x-\alpha_k|\sim\left|\frac{1}{2\alpha_k}\sum_{i=1}^{n}(\bar{a}_{ki}da_{ki}+a_{ki}\overline{da}_{ki})\right| \tag{5}$$

$$\leqq\frac{1}{\alpha_k}\sum_{i=1}^{n}|a_{ki}|\,|da_{ki}|,\quad \text{where}\quad \sum_{i=1}^{n}|a_{ik}|^2=\alpha_k^2.$$

The righthand side of (5) is the distance from a point $P(|da_{k1}|, |da_{k2}|, \dots, |da_{kn}|)$ to a plane $\sum_{i=1}^{n}|a_{ki}|\,|da_{ki}|=0$ passing through 0 in a n dimensional euclidean space. Hence it does not exceed the distance $\sqrt{\sum_{i=1}^{n}|da_{ki}|^2}$ between O and P and it attains the extremal value $\sqrt{\sum_{i=1}^{n}|da_{ki}|^2}$ only when the vector $\overrightarrow{OP}$ is perpendicular to the plane: $\frac{|da_{k1}|}{|a_{k1}|}=\frac{|da_{k2}|}{|a_{k2}|}=\cdots=\frac{|da_{kn}|}{|a_{kn}|}=q_k$.

Putting $da_{ki}=a_{ki}q_k\varepsilon_{ki}$, where $|\varepsilon_{ki}|=1$ and substituting them in (5) we know that the equality in (5) holds only when $\varepsilon_{ki}=\pm1$ and have the same sign so long as $a_{ki}\neq 0$. We call this case in which the relation $|x-\alpha_k|=\sqrt{\sum_{i=1}^{n}|da_{ki}|^2}$ holds, exceptional and the other ordinary. As the greatest proper value of a non-negative Hermitian matrix is not smaller than the diagonal elements we will prove in the ordinary case the following inequality instead of (4)

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1) During press I find some mistakes in the proof of (7) which I will correct in another place.

$$(6) \qquad \frac{1+a_k}{1-a_k} < \frac{1+x}{1-x}\,\frac{1+y}{1-y}\,, \quad (k=1,2,\dots,p).$$

Neglecting infinitesimals of the second order, (6) is reduced to

$$(7) \qquad y(1-a_k^2)+x > a^k .$$

But

$$(8) \qquad \frac{|x-a^k|}{1-a_k^2} < \frac{\sqrt{\sum_{i=1}^{n}|da_{ki}|^2}}{1-a_k^2} \leqq \sqrt{\sum_{i=1}^{n}\frac{|da_{ki}|^2}{(1-a_k^2)(1-a_i^2)}}\,,$$

because $0 < a_k < 1$ and $|x-a^k| < \sqrt{\sum_{i=1}^{n}|da_{ki}|^2}$ in the ordinary case.

Thus (6), and hence (4) holds in the ordinary case. They also hold in the exceptional case, so long as the equality in (8) does not occur. The inequality sign occurs in (4) in all these cases.

As the following relations however holds in the remaining case R

$$a_{ki}=0\,,\quad da_{ki}=0\,,\quad i>p\,;\quad da_{ki}=\rho_k a_{ki}\quad i\leqq p\,.$$

We can prove (4) directly: namely by a real orthogonal transformation of the form $\begin{pmatrix} D^{(p)} & 0 \\ 0 & E^{(n-p)} \end{pmatrix}$, which is a special displacement fixing 0, we make A of the form

$$(9) \qquad \begin{pmatrix} A_1^{(p_1)} & & & & 0 \\ & A_2^{(p_2)} & & & \\ & & \ddots & & \\ & & & A_s^{(p_s)} & \\ 0 & & & & F_1 \end{pmatrix},$$ where $p_1+p_2+\dots+p_s=p$, and at the same time $A+dA$ of the form $\begin{pmatrix} \lambda_1 A_1 & & & & 0 \\ & \lambda_2 A_2 & & & \\ & & \ddots & & \\ & & & \lambda_s A_s & \\ 0 & & & & F_1+dF_1 \end{pmatrix}$, where λ_i, $i=1,2,\dots,s$ mean real numbers.

Hence follows

$$C_1^{-1}B^*\bar{B}^*C_1 = dA\big(E-\bar{A}(A+dA)\big)^{-1}\overline{dA}\big(E-A(\overline{A+dA})\big)^{-1}$$

$$= \begin{pmatrix} (\lambda_1-1)A_1 & & & & 0 \\ & (\lambda_2-1)A_2 & & & \\ & & \ddots & & \\ & & & (\lambda_s-1)A_s & \\ 0 & & & & dF_1 \end{pmatrix} \begin{pmatrix} (1-\lambda_1 a_1^2)^{-1}E^{(p_1)} & & & & 0 \\ & (1-\lambda_2 a_1^2)^{-1}E^{(p_2)} & & & \\ & & \ddots & & \\ & & & (1-\lambda_s a_1^2)^{-1}E^{(p_s)} & \\ 0 & & & & F_2 \end{pmatrix}$$

$$\begin{pmatrix} (\lambda_1-1)\bar{A}_1 & & & & 0 \\ & (\lambda_2-1)\bar{A}_2 & & & \\ & & \ddots & & \\ & & & (\lambda_s-1)A_s & \\ 0 & & & & \overline{dF_1} \end{pmatrix} \begin{pmatrix} (1-\lambda_1 a_1^2)^{-1}E^{(p_1)} & & & & 0 \\ & (1-\lambda_2 a_1^2)^{-1}E^{(p_2)} & & & \\ & & \ddots & & \\ & & & (1-\lambda_i a_1^2)^{-1}E^{(p_s)} & \\ 0 & & & & \bar{F}_2 \end{pmatrix}$$

$$=\begin{pmatrix} (\lambda_1-1)^2(1-\lambda_1\alpha_1^2)^{-2}\alpha_1^2E^{(p_1)} & & & & 0 \\ & (\lambda_2-1)^2(1-\lambda_2\alpha_1^2)^{-2}\alpha_1^2E^{(p_2)} & & & \\ & & \ddots & & \\ & & & (\lambda_s-1)^2(1-\lambda_s\alpha_1^2)^{-2}\alpha_1^2E^{(p_s)} & \\ 0 & & & & F_3 \end{pmatrix}.$$

We take dA so small that $1-\lambda_i\alpha_1^2>0$ holds.

If the greatest proper value of $B^*\bar{B}^*$ is $\frac{(\lambda_i-1)^2\alpha_1^2}{(1-\lambda_i\alpha_1^2)^2}$, (4) is reduced to the case treated in the lemma.

If the greatest proper value comes from F_3, it is greater than $\frac{(\lambda_i-1)^2\alpha_1^2}{(1-\lambda_i\alpha_1^2)^2}$. In this case (4) also holds with the inequality sign, because $\frac{1+z}{1-z}$ is a monotone increasing function of z in the interval $<0\ 1>$. The equality in (4) holds only when $0\leqq\lambda_i\leqq 1$ and the greatest proper value of $B^*\bar{B}^*$ is $(\lambda_i-1)^2(1-\lambda_i\alpha_1^2)^{-2}\alpha_1^2$. By a displacements fixing 0, $W=U_1ZU_1'$, $U'\bar{U}=E$, we can make $UZU'\bar{U}\bar{Z}\bar{U}'=UZ\bar{Z}\bar{U}'$ into a diagonal form $\begin{pmatrix}\zeta_1^2 & & 0\\ & \ddots & \\ 0 & & \zeta_n^2\end{pmatrix}$ in which the greatest proper value is $\zeta_1^2=\zeta_2^2=\cdots=\zeta_p^2$ and by a special displacement fixing 0 we can make W of the form (9) in the case R. Hence (4) holds even when we omit the condition about A.

Let A and B be two arbitrary points. Draw a line l passing through A and B, which is the image of the euclidean straight line $W=\lambda B^*$, and divide it in points $A_0=A_1, A_2, \cdots, A_r=B$ so that each point lies sufficiently near to its neighbours, then $(0, A_k)\leqq(0, A_{k+1})+(A_k, A_{k+1})$; adding we get $(0A)\leqq(0B)+\sum(A_k, A_{k+1})=(0B)+(AB)$ by the lemma.

As our concept of the distance is invariant under displacements, we can reduce the general case to our case, if we transform a point C to 0 by a displacement.

Thus our definition of the distance satisfies the three conditions of the general idea of the distance.

The Fuclidean straight lines passing through 0 and their images by the displacements are geodesics in our space $\mathfrak{A}$ with the metric. We call them straight lines in $\mathfrak{A}$. Under the angle θ between two intersecting straight lines $\overline{CA}$, $\overline{CB}$ we mean the Euclidean angle enclosed by the images $\overline{0A^*}$, $\overline{0B^*}$ at 0 of these straight lines, namely

$$\cos\theta=\frac{1}{2}\frac{Sp(\bar{A}^*B^*+A^*\bar{B}^*)}{\sqrt{SpA^*\bar{A}^*}\sqrt{SpB^*\bar{B}^*}}.$$