33. *Mean Ergodic Theorem in Abstract (L)-Spaces.*

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Let us consider an abstract (L)-space (notation: (AL)). Under abstract (L)-space (AL) we mean a semi-ordered Banach space, whose norm is additive on positive elements. That is, (AL) is a Banach space, which satisfies the following axioms[1] (we denote by $x, y, \dots$ the elements of (AL), by $\|x\|$, $\|y\|, \dots$ their norm and by λ a real number):

(1) a semi-order relation $x > y$ is defined on (AL), and (AL) is a lattice with respect to this semi-ordering. That is:

(1–1) $x > y$ and $y > z$ imply $x > z$,

(1–2) $x > y$ and $\lambda > 0$ imply $\lambda x > \lambda y$,

(1–3) $x > y$ implies $x + z > y + z$ for any z,

(1–4) $x_n > y_n$, $x_n \to x$ (strongly) and $y_n \to y$ (strongly) imply $x \geqq y$ ($x \geqq y$ means $x > y$ or $x = y$),

(1–5) to any pair of elements x and y, there exists a maximum $z = x \vee y$ such that $z \geqq x$, $z \geqq y$ and $z \leqq z'$ for any z' with $z' \geqq x$, $z' \geqq y$,

(1–6) to any pair of elements x and y, there exists a minimum $w = x \wedge y$ such that $w \leqq x$, $w \leqq y$ and $w \geqq w'$ for any w' with $w' \leqq x$, $w' \leqq y$.

(2) norm is additive on positive elements: $x > 0$ and $y > 0$ imply $\|x + y\| = \|x\| + \|y\|$.

Such a space was discussed by Garrett Birkhoff.[2] He has introduced an abstract (L)-space as a generalization of a concrete (L)-space[3] (notation: (L)), and has discussed the iteration of positive[4] bounded linear operations in such an abstract (L)-space.[5]

The main result of G. Birkhoff may be stated as follows:

Theorem. Let T be a positive bounded linear operation which maps an abstract (L)-space (AL) into itself. If T is of norm 1, and if for any $x \in (AL)$ the sequence $\{T^n(x)\}$ $(n = 1, 2, \dots)$ is bounded from

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1) Separability is not assumed.

2) G. Birkhoff: Dependent probabilities and the space (L), Proc. Nat. Acad. U. S. A., **24** (1938), 154–159.

3) Under concrete (L)-space, we mean the ordinary Banach space (L), composed of all the measurable functions which are absolutely integrable in $0 \leqq x \leqq 1$. For any $f(x) \in (L)$, we define its norm by $\|f\| = \int_0^1 |f(x)|\, dx$. (L) is separable in this norm.

4) A linear transformation, which maps a semi-ordered linear space into itself, is called to be positive if $x \geqq 0$ implies $Tx \geqq 0$.

5) This may be considered as the most general formulation of Markoff's process, since, as is well known, the problem of Markoff's process is nothing but the investigation of the behaviour of the iteration of positive bounded linear operations of norm 1 in the concrete (L)-space (L) (or the space (V). See the footnote (3) on page 122).

above in the sense of lattice,[1] *then* $\lim_{n\to\infty} f(x_n)$, *where* $x_n = \frac{1}{n}\{T(x)+T^2(x)+\cdots+T^n(x)\}$, $n=1, 2, \ldots$, *exists for any bounded linear functional* $f(x)$ *defined on* (AL).

As a generalization of this result, we shall prove in the present paper the following

Theorem. Let T *be a bounded linear operation which maps an abstract* (L)*-space* (AL) *into itself. If there exists a constant* C *such that* $\| T^n \| \leqq C$ *for* $n=1, 2, \ldots$, *and if for any* $x \in (AL)$ *the sequence* $\{x_n\}$ $(n=1, 2, \ldots)$ *is bounded from above in the sense of lattice, then the sequence* $\{x_n\}$ $(n=1, 2, \ldots)$ *converges strongly to a point* $\bar{x} \in (AL)$, *that is, Mean Ergodic Theorem is valid in* (AL).

It is to be noted that the condition that T is positive is unnecessary, and that we can even prove the *strong* convergence. Our method of proof is entirely different from that of G. Birkhoff. The following proof is based upon the fact that the separable abstract (L)-space may be considered as a closed linear subspace of the concrete (L)-space (L). We shall sketch only the outline of the proof. The detail of the proof as well as the discussion of the allied problems will be given in another paper.

The proof is divided into the following five stages:

1. First we remark that the sequence $\{x_n\}$ $(n=1, 2, \ldots)$ is also bounded from below in the sense of lattice.

2. Next it is to be noted that in discussing the behaviour of the sequence $\{x_n\}$ $(n=1, 2, \ldots)$ we have only to consider the separable part of (AL). We have, indeed, only to consider the closed linear subspace (AL') of (AL), which is spanned by $\{x_n\}$ $(n=1, 2, \ldots)$. Moreover, we may assume that this closed linear subspace (AL') is also closed in the sense of lattice, that is, that this (AL') itself is a lattice with respect to the original semi-ordering of (AL). Indeed, we can easily construct a separable closed linear subspace (AL'') of (AL), which contains (AL') and which is closed in the sense of lattice. Hence we assume hereafter that (AL) itself is separable.

3. Every separable abstract (L)-space may be considered as a closed linear subspace of the concrete (L)-space (L), that is, a separable abstract (L)-space is simultaneously isometric and lattice-isomorphic to a closed linear subspace of the concrete (L)-space (L). This may be proved by using the spectral representation of elements of (AL), which was given by H. Freudenthal.[2] In the case, when (AL) is a separable closed linear subspace of a space of functions of bounded variation,[3] this theorem was already proved by S. Banach and S. Mazur,[4] and the same idea is also found in a recent paper of F.

1) A sequence of points $\{x_n\}$ $(n=1, 2, \ldots)$ in a semi-ordered space is called to be bounded from above in the sense of lattice, if there exists a point x_0 such that $x_0 \geqq x_n$ for $n=1, 2, \ldots$. The boundedness from below in the sense of lattice is analogously defined.

2) H. Freudenthal: Teilweise geordnete Moduln, Proc. Acad. Amsterdam, **39** (1936), 641–651.

3) The Banach space (V) of all the functions of bounded variation is a kind of abstract (L)-space.

4) S. Banach and S. Mazur: Zur Theorie der linearen Dimension, Studia Math., **4** (1933), 100–112.

Wecken.[1] Hence we may assume that (AL) is a closed linear subspace of the concrete (L)-space (L).

4. In the concrete (L)-space (L), we can choose from any sequence of elements, which is bounded both from above and from below in the sense of lattice, a subsequence which converges weakly to some element $\bar{x} \in (L)$. This fact is easy to prove and was noticed by many authors. In a recent paper of F. Riesz,[2] we find also the same idea. Since (AL) is a closed linear subspace of (L), (AL) is also weakly closed in (L), and consequently $\bar{x}$ belongs also to (AL).

5. Thus we have proved that there exists a subsequence $\{x_{n_\nu}\}$ $(\nu=1, 2, \dots)$ of $\{x_n\}$ $(n=1, 2, \dots)$ which converges weakly to a point $\bar{x} \in (AL)$. It is now an easy matter to deduce from this our final result. Indeed, returning to the original space (AL) (where the separability is not assumed), we see from the theorem of K. Yosida[3] that the sequence $\{x_n\}$ $(n=1, 2, \dots)$ converges strongly to $\bar{x}$, and hereby the proof of the theorem is completed.

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1) F. Wecken: Unitärinvarianten selbstadjungierter Operatoren, Math. Ann., **116** (1938), 422–455.

2) F. Riesz: Some Mean Ergodic Theorems, Journal of the London Math. Soc., **13** (1938), 274–278.

3) K. Yosida: Mean Ergodic Theorem in Banach spaces, Proc. **14** (1938), 292–294. See also: S. Kakutani: Iteration of linear operations in complex Banach spaces, Proc. **14** (1938), 295–300, and also the paper of F. Riesz cited in (2) above.