74. Mean Ergodic Theorem in Banach Spaces.

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§ 1. *Introduction and the theorem.*

The mean ergodic theorem of J. von Neumann reads as follows:

Let T be a unitary operator in the Hilbert space $\mathfrak{H}$. Then, for any $x \in \mathfrak{H}$, the sequence

$$(1) \qquad x_n = \frac{T\cdot x + T^2\cdot x + \cdots + T^n\cdot x}{n} \qquad (n=1, 2, \ldots)$$

converges strongly to a point $\in \mathfrak{H}$.

Neumann's proof is based upon Stone's theorem concerning the one-parameter group of unitary operators in $\mathfrak{H}$. We find F. Riesz's elementary proof in E. Hopf's book.[1]

Recently C. Visser[2] gave the following theorem:

Let a linear operator T in $\mathfrak{H}$ satisfy the condition; $\| T^n \| \leqq$ a constant for $n=1, 2, \ldots$. Then, for any $x \in \mathfrak{H}$, the sequence (1) *converges weakly to a point* $\in \mathfrak{H}$.

He also showed that the mean ergodic theorem is easily obtained from this theorem. Thus we have another elementary proof of the mean ergodic theorem.

In the present note I intend to give a more general

Theorem. Let a linear operator T in the (real or complex) Banach space $\mathfrak{B}$ satisfy the two conditions:

(2) $\| T^n \| \leqq$ *a constant C for* $n=1, 2, \ldots$,

(3) *T is weakly completely continuous, viz. T maps the unit sphere $\| x \| \leqq 1$ of $\mathfrak{B}$ on the point set which is weakly compact in $\mathfrak{B}$.*[3]

Then, for any $x \in \mathfrak{B}$, the sequence (1) *converges strongly to a point $\bar{x} \in \mathfrak{B}$. We have $T\cdot\bar{x} = \bar{x}$.*

As the existence of the inverse T^{-1} of T is not assumed, this theorem may be applied in the problem of the temporally homogeneous stochastic process.[4] The applications will be published elsewhere.

I here express my hearty thanks to S. Kakutani who kindly communicated me that Visser's weak convergence theorem can be extended to $\mathfrak{B}$.[5]

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1) Ergodentheorie, Berlin (1937), 23.

2) Proc. Amsterdam Acad. 16, 5 (1938), 487–495.

3) It is sufficient to assume that, for any $x \in \mathfrak{B}$, the sequence (1) is weakly compact in $\mathfrak{B}$. See the proof below. As the Hilbert space is weakly compact locally such conditions are not needed in Visser's theorem.

4) Cf. my preceding paper.

5) See the following paper of Kakutani, where we find his ingeneous arguments.

§ 2. *The proof of the theorem.*

Let x be any point of $\mathfrak{B}$. We have, by (2), $\left\|\frac{T\cdot x+T^2\cdot x+\cdots+T^n\cdot x}{n}\right\|$ $\leq C\,\|x\|$ for $n=1,2,\ldots$. Hence the sequence (1) is weakly compact in $\mathfrak{B}$ by (3). Thus there exist a partial sequence $\{x_{n'}\}$ and an element $\bar{x}\in\mathfrak{B}$ such that

$$\lim_{n'\to\infty} f(x_{n'})=f(\bar{x}) \tag{4}$$

for any *linear functional* f defined in $\mathfrak{B}$. We have

$$T\cdot\bar{x}=\bar{x}, \tag{5}$$

by
$$\|T\cdot x_{n'}-x_{n'}\|=\left\|\frac{T^{n'+1}\cdot x-T\cdot x}{n'}\right\|\leqq\frac{2C\|x\|}{n'}.$$

We put $x=\bar{x}+(x-\bar{x})$. By (5) we have $T^n\cdot\bar{x}=\bar{x}$ and hence we have $x_n=\bar{x}+z_n$ where $z_n=\frac{T+T^2+\cdots+T^n}{n}(x-\bar{x})$. Hence it is sufficient to prove that z_n converges *strongly* to zero.

Consider the linear *closed* subspace $\mathfrak{B}_0$ of $\mathfrak{B}$ spanned by the elements of the from $(y-T\cdot y)$, $y\in\mathfrak{B}$. Then, for any $w\in\mathfrak{B}_0$, $\frac{T+T^2+\cdots+T^n}{n}w$ converges strongly to zero. The proof runs as follows. If w is of the form $(y-T\cdot y)$, we have

$$\left\|\frac{T+T^2+\cdots+T^n}{n}w\right\|=\left\|\frac{T\cdot y-T^{n+1}\cdot y}{n}\right\|\leqq\frac{2C}{n}\|y\| \tag{6}$$

which converges strongly to zero. Let now w be not of the form $(y-T\cdot y)$, then for any $\varepsilon>0$ there exists $y\in\mathfrak{B}$ such that $\|w-(y-T\cdot y)\|$ $\leqq\varepsilon$. Thus, by (1),

$$\left\|\frac{T+T^2+\cdots+T^n}{n}w-\frac{T+T^2+\cdots+T^n}{n}(y-T\cdot y)\right\|\leqq C\varepsilon.$$

Therefore, at any $w\in\mathfrak{B}_0$, $\frac{T+T^2+\cdots+T^n}{n}w$ converges strongly to zero.

Next assume that $(x-\bar{x})$ does not belong to $\mathfrak{B}_0$. Then by S. Banach's theorem,[1] there exists a linear functional f_0 such that

$$f_0(x-\bar{x})=1,\qquad f_0(z)=0 \text{ for any } z\in\mathfrak{B}_0.$$

As $(T^m\cdot x-T^{m+1}\cdot x)\in\mathfrak{B}_0$ we have $f_0(T^m\cdot x)=f_0(T^{m+1}\cdot x)$. Hence $f_0\left(\frac{T+T^2+\cdots+T^n}{n}x\right)=f_0(x)$ for $n=1,2,\ldots$. Therefore, by (4), $f_0(\bar{x})=f_0(x)$ contrary to $f_0(x-\bar{x})=1$.

Thus $(x-\bar{x})\in\mathfrak{B}_0$ and hence x_n converges strongly to $\bar{x}$. Q. E. D.

Remark 1. The above proof shows that

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1) S. Banach: Théorie des opérations linéaires, Warszawa (1932), 57.

$$\left\| \frac{T+T^2+\cdots+T^n}{n} x - \bar{x} \right\| \leqq \frac{d(x)}{n} \qquad (n=1, 2, \ldots)$$

with a constant $d(x)$, if the image of $\mathfrak{B}$ by $(E-T)$[1] is closed.

Remark 2. The correspondence $x \rightarrow \bar{x}$ is given by a linear operator $T_1 : \bar{x} = T_1 \cdot x$. By (5) we have $TT_1 = T_1$. Thus $T^n T_1 = T_1$ for any n and hence

$$T_1^2 = T_1 . \tag{7}$$

From the inequalities $\left\| \frac{T+T^2+\cdots+T^n}{n} x - \frac{T+T^2+\cdots+T^n}{n} T \cdot x \right\| \leqq \frac{2C}{n} \| x \|$ we see that $T_1 T = T_1$. Thus

$$TT_1 = T_1 T = T_1 . \tag{8}$$

Hence $T(T_1 \cdot x) = T_1 \cdot x$ for any $x \in \mathfrak{B}$. If $T \cdot y = y$, we have $T^n \cdot y = y$ for any n and thus $T_1 \cdot y = y$.

Therefore T_1 is the *projection operator* which maps $\mathfrak{B}$ on the proper subspace (Eigenraum) of T belonging to the proper value 1.

By applying the theorem to (T/λ), $|\lambda| = 1$, we see that the strong limit T_λ of $T_{\lambda, n} = \frac{1}{n} \left\{ \frac{T}{\lambda} + \frac{T^2}{\lambda^2} + \cdots + \frac{T^n}{\lambda^n} \right\}$ satisfies

$$T_\lambda^2 = T_\lambda , \quad T_\lambda T = TT_\lambda = \lambda T_\lambda , \quad T_\lambda T_\mu = 0 \tag{9}$$

$$\text{for } \lambda \neq \mu \text{ and } |\lambda| = |\mu| = 1 .$$

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1) E is the identity operator in $\mathfrak{B}$.