37. *A Note on the Singular Integral.*

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In the present paper,[1] I will give a remark about the convergence of the integral

(1) $$T_m(x;f)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}K(x-u,\ m)f(u)du\,.$$

Mr. Northrop[2] gave the necessary and sufficient conditions in terms of Fourier transform of $K(x, m)$ for the convergence of $T_m(x;f)$ to $f(x)$ in the mean L_2 for every function $f(x)\in L_2(-\infty,\infty)$.[3] And recently he treated the same problem and has given sufficient conditions for the convergence in the mean L_q in the case where $f(x)$ is the Fourier transform in L_q of some function in L_p, and necessary conditions for the convergence in the mean L_p in the case where $K(x, m)$ is the Fourier transform in L_p of some function in L_p and $f(x)\in L_q$, where $1<p\leqq 2$ and $\frac{1}{p}+\frac{1}{q}=1$. The cases $p=1$, $q=\infty$ and $p=\infty$, $q=1$ were not treated. We here consider the case closely related to this.

H. Hahn[4] has previously given the sufficient conditions for the convergence in the mean L_1 of $\int_{-\infty}^{\infty}K(x,\ u;\ m)f(u)du$ to $f(x)\in L_1$, but not in terms of Fourier transform.

Now consider the integral

$$f(x,\ m)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\frac{\sin m(x-u)}{x-u}du=(2\pi)^{-\frac{1}{2}}\int_{-m}^{m}F(u)e^{ixu}du\,,$$

where $F(x)$ is the Fourier transform of $f(x)$, or $f(x)$ is the Fourier transform of $F(x)$. If $f(x)\in L_r(r)$ $(r>1)$, this converges in the mean L_r to $f(x)$. Northrop's theorem may be considered as the extension of this fact. But this fact does not hold when $f(x)\in L_1$. Therefore it will be natural to modify the mode of convergence when $f(x)\in L_1$. Concerning the above fact, I had reached the result[5] that if $f(x)\in L_1$, then

(2) $$\lim_{m\to\infty}\int_{-\infty}^{\infty}\phi\big\{f(x,\ m)-f(x)\big\}dx=0\,,$$

where $\phi(x)=\dfrac{|x|}{|\log|x||^{1+\varepsilon}+1}$ $(\varepsilon>0)$.

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1) My former name was Tatsuo Takahashi.

2) Northrop, Note on a singular integral, I, Bull. Amer. Math. Soc. **40** (1934); II, Duke Math. Journ., **2** (1936).

3) Hereafter we write L_p instead of $L_p(-\infty,\ \infty)$.

4) Hahn, Wiener Denkschriften, **93** (1917), 667.

5) T. Takahashi, On the conjugate function of an integrable function and Fourier series and Fourier transforms, Sci. Rep. Tôhoku Imp. Univ. Ser. I. **25** (1936).

By this reason, I will give sufficient conditions for the validity of

$$\lim_{m\to\infty}\int_{-\infty}^{\infty}\phi\Big\{T_m(x;f)-f(x)\Big\}dx=0\,. \tag{3}$$

But I could not succeed to prove (3) for all $f(x)\in L_1$, but for $f(x)$ in a subclass of L_1. Therefore the above Fourier transform theorem is not quitely generalized. Our theorem runs as follows:

Let $xf(x)$ be absolutely integrable in $(-\infty,\infty)$ and there is a function $k(x,m)\in L_1$ such that

$$K(x,m)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}k(u,m)\,e^{ixu}\,du\,,$$

(4) $\int_{-\infty}^{\infty}|\,dk(x,m)\,|<M$, *$M$ being independent of m,*

(5) $\int_{-\infty}^{\infty}|\,k(x,m)-p(x,m)\,|\,dx$ *is bounded and tends to zero as $m\to\infty$, where $p(x,m)=1$ for $|x|<m$ and $=0$ otherwise. Then*

$$\lim_{m\to\infty}\int_{-\infty}^{\infty}\phi\Big\{T_m(x;f)-f(x)\Big\}dx=0\,,$$

where
$$\phi(x)=\frac{|x|}{|\log|x||^{1+\epsilon}+1}\,.$$

Denote

$$\psi(x)=\phi(x),\quad\text{for}\quad 0\leqq x\leqq 1\quad\text{and}\quad e^{\epsilon}\leqq x<\infty\,,$$
$$=\text{linear for}\quad 1\leqq x\leqq e^{\epsilon}\,.$$

Then $\psi(x)$ is increasing and there exists a constant A such that

$$\phi(x)\leqq A\psi(x)\,,\qquad \psi(x)\leqq A\phi(x)\,.$$

In the following lines A may differ on each occurrence, but represents always a constant. Since $\psi(2x)\leqq A\psi(x)$, we have

$$\phi(x+y)\leqq A\psi(x+y)\leqq A\psi\Big\{2\,\mathrm{Max}\,(x,y)\Big\}$$
$$\leqq A\Big\{\psi(2x)+\psi(2y)\Big\}\leqq A\Big\{\psi(x)+\psi(y)\Big\}\leqq A\Big\{\phi(x)+\phi(y)\Big\}\,.$$

Thus we get

$$\phi(x+y)\leqq A\Big\{\phi(x)+\phi(y)\Big\}\,. \tag{6}$$

Further we have

$$\phi(Ax)\leqq A\phi(x)\,. \tag{7}$$

From the assumption on $K(x,m)$, we have

$$\int_{-\infty}^{\infty}K(x-u,m)f(u)\,du=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}f(u)\,du\int_{-\infty}^{\infty}k(v,m)\,e^{i(x-u)v}\,dv \tag{8}$$
$$=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}k(v,m)\,e^{ixv}\,dv\int_{-\infty}^{\infty}f(u)\,e^{-iuv}\,du\,.$$

The change of order of integration is legitimate from the absolute integrability of $k(x,m)$ and $f(x)$. Hence

$$\phi\Big\{T_m(x;f)-f(x)\Big\}$$

$$=\phi\Big\{T_m(x;f)-\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,\frac{\sin m(x-u)}{x-u}\,du$$

$$+\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,\frac{\sin m(x-u)}{x-u}\,du-f(x)\Big\}$$

which does not exceed by (6)

$$A\phi\Big\{T_m(x;f)-\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,\frac{\sin m(x-u)}{x-u}\,du\Big\}$$

$$+A\phi\Big\{\frac{1}{\pi}\int_{-\infty}^{\infty}f(u)\,\frac{\sin m(x-u)}{x-u}\,du-f(x)\Big\}$$

$$=AJ_1+AJ_2,\quad \text{say.}$$

By (2) we get

$$\lim_{m\to\infty}\int_{-\infty}^{\infty}J_2\,dx=0\,.$$

By (8), we have

$$J_1=\phi\Big\{\frac{1}{2\pi}\int_{-\infty}^{\infty}k(v,m)e^{ixv}dv\int_{-\infty}^{\infty}f(u)e^{-iuv}du-\frac{1}{2\pi}\int_{-m}^{m}e^{ixv}dv\int_{-\infty}^{\infty}f(u)e^{-iuv}du\Big\}$$

$$=\phi\Big\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(k(v,m)-p(v,m)\Big)e^{ixv}dv\int_{-\infty}^{\infty}f(u)e^{-iuv}du\Big\}\,.$$

But from (4) and (5), we see that $\lim_{|v|\to\infty} k(v,m)=0$. Using this, we have

$$\int_{-\infty}^{\infty}\Big(k(v,m)-p(v,m)\Big)e^{ixv}dv\int_{-\infty}^{\infty}f(u)e^{-iuv}du$$

$$=\frac{1}{ix}\int_{-\infty}^{\infty}e^{ixv}d\Big\{\Big(k(v,m)-p(v,m)\Big)\int_{-\infty}^{\infty}f(u)e^{-iuv}du\Big\}$$

$$=\frac{1}{ix}\int_{-\infty}^{\infty}e^{ixv}\Big(k(v,m)-p(v,m)\Big)\,dv\int_{-\infty}^{\infty}uf(u)e^{-iuv}du$$

$$+\frac{1}{ix}\int_{-\infty}^{\infty}e^{ixv}\Big(\int_{-\infty}^{\infty}f(u)e^{iuv}dv\Big)d\Big(k(v,m)-p(v,m)\Big)$$

$$=O\Big(\frac{1}{x}\Big),$$

from (4) and (5).

Let

$$\int_{-\infty}^{\infty}J_1(x)=\int_{B}^{\infty}+\int_{-B}^{B}+\int_{-\infty}^{-B}=S_1+S_2+S_3,\ \text{say}\ (B>1)\,.$$

Then by (9) and (7), we have

$$S_1\leqq A\int_{B}^{\infty}\phi\Big(\frac{1}{x}\Big)dx=A\int_{B}^{\infty}\frac{dx}{x(\log x)^{1+\epsilon}+1}\,,$$

which becomes as small as we please if we take B very large. The same holds also for S_3.

$$S_2=\int_{-B}^{B}\phi\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\big(k(v,m)-p(v,m)\big)e^{ixv}dv\int_{-\infty}^{\infty}f(u)e^{-iuv}du\right\}dx$$

$$\leqq AB\left(\int_{-\infty}^{\infty}|k(v,m)-p(v,m)|dv\right)$$

which tends to zero as $m\to\infty$. Combining these estimations we get the required result.