PAPERS COMMUNICATED

54. *On Analytic Functions Regular in the Half-plane.*

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(Comm. by T. YOSIE, M.I.A., June 12, 1936.)

1. Let $f(z)=f(x+iy)$ be analytic in the half-plane $y>0$. If, for every positive y, there exists a constant C depending only on p such that

$$\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \leqq C^p , \tag{1}$$

where $p>0$, then we say that $f(z)$ belongs to the class $\mathfrak{H}_p$ and we write as $f(z) \in \mathfrak{H}_p$.

Theorem 1. If $p>0$, then

(i) *A function $f(z) \in \mathfrak{H}_p$, for almost all x, has a limit function $f(x)$ to which it tends along any non-tangential path,*

(ii) *Any $f(z) \in \mathfrak{H}_p$ tends to its limit function $f(x)$ in the mean of order p,*

(iii) *As $y \downarrow 0$,*

$$\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \uparrow \int_{-\infty}^{\infty} |f(x)|^p \, dx .$$

Theorem 2. If $p>0$, then a function $f(z) \in \mathfrak{H}_p$ can be represented as a product

$$f(z)=b_f(z)\, h(z) ,$$

where

$$b_f(z)=\prod_{(\nu)} \frac{z-z_\nu}{z-\bar{z}_\nu} \cdot \frac{\bar{z}_\nu - i}{z_\nu + i}$$

is the Blaschke function associated with $f(z)$ and $h(z) \in \mathfrak{H}_p$ and $h(z)$ does not vanish in the half-plane $y>0$. Here $\{z_\nu\}$ is the sequence of zeros of $f(z)$ in the half-plane $y>0$. And

$$|b_f(z)|<1, \qquad \textit{for} \quad y>0,$$
$$|b_f(x)|=1, \qquad \textit{almost everywhere.}$$

The case $p\geqq 1$ in these theorems was recently obtained by Professors Hille and Tamarkin.[1] They proved the theorem for the case by applying a lemma due to Mr. Gabriel[2] and by representing $f(z)$ as the Poisson integral associated with its limit function $f(x)$. But their method is not applicable to the case $0<p<1$. And the problem has been left open. But by a theorem due to Hardy, Ingham and Pólya,[3] if $p>0$ and for $y\geqq y_0>0$,

1) Hille and Tamarkin, On the absolute integrability of Fourier transforms, Fundamenta Math., **25** (1935).

2) See Hille and Tamarkin, loc. cit., Lemma 2.1.

3) Hardy, Ingham and Pólya, Theorems concerning mean values of analytic functions, Proc. of the Royal Soc. (A) **113** (1927).

$$\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \leqq C^p ,$$

where C depends only on y_0, then $f(z)$ is bounded in the closed half-plane $y \geqq y_0+\varepsilon$, ε being an arbitrary but fixed positive number and we can prove that $\lim_{z\to\infty} f(z)=0$ uniformly for $y \geqq y_0+2\varepsilon$. Thus we can apply Gabriel's lemma and by little devices we can prove the theorem in consideration.

2. Let $f(z)=f(x+iy)$ be regular in the half-plane $y>0$. If, for every positive number y_0,

$$\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \leqq C^p , \qquad (p>0) ,$$

provided $y \geqq y_0$, where C may depend on y_0 but be independent of y, then we say that $f(z)$ belongs to the class $\mathfrak{G}_p$ and we write $f(z) \in \mathfrak{G}_p$. Put

$$f(x+iy)=u(x, y)+iv(x, y) ,$$

$$M(f, y)= \max_{-\infty<x<\infty} |f(x+iy)| .$$

Theorem 3. Let $f(z) \in \mathfrak{G}_p$. If $p>0$ and $a \geqq 0$ and

$$M_p(u, y) \leqq Cy^{-a} , \qquad C \text{ a constant,}$$

then

$$M(f, y) \leqq K\, Cy^{-a-\frac{1}{p}} ,$$

where K is independent of y.

The analogous theorem for the function regular in the unit circle is due to Professors Hardy and Littlewood.[1]

Our argument depends upon the use of Fourier transform which plays a rôle similar to the power series in the unit circles, and other devices due to Hardy and Littlewood.

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1) Hardy and Littlewood, Some properties of the conjugate functions, Crelle, Hensel Festschrift.