15. *On the Convergence Factor of the Fourier-Denjoy Series.*

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Hardy has shown that $(\log n)^{-1}$ is a convergence factor of the Fourier-Lebesgue series. The object of this paper is to show that n^{-1} is a convergence factor of the Fourier-Denjoy series, and to construct an example such that $n^{-\delta}$ $(0<\delta<1)$ is not the convergence factor of the Fourier-Denjoy series.

1. Let $f(x)$ be a function, integrable in Denjoy-Perron's sense and periodic, with period 2π. And let

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\,. \tag{1·1}$$

Then we have

Theorem. n^{-1} *is a convergence factor of the Fourier-Denjoy series* (1·1). *In fact,*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{n} \tag{1·2}$$

converges almost everywhere.

In order to prove the theorem, we require the following

Lemma.[1] *The Fourier-Denjoy series* (1. 1) *is summable* $(C, 1+\delta)$ $(\delta>0)$ *almost everywhere.*

Put
$$s_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)\,,$$

$$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}\,,$$

and
$$\phi_1(t)=\int_0^t\phi(u)du\,.$$

Then
$$\phi_1(t)=o(t)^{2)}$$

for almost all values of x in $(-\pi, \pi)$, and then

1) Priwalof: Rend. di Palermo, **41** (1916).
c.f. Bosanquet, Proc. London math. soc. 31.
2) Hobson: Theory of function, vol. I (1921), p. 642.

$$s_n(x)=f(x)+\frac{1}{\pi}\int_0^\pi \phi(t)\frac{\sin(2n+1)\frac{t}{2}}{\sin\frac{t}{2}}dt$$

$$=f(x)+\frac{1}{\pi}\phi_1(\pi)\frac{\sin(2n+1)\frac{\pi}{2}}{\sin\frac{\pi}{2}}-\frac{1}{\pi}\int_0^\pi \phi_1(t)\frac{d}{dt}\left(\frac{\sin(2n+1)\frac{t}{2}}{\sin\frac{t}{2}}\right)dt$$

$$=o(n)+\int_0^\pi o(t)O\left(\frac{n}{t}\right)dt=o(n)\,,$$

almost everywhere in $(-\pi, \pi)$.

Therefore

$$\sum_{k=1}^{n}\frac{a_k\cos kx+b_k\sin kx}{k}k=s_n(x)-\frac{a_0}{2}=o(n) \qquad (1\cdot3)$$

almost everywhere in $(-\pi, \pi)$.

Now, by Hardy and Littlewood's theorem,[1] the series $(1\cdot2)$ is summable (C, δ) $(\delta>0)$ almost everywhere. On the other hand, if $(1\cdot2)$ is summable (C, δ), then it is convergent, provided that $(1\cdot3)$ is satisfied. Hence the theorem is proved.

2. We will construct an example such that $(1\cdot1)$ is the Fourier-Denjoy series and

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{n^\delta}$$

diverges almost everywhere for $0<\delta<1$.

Let r be a positive integer such that $r>(1-\delta)^{-1}$, and let

$$r\delta<p<r-1\,. \qquad (2\cdot1)$$

We put $\alpha_k=\frac{\pi}{(k\,!)^{r-r\delta}}\,, \qquad (k=1, 2, \ldots\ldots)\,,$

and take c_k such that

$$0\leqq c_k\leqq (k\,!)^r k^{p-r}\,, \qquad (k=1, 2, \ldots\ldots)\,.$$

Now, we define $\phi(t)$ by

$$\phi(t)=c_k\cos\{(k\,!)^r t\}\,, \quad \text{for } t \text{ in } (\alpha_k\alpha_{k-1})\ (k=2, 3, \ldots\ldots)\,,$$

and $\phi(t)=\phi(-t)$. Then $\phi(t)$ is an even function, integrable in Lebesgue's sense in any interval, excluding the origin.

1) Hardy and Littlewood: Math. Zeits., **19** (1924).
2) Knopp: Rend. di Palermo, **25** (1907).

Now $$I_k=\int_{\alpha_k}^{\alpha_{k-1}}\phi(t)dt=c_k\int_{\alpha_k}^{\alpha_{k-1}}\cos\{(k\,!)^r t\}dt$$
$$=\frac{c_k}{(k\,!)^r}\Big[\sin\{(k\,!)^r t\}\Big]_{\alpha_k}^{\alpha_{k-1}}=O(k^{p-r}).$$

If x' lies in $(\alpha_i\alpha_{i-1})$ and x'' in $(\alpha_j\alpha_{j-1})$, and $x''>x'>0$ then
$$\left|\int_{x'}^{x''}\phi(t)dt\right|\leq\sum_j^i|I_k|=O(\sum_j^i k^{p-r}).$$

By (2·1), $\sum_j^i k^{p-r}=o(1)$, for $i,j\to\infty$, hence $\phi(t)$ is integrable in Denjoy-Perron's sense, and the point $t=0$ is the only point of non-summability.

Let $$\phi(t)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nt, \tag{2·2}$$

where $$a_n=\frac{2}{\pi}\int_0^{\pi}\phi(t)\cos ntdt. \tag{2·3}$$

First, we take c_2 arbitrarily, then we can find a positive integer $k_1(>2)$ such that $\left|\int_{\alpha_2}^{\pi}\phi(t)\cos\{(k_1\,!)^r t\}dt\right|<1$;

then put $c_k=0$ for $2<k<k_1$ and $c_{k_1}=(k_1\,!)^r k_1^{p-r}$.

Next, we can find $k_2(>k_1)$ such that
$$\left|\int_{\alpha_{k_1}}^{\pi}\phi(t)\cos\{(k_2\,!)^r t\}dt\right|<1;$$

then put $c_k=0$ for $k_1<k<k_2$ and $c_{k_2}=(k_2\,!)^r k_2^{p-r}$, and so on.

Proceeding in this way we get a sequence of positive integers $\{k_i\}$ such that $\left|\int_{\alpha_{k_{i-1}}}^{\pi}\phi(t)\cos\{(k_i\,!)^r t\}dt\right|<1$,

where $c_k=0$ for $k_{i-1}<k<k_i$ and $c_{k_i}=(k_i\,!)^r k_i^{p-r}$.

Hence $\phi(t)$ is completely determined in $(-\pi,\pi)$.

Now $$a_{(k_i)^r}=\frac{2}{\pi}\int_0^{\pi}\phi(t)\cos\{(k_j\,!)^r t\}dt$$
$$=\frac{2}{\pi}\int_0^{\alpha_{k_i}}+\frac{2}{\pi}\int_{\alpha_{k_i}}^{\alpha_{k_{i-1}}}+\frac{2}{\pi}\int_{\alpha_{k_{i-1}}}^{\pi}$$
$$=\frac{2}{\pi}J_1+\frac{2}{\pi}J_2+\frac{2}{\pi}J_3. \tag{2·4}$$

Then $$J_3=O(1). \tag{2·5}$$

Let $$\phi_1(t)=\int_0^t \phi(u)du=O(1)$$
for $0<t<\pi$, then

$$J_1=\Big[\phi_1(t)\cos\{(k_i!)^r t\}\Big]_0^{\alpha_{k_i}}+(k_i!)^r\int_0^{\alpha_{k_i}}\phi_1(t)\sin\{(k_i!)^r t\}dt$$
$$=O(1)+O[(k_i!)^r\alpha_{k_i}]=O[(k_i!)^{\delta r}]. \tag{2·6}$$

At last, we have

$$J_2=\int_{\alpha_{k_i}}^{\alpha_{k_i-1}}\phi(t)\cos\{(k_i!)^r t\}dt=\int_{\alpha_{k_i}}^{\alpha_{k_i-1}}\phi(t)\cos\{(k_i!)^r t\}dt$$
$$=c_{k_i}\int_{\alpha_{k_i}}^{\alpha_{k_i-1}}\cos^2\{(k_i!)^r t\}dt$$
$$=\frac{1}{2}c_{k_i}(\alpha_{k_{i-1}}-\alpha_{k_i})+\frac{1}{2}c_{k_i}\int_{\alpha_{k_i}}^{\alpha_{k_i-1}}\cos\{2(k_i!)^r t\}dt$$
$$=\frac{1}{2}(k_i!)^r k_i^{p-r}\frac{\pi(k_i^{r-r\delta}-1)}{(k_i!)^{r-r\delta}}+\frac{c_{k_i}}{2(k_i!)^r}\Big[\sin\{2(k_i!)^r t\}\Big]_{\alpha_{k_i}}^{\alpha_{k_i-1}}$$
$$=\frac{\pi}{2}(k_i!)^{r\delta}(k_i^{p-r\delta}-k_i^{p-r})+o(1). \tag{2·7}$$

By (2·4), (2·5), (2·6), (2·7) we have

$$a_{(k_i!)^r}=(k_i!)^{r\delta}k_i^{p-r\delta}+O[(k_i!)^{r\delta}].$$

Hence $\overline{\lim\limits_{n\to\infty}}\dfrac{a_n}{n^\delta}=\infty$, when $0<\delta<1$. By a theorem due to Steinhaus,[1]

$$\overline{\lim_{n\to\infty}}\left|\frac{a_n}{n^\delta}\cos nx\right|=\infty,$$

almost everywhere in $(-\pi,\pi)$. Thereforethe series $\sum\limits_{n=1}^{\infty}\dfrac{a_n}{n^\delta}\cos nx$ is divergent almost everywhere.

Lastly, by a Riesz's theorem,[2] the Fourier-Denjoy series (2·2) just defined is not summable (C,δ) $(0<\delta<1)$ almost everywhere, while it is summable $(C, 1)$ almost everywhere.[3][4]

1) Rajchman: Fund. math., **3** (1922), 301.

2) Hardy and Riesz: General theory of Dirichlet's series, p. 33.

3) Hobson: Theory of function, vol. II, p. 573.

4) Since I have written this paper, I found that Prof. Titchmarsh (Proc. London Math. Soc., 22 (1924), p. XXV.) constructed an example such that the coefficients of Fourier-Denjoy series of an even function satisfy $a_n\neq o\{n\lambda(n)\}$, where $\lambda(n)$ is any positive sequence, such that $\lambda(n)\to 0$ and $n\lambda(n)\to\infty$. By this example, we can assert that n^{-1} is the "best possible" convergence factor of the Fourier-Denjoy series.