A review of exact results for fluctuation formulas in random matrix theory[*]

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Abstract: Covariances and variances of linear statistics of a point process can be written as integrals over the truncated two-point correlation function. When the point process consists of the eigenvalues of a random matrix ensemble, there are often large N universal forms for this correlation after smoothing, which results in particularly simple limiting formulas for the fluctuation of the linear statistics. We review these limiting formulas, derived in the simplest cases as corollaries of explicit knowledge of the truncated two-point correlation. One of the large N limits is to scale the eigenvalues so that limiting support is compact, and the linear statistics vary on the scale of the support. This is a global scaling. The other, where a thermodynamic limit is first taken so that the spacing between eigenvalues is of order unity, and then a scale imposed on the test functions so they are slowly varying, is the bulk scaling. The latter was already identified as a probe of random matrix characteristics for quantum spectra in the pioneering work of Dyson and Mehta.

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1. Introduction

The formulation of random matrix theory for applications to the spectra of complex quantum systems was laid out in pioneering works of Wigner, Gaudin, Mehta and Dyson. Preprints of the original papers, dating from the late 1950's and early 1960's, are conveniently bundled in a book edited by Porter [147], itself published in 1965, which contains too a review of this early literature. During the 1980's, as a fundamental contribution to the subject of quantum chaos, work of Bohigas et al. [25] made it clear that the correct meaning to give to a "complex quantum system" is any quantum system for which the underlying classical dynamics is chaotic. Single particle quantum systems in this class, such as kicked tops and irregular billiard domains, were subsequently studied intensely; see [100].

To test against random matrix predictions two statistical quantities, both of which were prominent in the works of the pioneering researchers cited above, were preferred. Assuming an unfolding of the energy levels so that their mean spacing becomes unity, one is the distribution of the spacing between consecutive eigenvalues, while the other is the so-called number variance, $\Sigma^2(L)$ say, corresponding to the fluctuation of the number of eigenvalues in an interval of length L, assumed large. From a theoretical viewpoint, these statistics have distinct characteristics.

Define

$$N_L = \sum_l \chi_{\lambda_l \in [0,L]}, \tag{1.1}$$

where $\{\lambda_l\}$ denotes the unfolded eigenvalues labelled from some origin in the bulk. Then, by the unfolding assumption,

$$\langle N_L \rangle = L,$$

while the variance is precisely the number variance

$$\langle (N_L - L)^2 \rangle = \Sigma^2(L). \tag{1.2}$$

Generally a statistic $A = \sum_l a(\lambda_l)$ is referred to as a linear statistic. Thus $\Sigma^2(L)$ is the variance of the particular linear statistic (1.1). In contrast the spacing

distribution, being a function of consecutive eigenvalues, does not relate to the structure of a linear statistic.

The present review focusses attention on fluctuation formulas associated with linear statistics. In addition to the quantity N_L, there are now many linear statistics and random matrix ensembles for which knowledge of the corresponding distributional properties is of applied interest. Moreover, consideration of the calculation of these distributions involves rich mathematical structures, with the scope for further research. At the same time the existing literature is vast, necessitating some restriction to the scope of the review. Thus for the most part we consider only the formulas for covariances and variances of linear statistics. Where possible we relate these formulas to the corresponding smoothed truncated two-point correlation function.

Throughout Section 2 we identify models in random matrix theory for which the truncated two-point correlation function is known explicitly and has a sufficiently simple analytic structure to allow large N analysis of the covariances and variances of linear statistics. There are two types of large N limits which lead to structured results. One is when the eigenvalues are scaled to have a limiting compact support, with the test functions by way of the linear statistics varying on the scale of the support. The other is when the test functions are first chosen to vary on the scale of the mean spacing between eigenvalues, and a scaling is chosen so that in the large N limit the eigenvalues are on average a unit distance apart. Next a scale L is introduced into the test functions, and the limit $L \to \infty$ is taken. Some understanding of the structures found can be given by adapting a log-gas viewpoint, which we do in subsection 2.10.2.

In Section 3 we review fluctuation formulas which are generalisations of those encountered in Section 2, but which require a more challenging analysis. First considered are the classical β-ensembles, where a loop equation analysis suffices to obtain the fluctuation formulas in a global scaling. Most prominent in this class are the Gaussian orthogonal and unitary ensemble cases. They permit numerous generalisations, and we take note of the corresponding fluctuation formulas of a number of them. One of these generalisations is to Wigner matrices, where the independent Gaussian entries are replaced by a more general zero mean, fixed standard deviation random variable. The classical Laguerre ensembles, realised in the case of orthogonal and symplectic symmetry by the matrix structure $W^\dagger W$ for W a rectangular Gaussian matrix, also permit generalisations. One is to consider the eigenvalues of $W_M^\dagger W_M$, where $W_M = G_M G_{M-1} \cdots G_1$, with each G_j an $N_j \times N_{j-1}$ a rectangular complex Gaussian matrix. In the global scaling limit, there is a simple formula for the variance of a polynomial linear statistic. The predictions of this formula can be compared with that obtained from a loop equation analysis. The final topic considered is that of variance formulas associated with the eigenvalues of Ginibre matrices, i.e. non-Hermitian square Gaussian random matrices, in the global scaling limit.

2. Formalism and simple examples

2.1. Covariance, variance and correlation functions

We take the viewpoint of the eigenvalues for a random matrix ensemble as an example of a continuous point or particle process. For notational convenience we regard the points as confined to an interval I of the real line, although this in not necessary — the domain may as well be in higher dimensions. For N particles we let $p_N(x_1, \ldots, x_N)$ denote their joint probability density function. Integrating out all but one, respectively two, particles gives the corresponding one and two point correlations

$$\rho_{(1),N}(x_1) = N \int_I dx_2 \cdots \int_I dx_N \, p_N(x_1, \ldots, x_N)$$

$$\rho_{(2),N}(x_1, x_2) = N(N-1) \int_I dx_3 \cdots \int_I dx_N \, p_N(x_1, \ldots, x_N). \tag{2.1}$$

Equivalently

$$\rho_{(1),N}(x) = \Big\langle \sum_{l=1}^{N} \delta(x - x_l) \Big\rangle$$

$$\rho_{(2),N}(x, x') = \Big\langle \sum_{\substack{l,l'=1 \\ l \neq l'}}^{N} \delta(x - x_l)\delta(x' - x_{l'}) \Big\rangle. \tag{2.2}$$

For large separation we expect $\rho_{(2),N}(x_1, x_2) \approx \rho_{(1),N}(x_1)\rho_{(1),N}(x_2)$ which motives introducing the truncated (or connected) two point correlation

$$\rho_{(2),N}^T(x_1, x_2) = \rho_{(2),N}(x_1, x_2) - \rho_{(1),N}(x_1)\rho_{(1),N}(x_2). \tag{2.3}$$

A simple but fundamental result is that these correlations relate to the covariance between two linear statistics, defined by

$$\mathrm{Cov}\Big(\sum_{l=1}^{N} f(x_l), \sum_{l=1}^{N} g(x_l)\Big) := \Big\langle \sum_{l,l'=1}^{N} f(x_l)g(x_{l'})\Big\rangle - \Big\langle \sum_{l=1}^{N} f(x_l)\Big\rangle\Big\langle \sum_{l=1}^{N} g(x_l)\Big\rangle. \tag{2.4}$$

Proposition 2.1. *We have*

$$\mathrm{Cov}\Big(\sum_{l=1}^{N} f(x_l), \sum_{l=1}^{N} g(x_l)\Big)$$

$$= \int_I dx \int_I dx' \, f(x)g(x')\Big(\rho_{(2),N}^T(x, x') + \rho_{(1),N}(x)\delta(x - x')\Big) \tag{2.5}$$

$$= -\frac{1}{2} \int_I dx \int_I dx' \, (f(x) - f(x'))(g(x) - g(x'))\rho_{(2),N}^T(x, x'). \tag{2.6}$$

Proof. In relation to the first term on the RHS of (2.4) we have

$$\Big\langle \sum_{l,l'=1}^{N} f(x_l)g(x_{l'}) \Big\rangle = \int_I dx \int_I dx' \, f(x)g(x') \Big\langle \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \delta(x - x_j)\delta(x' - x_k) \Big\rangle$$

$$+ \int_I f(x)g(x) \Big\langle \sum_{j=1}^{N} \delta(x - x_j) \Big\rangle dx$$

$$= \int_I dx \int_I dx' \, f(x)g(x') \Big(\rho_{(2),N}(x,x') + \delta(x - x')\rho_{(1),N}(x) \Big),$$

$$(2.7)$$

while the second term allows the simple rewrite

$$\Big\langle \sum_{l=1}^{N} f(x_l) \Big\rangle \Big\langle \sum_{l=1}^{N} g(x_l) \Big\rangle = \int_I dx \int_I dx' \, f(x)g(x')\rho_{(1),N}(x)\rho_{(1),N}(x'). \quad (2.8)$$

Subtracting (2.8) from (2.7), substituting in (2.4) and recalling the definition (2.3) gives (2.5).

In relation to (2.6), comparing with (2.5) we see that it suffices to show

$$- \int_I dx \, f(x)g(x) \int_I dx' \, \rho_{(2),N}^{T}(x,x') = \int_I f(x)g(x)\rho_{(1),N}(x) \, dx.$$

This is true since we can check from the definitions that

$$\int_I \rho_{(2),N}^{T}(x,x') \, dx' = -\rho_{(1),N}(x). \quad (2.9)$$

$$\square$$

We remark that a simple consequence of the first equation in (2.2) is the formula for the mean

$$\mathbb{E}\Big(\sum_{l=1}^{N} f(x_l) \Big) := \Big\langle \sum_{l=1}^{N} f(x_l) \Big\rangle = \int_I f(x)\rho_{(1),N}(x) \, dx, \quad (2.10)$$

as already used in (2.8), and setting $f = g$ in Proposition 2.1 gives the corresponding result for the variance. We note too that the quantity

$$C_{(2),N}(x,x') := \rho_{(2),N}(x,x') + \delta(x - x')\rho_{(1),N}(x) \quad (2.11)$$

has the interpretation of a response density which is induced by there being an eigenvalue at point x'.

2.2. *Global scaling limit of CUE random matrices*

Haar distributed random unitary matrices, or equivalently the circular unitary ensemble (CUE), provides the most straightforward example within random matrix theory of a significant simplification of the formulas in Proposition 2.1 for the covariances. First, with the eigenvalues of the CUE all being on the unit circle in the complex plane, we interpret the x_l as the angle parametrising the eigenvalues, and so take $I = [0, 2\pi)$. For the density we then have $\rho_{(1),N}(x) = N/2\pi$ independent of the angle x, and $C_{(2),N}(x, x')$ (recall the notation (2.11)) is a function of $x - x'$ which is periodic of period 2π in this variable. The corresponding Fourier series is well known (see e.g. [168, §5.2]) to have the simple form

$$C_{(2),N}(x, x') = \frac{1}{(2\pi)^2} \sum_{l=-\infty}^{\infty} m_l^{\mathrm{CUE}} e^{il(x-x')}, \quad m_l^{\mathrm{CUE}} = \begin{cases} |l|, & |l| < N \\ N, & |l| \geq N. \end{cases} \quad (2.12)$$

Substituting in (2.5) allows for simplification to a single sum.

Proposition 2.2. *We have*

$$\mathrm{Cov}^{\mathrm{CUE}} \Big(\sum_{l=1}^{N} f(x_l), \sum_{l=1}^{N} g(x_l) \Big) = \sum_{l=-\infty}^{\infty} m_l^{\mathrm{CUE}} f_l g_{-l}, \quad (2.13)$$

where $f_l = (1/2\pi) \int_0^{2\pi} f(x) e^{ilx} \, dx$ and similarly the meaning of g_{-l}. Moreoever, if f and g are differentiable on $[0, 2\pi)$ with f', g' Hölder continuous of order $\alpha > 0$ then

$$\lim_{N \to \infty} \mathrm{Cov}^{\mathrm{CUE}} \Big(\sum_{l=1}^{N} f(x_l), \sum_{l=1}^{N} g(x_l) \Big) = \sum_{l=-\infty}^{\infty} |l| f_l g_{-l}, \quad (2.14)$$

while if $f = g = \chi_{[0,L]}$ $(0 < L < 2\pi)$ and N_L is specified by (1.1) then

$$\lim_{N \to \infty} \frac{1}{\log N} \mathrm{Var}^{\mathrm{CUE}}(N_L) = \frac{1}{\pi^2}. \quad (2.15)$$

Proof. It remains to consider (2.14) and (2.15). To deduce (2.14) from (2.13), the essential point is that the stated conditions on f and g imply that for large l the decay of $f_l g_{-l}$ is $O(1/l^{2(1+\alpha)})$. This allows the sum over l on the RHS of (2.13) to be truncated at $|l| < N$ in the large N limit and tells us too that the sum on the RHS of (2.14) converges.

In relation to (2.15), with $f = g = \chi_{[0,L]}$, $0 < L < 2\pi$, we compute that for $l \neq 0$, $f_l = (e^{ilL} - 1)/(2\pi il)$. This substituted in (2.13) implies the stated result. $\square$

Let $0 < L_1, L_2 < 2\pi$ with $L_1 \neq L_2$. We see from (2.13) and the derivation of (2.15) that

$$\lim_{N \to \infty} \mathrm{Cov}^{\mathrm{CUE}}(N_{L_1}, N_{L_2}) = \frac{1}{\pi^2} \sum_{l=1}^{\infty} \frac{1}{l} \Big(\cos l(L_1 - L_2)/2 - \cos l(L_1 + L_2)/2 \Big)$$

$$= \frac{1}{\pi^2} \Big(\log|\sin(L_1 + L_2)/2| - \log|\sin(L_1 - L_2)/2| \Big). \tag{2.16}$$

Note that this diverges for $L_1 = L_2$, which is in keeping with (2.15). We refer to [155] for the analogous result in the case of complex Gaussian Hermitian random matrices, and a discussion of further context.

Remark 2.3. 1. The characteristic function, $\hat{P}_{N,f}(t)$ say, for the distribution of the linear statistic $\sum_{l=1}^{N} f(x_l)$ in the setting of the first paragraph of §2.1 is given by

$$\hat{P}_N(t; f) = \int_I dx_1 \cdots \int_I dx_N \Big(\prod_{l=1}^{N} e^{itf(x_l)} \Big) p_N(x_1, \ldots, x_N). \tag{2.17}$$

In the case of the CUE, $I = [0, 2\pi)$ and

$$p_N(x_1, \ldots, x_N) = \frac{1}{(2\pi)^N N!} \prod_{1 \le j < k \le N} |e^{ix_k} - e^{ix_j}|^2; \tag{2.18}$$

see [75, Prop. 2.2.5 with $\beta = 2$]. A well known variant of the Andréief identity [10, 77] allows (2.17) to be written as the Toeplitz determinant

$$\hat{P}_N(t; f) = \det \left[\frac{1}{2\pi} \int_0^{2\pi} e^{itf(x)} e^{i(j-k)x} \, dx \right]_{j,k=1}^{N}. \tag{2.19}$$

For f satisfying the conditions of Proposition 2.2, the celebrated strong Szegö theorem [19] gives

$$\lim_{N \to \infty} e^{-itNf_0} \hat{P}_N(t; f) = \exp\Big(-t^2 \sum_{l=1}^{\infty} l f_l f_{-l} \Big). \tag{2.20}$$

On the other hand, according to the cumulant expansion, for small t

$$\hat{P}_{N,f}(t) = e^{it\mathbb{E}(\sum_{j=1}^{N} f(x_j)) - (t^2/2)\mathrm{Var}(\sum_{j=1}^{N} f(x_j)) + O(t^3)}. \tag{2.21}$$

Comparing (2.20) to (2.21) shows consistency with (2.13) in the case $f = g$. Moreover (2.20) gives that the limiting distribution of the centred linear statistic under the conditions of Proposition 2.2 is a Gaussian. This interpretation of the strong Szegö theorem was first given by Johansson [109].

2. We see from (2.12) that m_l^{CUE} is independent of N for all $|l| < N$. Closely related to this is the fact that the distribution of $|\mathrm{Tr}\, U^k|^2 = |\sum_{j=1}^{N} e^{ix_j k}|^2$, for k a positive integer less than or equal to N, is such that its first N moments coincide with the corresponding moments of $\sqrt{k}$ times a standard complex Gaussian random variable for $k \le N$ [110], [56].

3. Consistent with (2.12) is the functional form [64]

$$\rho_{(2),N}^{T}(x, x') = -\left(\frac{\sin N(x - x')/2}{2\pi \sin(x - x')/2} \right)^2 = -\frac{1}{8\pi^2} \left(\frac{1}{\sin^2(x - x')/2} - \frac{\cos N(x - x')}{\sin^2(x - x')/2} \right). \tag{2.22}$$

Substituting in (2.6) then shows

$$
\begin{aligned}
\lim_{N \to \infty} &\operatorname{Cov}^{\mathrm{CUE}} \left(\sum_{l=1}^{N} f(x_l), \sum_{l=1}^{N} g(x_l) \right) \\
&= \frac{1}{16\pi^2} \int_0^{2\pi} dx \int_0^{2\pi} dx' \, \frac{(f(x) - f(x'))(g(x) - g(x'))}{\sin^2(x - x')/2} \\
&= -\frac{1}{2\pi^2} \int_0^{2\pi} dx \int_0^{2\pi} dx' \left(\frac{d}{dx} f(x) \right) \left(\frac{d}{dx'} g(x') \right) \log |\sin(x - x')/2)|,
\end{aligned}
\tag{2.23}
$$

valid provided f and g are Hölder continuous of order $\alpha > 1/2$ in the first expression, and satisfy the conditions required for (2.14) in the second (these conditions ensure the double integrals converge). The second formula follows from the first upon using the identity

$$
\frac{1}{4} \frac{1}{\sin^2(x - x')/2} = \frac{\partial^2}{\partial x \partial x'} \log \left| \sin \left(\frac{x - x'}{2} \right) \right|.
\tag{2.24}
$$

and integrating by parts. The Fourier expansion

$$
\log \left| \sin \left(\frac{x - x'}{2} \right) \right| = \sum_{p=-\infty}^{\infty} \alpha_p e^{ip(x - x')},
\tag{2.25}
$$

where

$$
\alpha_p = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \sin \left(\frac{x}{2} \right) \right| e^{-ipx} \, dx = -\frac{1}{2|p|} \ (p \neq 0),
$$

provides a direct transformation from the second formula in (2.23), to the formula of (2.13), upon integration by parts.

For a given N it is straightforward to sample $N \times N$ CUE matrices — indeed this is now an inbuilt function in the Mathematica computer algebra package — and to numerically compute the corresponding eigenangles $\{x_j\}_{j=1}^{N}$. This allows the result (2.14) to be illustrated for a particular linear statistic. We choose $f(x) = g(x) = \cos 2x$. The corresponding random function $\sum_{j=1}^{N} \cos 2x_j$ then has mean zero, and according to (2.14) has variance equal to 2. A plot of the value of a single realisation of this random function for $N = 1, 2, \ldots, 150$, and with successive values joined by straight lines as a visual aid, is given in Figure 2.1.

2.3. *Global scaling limit for a deformation of the CUE*

The non-oscillatory term in (2.22) is independent of N, which according to (2.6) is responsible for the leading large N covariance itself being independent of N, as seen in (2.22). In contrast, for eigenvalues behaving as a perfect gas of non-interacting particles on $[0, 2\pi)$, the joint eigenvalue density is
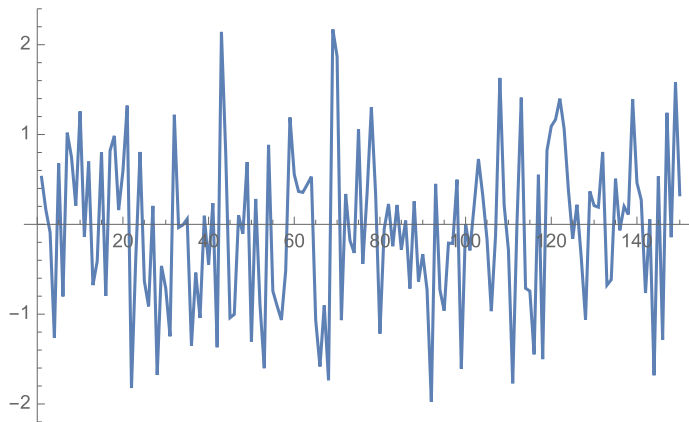
$$
p_N(x_1, \ldots, x_N) = \frac{1}{(2\pi)^N},
\tag{2.26}
$$

FIG 2.1. *Plot of values of a single realisation of the random function $\sum_{j=1}^{N} \cos 2x_j$ for $\{x_j\}$ the eigenangles of an $N \times N$ CUE matrix, for N from 1 up to 150.*

and we see from (2.2) and (2.3) that the truncated two-particle correlation is now proportional to N,

$$\rho_{(2),N}^{T}(x_1, x_2) = -\frac{N}{(2\pi)^2}. \tag{2.27}$$

Substituting in (2.6) gives that the corresponding covariance is similarly proportional to N,

$$\text{Cov}\left(\sum_{l=1}^{N} f(x_l), \sum_{l=1}^{N} g(x_l)\right) = \frac{N}{8\pi^2} \int_0^{2\pi} dx \int_0^{2\pi} dx' \, (f(x) - f(x'))(g(x) - g(x')). \tag{2.28}$$

In the mid 1960's Gaudin [93] introduced into random matrix theory an eigenvalue PDF interpolating between the CUE and Poisson functional forms, (2.18) and (2.26) respectively,

$$\frac{\alpha^{N(N-1)/2}}{Q_N} \prod_{1 \leq j < k \leq N} \left|\frac{e^{ix_j} - e^{ix_k}}{e^{ix_j} - \alpha e^{ix_k}}\right|^2 = \frac{1}{Q_N} \prod_{1 \leq j < k \leq N} \left(1 + \frac{\sinh^2 \gamma}{\sin^2(x_j - x_k)/2}\right)^{-1}, \tag{2.29}$$

with $0 < x_l < 2\pi$ $(l = 1, \dots, N)$, $\alpha := e^{-2\gamma}$ and

$$Q_N = (2\pi)^N N! \alpha^{N(N-1)/2} \prod_{k=1}^{N} \frac{1 - \alpha}{1 - \alpha^k}. \tag{2.30}$$

Thus taking the limit $\alpha \to 1$, or equivalently $\gamma \to 0$ reclaims (2.26), while taking $\alpha \to 0$, or equivalently $\gamma \to \infty$ reclaims (2.18). We will refer to this ensemble as the CUE$_\alpha$. In [72] the CUE$_\alpha$ was related to the theory of parametric eigenvalue motion due to Pechukas [145] and Yukawa [169], and most recently it was placed within the theory of circulant L-ensembles [79].

As an interpolation between (2.22) and (2.27) it was found in [93, §4.3] that

$$\rho^T_{(2)}(x_1, x_2; \alpha) = -\frac{N}{(2\pi)^2} - \frac{2}{(2\pi)^2} \operatorname{Re} \sum_{0 \le \mu_1 < \mu_2 \le N} \prod_{k=\mu_1}^{\mu_2 - 1} \frac{1 - \alpha^k}{e^{i(x_1 - x_2)} - \alpha^k}. \quad (2.31)$$

Thus from this we see that

$$\lim_{\alpha \to 0} \rho^T_{(2)}(x_1, x_2; \alpha) = -\frac{N}{(2\pi)^2} - \frac{1}{(2\pi)^2} \sum_{\substack{\mu_1, \mu_2 = 0 \\ \mu_1 \ne \mu_2}} e^{i(\mu_1 - \mu_2)(x_1 - x_2)}(N - |\mu_1 - \mu_2|), \quad (2.32)$$

which is equivalent to the Fourier expansion (2.12). We read off too the limiting behaviour

$$\lim_{\alpha \to 1} \rho^T_{(2)}(x_1, x_2; \alpha) = -\frac{N}{(2\pi)^2}, \quad (2.33)$$

in agreement with (2.27).

Of interest is the analogue of (2.14) for $0 < \alpha < 1$. In relation to this, and with $[w^{-p}]f(w)$ denoting the coefficient of w^{-p} in the Laurent expansion of $f(w)$, for $p \in \mathbb{Z}^+$ set

$$m_p^{(\alpha)} = m_{-p}^{(\alpha)} = \lim_{N \to \infty} [w^{-p}]\left(N - \sum_{0 \le \mu_1 < \mu_2 \le N} \prod_{k=\mu_1}^{\mu_2 - 1} \frac{1 - \alpha^k}{w - \alpha^k} \right). \quad (2.34)$$

Consideration of the small α expansion of this quantity shows it to be well defined. While the Laurent expansion of the quantity in brackets in (2.34) is complicated for finite N and general p, the coefficients greatly simplify in the limit $N \to \infty$. The mechanism is that the terms for which it is difficult to predict their coefficients do not occur until the order of $\alpha^{O(N)}$ in the small α expansion, and thus vanish in the limit. The terms before that have a regular pattern, allowing us to conclude

$$m_p^{(\alpha)} = |p| \sum_{j=0}^{\infty} \alpha^{|p|j} = \frac{|p|}{1 - \alpha^{|p|}}. \quad (2.35)$$

Consequently, with f and g in the class of functions specified in the statement of (2.14), the latter generalises to read

$$\lim_{N \to \infty} \operatorname{Cov}^{\mathrm{CUE}_\alpha}\left(\sum_{l=1}^{N} f(x_l), \sum_{l=1}^{N} g(x_l) \right) = \sum_{\substack{l=-\infty \\ l \ne 0}}^{\infty} \frac{|l|}{1 - \alpha^{|l|}} f_l g_{-l}. \quad (2.36)$$

Let $C^{(\alpha)}_{(2),N}$ be specified by (2.11) as it applies to the CUE_α. It follows from (2.31), (2.34) and (2.35) that we have

$$C^{(\alpha)}_{(2),\infty}(x, x') := \lim_{N \to \infty} C^{(\alpha)}_{(2),N}(x, x') = \frac{1}{(2\pi)^2} \sum_{\substack{p=-\infty \\ p \ne 0}}^{\infty} \frac{|p|}{1 - \alpha^{|p|}} e^{ip(x - x')}. \quad (2.37)$$

This functional form permits the alternative expression

$$C_{(2),\infty}^{(\alpha)}(x,x') = -\frac{1}{(2\pi)^2}\frac{\partial^2}{\partial x^2}\log\Big(\prod_{l=0}^{\infty}(1-\alpha^l e^{i(x-x')})(1-\alpha^l e^{-i(x-x')})\Big)$$

$$= -\frac{1}{(2\pi)^2}\frac{\partial^2}{\partial x^2}\log|\theta_1(x-x';\alpha^{1/2})|, \qquad (2.38)$$

where

$$\theta_1(z;q) := i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+1/2)^2}e^{(2n+1)iz}.$$

Here the first equality can be seen to agree with (2.37) by a direct calculation, while the second equality requires knowledge of the product formula for the Jacobi theta function θ_1. Hence, analogous to the second line of (2.23), in addition to (2.36) we have

$$\lim_{N\to\infty}\mathrm{Cov}^{\mathrm{CUE}_\alpha}\Big(\sum_{l=1}^{N}f(x_l),\sum_{l=1}^{N}g(x_l)\Big)$$

$$= -\frac{1}{2\pi^2}\int_0^{2\pi}dx\int_0^{2\pi}dx'\Big(\frac{d}{dx}f(x)\Big)\Big(\frac{d}{dx'}g(x')\Big)\log|\theta_1(x-x';\alpha^{1/2})|. \quad (2.39)$$

2.4. Bulk scaling limit of CUE and CUE$_\alpha$ random matrices

The bulk scaled limit is a rescaling of the coordinates of the particles in the point process so that they have an order unity density (taken to be unity for convenience). For the CUE, which has N eigenvalues with coordinates x_j between 0 and 2π, this is done by changing variables $x_j - \pi \mapsto 2\pi X_j/N$. Applying the corresponding change of variables to (2.5) with $f(2\pi X/N + \pi) = F(X)$, $g(2\pi X/N + \pi) = G(X)$ shows

$$\lim_{N\to\infty}\mathrm{Cov}^{\mathrm{CUE}}\Big(\sum_{l=1}^{N}F(X_l),\sum_{l=1}^{N}G(X_l)\Big)$$

$$= \int_{-\infty}^{\infty}dX\int_{-\infty}^{\infty}dX'\,F(X)G(X')\Big(\rho_{(2),\infty}^T(X,X')+\delta(X-X')\Big), \quad (2.40)$$

where, from (2.22),

$$\rho_{(2),\infty}^T(X,X') = -\frac{\sin^2\pi(X-X')}{(\pi(X-X'))^2}. \qquad (2.41)$$

The convergence of the double integral in (2.40) requires that both $F(X)$ and $G(X)$ be integrable at infinity. Noting that (2.41) is a function of the difference $X - X'$ allows the double integral in (2.40) to be reduced to a single integral involving Fourier transforms. And associating with $F(X)$ and $G(X)$ a length scale L, an analogue of (2.14) can be deduced, as first made explicit by Dyson and Mehta [67].

Proposition 2.4. *Introduce the structure function (also referred to as the spectral form factor)*

$$S_\infty^{\mathrm{CUE}}(k) := \int_{-\infty}^{\infty} C_{(2),\infty}(X,0)e^{ikX}\,dX = \begin{cases} \dfrac{|k|}{2\pi}, & 0 < k < 2\pi \\ 1, & |k| \geq 2\pi, \end{cases} \tag{2.42}$$

where the equality follows from the definition (2.11) of $C_{(2),\infty}$ and the functional form (2.41). We have

$$\lim_{N\to\infty} \mathrm{Cov}^{\mathrm{CUE}}\Big(\sum_{l=1}^{N} F(X_l), \sum_{l=1}^{N} G(X_l)\Big) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k)\hat{G}(-k)S_\infty^{\mathrm{CUE}}(k)\,dk, \tag{2.43}$$

where $\hat{F}(k)$ denotes the Fourier transform of $F(X)$, and similarly the meaning of $\hat{G}(-k)$. Furthermore, replacing $F(X)$ by $F_L(X) = F(X/L)$, and similarly replacing $G(X)$, we have that

$$\lim_{L\to\infty}\lim_{N\to\infty} \mathrm{Cov}^{\mathrm{CUE}}\Big(\sum_{l=1}^{N} F_L(X_l), \sum_{l=1}^{N} G_L(X_l)\Big) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \hat{F}(k)\hat{G}(-k)|k|\,dk, \tag{2.44}$$

assuming $\hat{F}(k)\hat{G}(-k)$ decays sufficiently fast for the integral to converge.

Proof. It remains to justify (2.44), starting from (2.43) with F, G replaced by F_L, G_L. From the definitions we have $\hat{F}_L(k) = L\hat{F}(Lk)$, and similarly $\hat{G}_L(k) = L\hat{G}(Lk)$. Changing variables $k \mapsto k/L$ and taking into consideration (2.42) shows that for large L the RHS of (2.43) reduces to the RHS of (2.44). $\square$

It was noted in [67] that choosing $F_L(X) = G_L(X) = \chi_{X\in[0,L]}$ gives $\hat{F}_L(k) = \frac{1}{ik}(1 - e^{ikL})$, which does not permit the passage from (2.43) to (2.44). By considering the functional form of (2.43) in this case, it was shown instead that for large L

$$\lim_{N\to\infty} \mathrm{Cov}^{\mathrm{CUE}}\Big(\sum_{l=1}^{N} \chi_{X_l\in[0,L]}\Big) \underset{L\to\infty}{\sim} \frac{1}{\pi^2}\log L + B_2, \tag{2.45}$$

where, with C denoting Euler's constant,

$$B_2 = \frac{1}{\pi^2}C + \frac{1}{\pi^2}(1 + \log 2\pi). \tag{2.46}$$

The essential step in obtaining the leading term is to consider the contribution to the integral (2.43) in the range $|k| < 1$. Making use of (2.42) and the functional form of $\hat{F}_L(k)$ as noted above, then changing variables in the integrand $kL/2 \mapsto k$ gives for the leading large L form

$$\frac{2}{\pi^2}\int_0^{L/2} \frac{\sin^2 k}{k}\,dk \sim \frac{1}{\pi^2}\log L, \tag{2.47}$$

in agreement with (2.45).

Remark 2.5. 1. An alternative functional form to the RHS of (2.44) is [67]

$$-\frac{1}{2\pi^2} \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dY \, F'(X)G'(Y) \log|X - Y|, \tag{2.48}$$

which at a formal level follows from the generalised Fourier transform

$$-\int_{-\infty}^{\infty} \log(|x|)e^{ikx} \, dx = \frac{\pi}{|k|}. \tag{2.49}$$

Furthermore, noting $\frac{\partial^2}{\partial X \partial Y} \log|X - Y| = 1/(X - Y)^2$, integration by parts of the expression

$$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dY \, \frac{(F(X) - F(Y))(G(X) - G(Y))}{(X - Y)^2} \tag{2.50}$$

shows it is equal to (2.48); cf. the equality in (2.23).

2. In the case $F_L = G_L$ (2.44) specifies the large L form of the bulk scaled variance of the linear statistic $\sum_{l=1}^{N} F_L(X_l)$. It was established by Soshnikov [157] that the distribution of the centred linear statistic $\sum_{l=1}^{N} F_L(X_l) - L \int_{-\infty}^{\infty} F(X) \, dX$ converges to a zero mean Gaussian with this bulk scaled variance; see also [36].

In the case of the bulk scaling of Gaudin's deformation of the CUE as specified by the probability density function (PDF) (2.29), taking the place of (2.41) is the functional form [93]

$$\lim_{N\to\infty} \left(\frac{N}{2\pi}\right)^2 \rho_{(2),N}^T (2\pi X/N, 2\pi X'/N; \alpha)\Big|_{\alpha=e^{-2\pi a/N}} := \rho_{(2),\infty}^T(X, X'; a)$$
$$= -\left(\frac{1}{2\pi a}\right)^2 \left| \int_{\nu}^{\infty} \frac{e^{-i\omega(X-X')/a}}{e^{\omega} + 1} \right|^2 d\omega, \quad (2.51)$$

where $\nu = -\log(e^{2\pi a} - 1)$. Changing variables $\omega/a = \omega'$, we note that for $a \to \infty$ only the integration region from $\omega' = -2\pi$ to 0 contributes, and (2.41) is reclaimed.

It was shown in [93] that the Fourier transform of (2.51) can be computed explicitly. Forming from this the structure function (recall the definition in (2.42)) gives

$$S_{\infty}(k; a) = 1 - \frac{1}{2} \frac{1}{\sinh(|k|a/2)} \left(\frac{e^{|k|a/2}}{2\pi a} \log\left(1 + e^{-|k|a}(e^{2\pi a} - 1)\right) - e^{-|k|a/2} \right). \tag{2.52}$$

Most significant in the context of fluctuation formulas with a scale parameter L is that the small k limit results in a constant [93]

$$S_{\infty}(k; a) \underset{k\to 0}{\sim} \frac{1}{2\pi a}(1 - e^{-2\pi a}), \tag{2.53}$$

in distinction to the behaviour of the functional form in (2.42). Hence, the analogue of (2.44) is now

$$\lim_{N \to \infty} \mathrm{Cov}^{\mathrm{CUE}_\alpha} \Big(\sum_{l=1}^{N} F_L(X_l), \sum_{l=1}^{N} G_L(X_l) \Big) \Big|_{\alpha=e^{-2\pi a/N}}$$

$$\underset{L \to \infty}{\sim} \frac{L}{2\pi} S_\infty(0; a) \int_{-\infty}^{\infty} \hat{F}(k)\hat{G}(-k) \, dk, \quad (2.54)$$

telling us in particular that the fluctuations are now of order L. This contrasts with the order unity fluctuation for the bulk scaled CUE seen in (2.44), and the order $\log L$ of (2.45). It contrasts too with the order unity fluctuation for the global scaled limit fluctuation of the CUE_α seen in (2.36). With regards to this latter point, note that setting $\alpha = e^{-2\pi a/N}$ for N large, as done in the bulk scaling limit (2.51), would lead to all the coefficients in the formula of (2.36) diverging. We observe too that taking $a \to \infty$ in (2.53), when we know (2.51) limits to the bulk scaled CUE result, gives that $S_\infty(k; a)$ tends to zero as k tends to zero, and so (2.54) breaks down.

Remark 2.6. 1. A behaviour analogous to (2.54) can be obtained in the context of the bulk scaled CUE, modified so that a fraction $(1 - \zeta)$, $0 < \zeta < 1$ of the eigenvalues have been deleted uniformly at random. We will denote this ensemble by $\widehat{\mathrm{CUE}}_\zeta$. Such a model was first considered in detail in [26]; for applications to Odlyzko's data set for the Riemann zeros see [85, 28]. Generally in this setting the density is multiplied by ζ and the two-point function by ζ^2. Hence in place of (2.40) we have

$$\lim_{N \to \infty} \mathrm{Cov}^{\widehat{\mathrm{CUE}}_\zeta} \Big(\sum_{l=1}^{N} F(X_l), \sum_{l=1}^{N} G(X_l) \Big)$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \, F(x)G(x')\Big(\zeta^2 \rho_{(2),\infty}^{T}(X, X') + \zeta\delta(X - X')\Big). \quad (2.55)$$

Furthermore

$$S_\infty^{\widehat{\mathrm{CUE}}_\zeta}(k) = \zeta^2 S_\infty^{\mathrm{CUE}}(k) + (\zeta - \zeta^2),$$

telling us in particular that $S_\infty^{\widehat{\mathrm{CUE}}_\zeta}(0) = \zeta - \zeta^2 \neq 0$. Thus the analogue of (2.54) holds in this setting. A detailed analysis is given in [24], which includes consideration of the critical setting specified by $\zeta = 1 - c/L$, $(c > 0)$, for which the analogue of the RHS of (2.54) is again of order unity but the corresponding distribution is no longer Gaussian.

2. The structure function taking a non-zero value at $k = 0$ as in (2.53) is also a feature of the statistics of the real eigenvalues in the ensemble of $N \times N$ real Gaussian matrices. For such matrices the eigenvalues occur in complex conjugate pairs and moreover there are $O(\sqrt{N})$ real eigenvalues which to leading order have uniform density in the interval $(-\sqrt{N}, \sqrt{N})$ [70]. Results from [87], [33] give that in the bulk scaled limit the truncated two-point correlation has the explicit form

$$\rho_{(2),\infty}^{T}(x, x') = -\frac{1}{2\pi} e^{-(x-x')^2} + \frac{1}{2}\frac{1}{\sqrt{2\pi}}|x - x'|e^{-(x-x')^2/2}\mathrm{erfc}(|x - x'|/\sqrt{2}). \quad (2.56)$$

Defining the structure function as in (2.42), a computer algebra assisted calculation gives $S_\infty(0) = (\sqrt{2} - 1)/\sqrt{\pi}$. Hence the covariance of the scaled linear statistics F_L, G_L is proportional to L as specified by (2.54).

2.5. Dyson Brownian motion for the CUE

Haar distributed matrices $U \in U(N)$ admit a generalisation involving a parameter τ, relating to the heat equation on the corresponding symmetric space; see e.g. [121, §2]. We will refer to this ensemble as $U(N; \tau)$. As first determined by Dyson [65], the eigenvalues $\{e^{i\theta_j(\tau)}\}_{j=1}^N$ then execute a particular Brownian dynamics with corresponding PDF obeying the Fokker-Planck equation

$$\frac{\partial p_t}{\partial \tau} = \mathcal{L} p_t, \qquad \mathcal{L} = \sum_{j=1}^N \frac{\partial}{\partial \theta_j} \left(\frac{\partial W}{\partial \theta_j} + \frac{1}{\beta} \frac{\partial}{\partial \theta_j} \right), \tag{2.57}$$

with $\beta = 2$, $W = -\sum_{1 \le j < k \le N} \log |e^{i\theta_k} - e^{i\theta_j}|$ and subject to a prescribed initial condition. Moreover, as a consequence of an observation of Sutherland [159] that provides a similarity transformation of $\mathcal{L}$ with $\beta = 2$ to a free quantum Hamiltonian, (2.57) can be exactly solved in a determinant form [141]. And moreover, there are a number of initial conditions for which the corresponding dynamical correlations can be written in a structured way.

One such initial condition is the equilibrium solution of (2.57), $p_0 = p_t|_{t\to\infty} \propto e^{-\beta W}$. In particular, the truncated two-point correlation for two different parameters, $\rho_{(1,1)}^T((x, \tau_x), (y, \tau_y))$ say, then has the explicit functional form

$$\rho_{(1,1)}^T((x, \tau_x), (y, \tau_y))$$
$$= \left(\frac{1}{2\pi}\right)^2 \left(\sum_{|n| \le N/2} \left(\frac{w}{z}\right)^n e^{\gamma_n(\tau_y - \tau_x)} \right) \left(\sum_{|n| \ge N/2+1} \left(\frac{z}{w}\right)^n e^{-\gamma_n(\tau_y - \tau_x)} \right). \tag{2.58}$$

Here N is assumed even for convenience, $w = e^{iy}$, $z = e^{ix}$ and $\gamma_n = (n - (1/2)^2)/2$. In the setting of two distinct parameters, replacing (2.5) is the covariance formula

$$\left\langle \sum_{j=1}^N f(x_j), \sum_{j=1}^N g(y_j) \right\rangle^{(\tau_x, \tau_y)} = \int_0^{2\pi} dx \, f(x) \int_0^{2\pi} dy \, g(y) \rho_{(1,1)}^T((x, \tau_x), (y, \tau_y)). \tag{2.59}$$

According to (2.58) we have for the limiting Fourier series form of the truncated two-point correlation

$$\lim_{N\to\infty} \rho_{(1,1)}^T((x, \tau_x), (x, \tau_y)) \Big|_{\tau_y - \tau_x = t/N} = \left(\frac{1}{2\pi}\right)^2 \sum_{n=-\infty}^\infty \left(\frac{z}{w}\right)^n \sum_{q=0}^{|n|-1} e^{-(|n|-2q)t} \tag{2.60}$$

(cf. the $N \to \infty$ form of (2.12)), and consequently

$$\lim_{N \to \infty} \left\langle \sum_{j=1}^{N} f(x_j), \sum_{j=1}^{N} g(x_j) \right\rangle^{U(N;\tau)} \bigg|_{\tau_y - \tau_x = t/N} = \sum_{n=-\infty}^{\infty} \left(\sum_{q=0}^{|n|-1} e^{-(|n|-2q)t} \right) f_n g_{-n}, \tag{2.61}$$

which we see reduces to (2.14) for $t = 0$. The work [121, §9] contains an analogous, but less explicit, formula to (2.61) in the case that Dyson Brownian motion is initialised from the identity.

Also of interest is the bulk scaling limit. In relation to the truncated two-point correlation it follows from (2.58) that [132]

$$\rho_{(1,1)}^{T,\text{bulk}}((X,0),(Y,t)) := \lim_{N \to \infty} (2\pi/N)^2 \rho_{(1,1)}^{T,\text{bulk}}((2\pi X/N, 0), (2\pi Y/N, 4\pi^2 t/N^2))$$

$$= \left(\int_0^1 e^{t(\pi u)^2/2} \cos \pi(Y - X)u \, du \right) \left(\int_1^\infty e^{-t(\pi u)^2/2} \cos \pi(Y - X)u \, du \right). \tag{2.62}$$

Defining the Fourier transform

$$S(k;t) := \int_{-\infty}^{\infty} \rho_{(1,1)}^{T,\text{bulk}}((X,0),(Y,t)) e^{i(X-Y)k} \, dt$$

we can calculate from (2.62) that for small $|k|$ [158], [75, Eq. (13.228)]

$$S(k;t) \sim \frac{|k|}{2\pi} e^{-\pi |k| t}, \tag{2.63}$$

and hence for the parameter dependent extension of (2.44) one obtains

$$\lim_{L \to \infty} \lim_{N \to \infty} \text{Cov}^{U(N;\tau)} \left(\sum_{l=1}^{N} F_L(X_l), \sum_{l=1}^{N} G_L(X_l) \right) \bigg|_{t=LT}$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \hat{F}(k) \hat{G}(-k) |k| e^{-\pi |k| T} \, dk. \tag{2.64}$$

2.6. *Global scaling of Haar distributed real orthogonal random matrices*

Closely related to the CUE is the ensemble of real orthogonal matrices — further distinguished by the determinant equalling plus 1 or minus 1, and the parity of N — chosen with Haar measure. Here the eigenvalues occur in complex conjugate pairs, so the eigenvalues on the unit circle in the lower half plane are not independent. The corresponding point process can then be considered as being restricted to the upper half plane. For the ensemble obtained by taking N even and the determinant to equal plus one for definiteness to give the matrix group $O^+(N)$ we have (see e.g. [75, §5.5.2])

$$\rho_{(2),N}^T(x, x') + \rho_{(1),N}(x)\delta(x - x')$$

$$= -\Big(\frac{1}{\pi} + \frac{2}{\pi}\sum_{l=1}^{N/2}\cos lx \cos lx'\Big)^2 + \delta(x - x')\Big(\frac{1}{\pi} + \frac{2}{\pi}\sum_{l=1}^{N/2}\cos^2 lx\Big). \quad (2.65)$$

Direct calculation reveals that for p, q non-negative integers less than or equal to $N/2$,

$$\int_0^\pi dx \int_0^\pi dx' \left(\rho_{(2),N}^T(x, x') + \rho_{(1),N}(x)\delta(x - x')\right)\cos px \cos qx' = \frac{p}{4}\delta_{p,q}. \quad (2.66)$$

Outside this range, the integral always vanishes if p, q are of distinct parity; it gives the value $1/4$ for p, q of the same parity and not equal, and the value $\min\{(p + 1)/4, (N + 2)/4\}$ for $p = q$. In keeping with the derivation of (2.14), this implies a simple expression for the limiting covariance.

Proposition 2.7. *Let*

$$f(x) = f_0^{\mathrm{c}} + 2\sum_{n=1}^\infty f_n^{\mathrm{c}}\cos(nx), \quad f_n^{\mathrm{c}} = \frac{1}{\pi}\int_0^\pi f(x)\cos(nx)\,dx, \quad (2.67)$$

and similarly for $g(x)$. If f and g are differentiable on $[0, \pi]$ with f', g' Hölder continuous of order $\alpha > 0$, then for N even

$$\lim_{N\to\infty}\mathrm{Cov}^{O^+(N)}\Big(\sum_{l=1}^{N/2}f(x_l), \sum_{l=1}^{N/2}g(x_l)\Big) = \sum_{n=1}^\infty n f_n^{\mathrm{c}}g_n^{\mathrm{c}}. \quad (2.68)$$

Furthermore, if $f = g = \chi_{[L_0,L_1]}$ $(0 < L_0 < L_1 < \pi)$ and $N_{[L_0,L_1]} = \sum_l \chi_{x_l \in [L_0,L_1]}$, then

$$\lim_{N\to\infty}\frac{1}{\log N}\mathrm{Var}^{O^+(\mathrm{N})}\left(N_{[L_0,L_1]}\right) = \frac{1}{\pi^2}. \quad (2.69)$$

Proof. The reasoning relating to (2.68) has already been given. In relation to (2.69), with $f = g = \chi_{[L_0,L_1]}$, $0 < L_0 < L_1 < \pi$, we compute that for $n \neq 0$, $f_n^{\mathrm{c}} = (\sin(L_1 n) - \sin(L_0 n))/(\pi n)$. Substituting in the RHS of (2.68), with the sum truncated at n equals order N as is justified by (2.66) and surrounding text, gives (2.69). $\square$

Remark 2.8. 1. The centred characteristic function associated with the linear statistic $\sum_{l=1}^{N/2} f(x_l)$, $0 < x_l < \pi$, for the ensemble of random real orthogonal matrices $O^+(N/2)$ has the limiting Gaussian form

$$\lim_{N\to\infty}\Big\langle\prod_{l=1}^{N/2}e^{it(f(x_l)-f_0^{\mathrm{c}})}\Big\rangle^{O^+(N)} = \exp\Big(-\frac{t^2}{2}\sum_{n=1}^\infty n(f_n^{\mathrm{c}})^2\Big) \quad (2.70)$$

(cf. (2.21)). This was first established by Johansson [110] for f polynomial; [49] extends the validity to $f = e^{V(e^{ix})} = e^{V(e^{-ix})}$ for V analytic in a neighbourhood of the unit circle.

2. For $R \in O^+(N/2)$ we have $\operatorname{Tr} R^k = \sum_{p=1}^{N/2} \cos kpx$. This linear statistic has a property analogous to that of $|\operatorname{Tr} U^k|^2$ for $U \in \text{CUE}$ noted in Remark 2.3.2. Thus the first $N/2$ moments coincide with those of $\sqrt{k}$ times a standard real Gaussian random variable [56], a fact which is related to the RHS of (2.66) being independent of N for all non-negative integers $p, q \leq N/2$.

2.7. The COE and CSE

Starting with matrices $U_N \in U(N)$ chosen with Haar measure, then forming symmetric unitary matrices $U_N^T U_N$ gives Dyson's circular orthogonal ensemble (COE). A variation is to begin with matrices $U_{2N} \in U(2N)$ chosen with Haar measure. Forming the self dual quaternion unitary matrices $Z_{2N}^{-1} U_{2N}^T Z_{2N} U_{2N}$, where $Z_{2N} = \mathbb{I}_N \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, gives Dyson's circular symplectic ensemble (CSE).

Define

$$S_N(\theta) = \frac{1}{2\pi} \frac{\sin(N\theta/2)}{\sin(\theta/2)}, \qquad D_N(\theta) = \frac{d}{d\theta} S_N(\theta),$$

$$I_N(\theta) = \int_0^\theta S_N(\theta') \, d\theta', \qquad J_N(\theta) = I_N(\theta) - \frac{1}{2}\operatorname{sgn}(\theta). \tag{2.71}$$

In terms of these quantities the corresponding two-point correlation functions read [68, 162]

$$\rho_{(2),N}^{\text{COE}}(\theta, \theta') = \left((S_N(\theta - \theta'))^2 - D_N(\theta - \theta') J_N(\theta - \theta') \right), \tag{2.72}$$

$$\rho_{(2),N}^{\text{CSE}}(\theta, \theta') = \frac{1}{4} \left((S_{2N}(\theta - \theta'))^2 - D_{2N}(\theta - \theta') I_{2N}(\theta - \theta') \right). \tag{2.73}$$

Starting from these expressions and defining the Fourier coefficients m_l^{COE} and m_l^{CSE} as in (2.12), we know from [140, 168] that

$$m_l^{\text{COE}} = N - (N - |l|)\chi_{N-|l|>0} + \min(|l|, N) - 2l \left(\sum_{s=M_-}^{M_+} \frac{1}{2s-1} \right) \tag{2.74}$$

$$m_l^{\text{CSE}} = \begin{cases} \dfrac{|l|}{2} + \dfrac{|l|}{2} \left(\dfrac{1}{2N-1} + \dfrac{1}{2N-3} + \cdots + \dfrac{1}{2N-(2|l|-1)} \right), & |l| \leq 2N-2 \\ N, & |l| > 2N-2, \end{cases} \tag{2.75}$$

where $M_- := \frac{1}{2}(N+1) + \max(0, |l| - N) + 1$, $M_+ := \frac{1}{2}(N+1) + |l|$. For $l > 0$ the quantity m_l^{COE} monotonically increases to the value N, while m_l^{CSE} has a single maximum at $l = N$, which to leading order in N is equal to $(N/4) \log N$.

From the exact results (2.74) we can deduce the analogues of (2.14) and (2.15).

Proposition 2.9. *Label the COE and CSE by $\beta = 1$ and $\beta = 4$ respectively. We have*

$$\lim_{N \to \infty} \mathrm{Cov}^{(\beta)} \left(\sum_{l=1}^{N} f(x_l), \sum_{l=1}^{N} g(x_l) \right) = \frac{2}{\beta} \sum_{l=-\infty}^{\infty} |l| f_l g_{-l}, \qquad (2.76)$$

while if $f = g = \chi_{[0,L]}$ $(0 < L < 2\pi)$ and N_L is specified by (1.1) then

$$\lim_{N \to \infty} \frac{1}{\log N} \mathrm{Var}^{(\beta)} (N_L) = \frac{1}{\beta \pi^2}. \qquad (2.77)$$

Proof. We consider (2.76) only; the working required in relation to (2.77) uses the same arguments with $f_l = g_l$ the explicit functional form noted in the proof of Proposition 2.2. We see from the above results that the Fourier coefficients have the functional form

$$m_l^{(\beta)} = \frac{2l}{\beta} + r_{l,N}^{(\beta)}, \quad |l| \le N, \qquad (2.78)$$

where $\lim_{N \to \infty} \sum_{l=1}^{N} f_l g_{-l} r_{l,N}^{(\beta)} = 0$ provided $f_l g_{-l} = O(1/l^{2+\epsilon})$, $\epsilon > 0$. We see too that outside this range, and with the same assumed decay of $f_l g_{-l}$, we have $\lim_{N \to \infty} \sum_{l=N+1}^{\infty} f_l g_{-l} m_l^{(\beta)} = 0$. The limit formula (2.76) now follows. $\qquad \square$

The bulk scaled limit is also of interest. For this we introduce the appropriate bulk scaling of the quantities (2.71),

$$S_\infty(X) = \frac{\sin \pi X}{\pi X}, \qquad D_\infty(X) = \frac{d}{dX} S_\infty(X),$$

$$I_\infty(X) = \int_0^X S_\infty(X') \, dX', \qquad J_\infty(X) = I_\infty(X) - \frac{1}{2} \mathrm{sgn}(X). \qquad (2.79)$$

We then see from (2.72), (2.73)

$$\lim_{N \to \infty} \left(\frac{2\pi}{N} \right)^2 \rho_{(2),N}^{\mathrm{COE}}(2\pi X/N, 2\pi X'/N) =: \rho_{(2),\infty}^{\mathrm{COE}}(X, X')$$

$$= (S_\infty(X - X'))^2 - D_\infty(X - X') J_\infty(X - X'), \quad (2.80)$$

$$\lim_{N \to \infty} \left(\frac{2\pi}{N} \right)^2 \rho_{(2),N}^{\mathrm{CSE}}(2\pi X/N, 2\pi X'/N) =: \rho_{(2),\infty}^{\mathrm{CSE}}(X, X')$$

$$= (S_\infty(2(X - X')))^2 - D_\infty(2(X - X')) \frac{1}{2} I_\infty(2(X - X')), \quad (2.81)$$

as first deduced in the work of Dyson [64] and Mehta–Dyson [127] respectively, although the functional form (2.81) is also contained in [64]. It results there from the computation of the bulk scaled two-point correlation function of every second eigenvalue in the COE; see [75, §4.2.3] for more on the implied inter-relationship. The work [64] also contains the computation of the Fourier

transform of the corresponding truncated two-point correlations, which gives for the corresponding structure functions

$$S_\infty^{\mathrm{COE}}(k) = \begin{cases} \frac{|k|}{\pi} - \frac{|k|}{2\pi}\log\left(1 + \frac{|k|}{\pi}\right), & |k| \le 2\pi, \\ 2 - \frac{|k|}{2\pi}\log\frac{|k|/\pi+1}{|k|/\pi-1}, & |k| \ge 2\pi. \end{cases} \tag{2.82}$$

$$S_\infty^{\mathrm{CSE}}(k) = \begin{cases} \frac{|k|}{4\pi} - \frac{|k|}{8\pi}\log|1 - \frac{|k|}{2\pi}|, & |k| \le 4\pi, \\ 1, & |k| \ge 4\pi. \end{cases} \tag{2.83}$$

Consequently, as made explicit in [67] in the case of the bulk scaled COE, the leading small $|k|$ term from these functional forms implies that the limiting variance formula (2.44) as obtained for the bulk scaled CUE is modified only by a simple proportionality,

$$\lim_{L\to\infty}\lim_{N\to\infty}\mathrm{Cov}^{(\beta)}\left(\sum_{l=1}^N F_L(X_l), \sum_{l=1}^N G_L(X_l)\right) = \frac{2}{\beta}\frac{1}{(2\pi)^2}\int_{-\infty}^\infty \hat{F}(k)\hat{G}(-k)|k|\,dk, \tag{2.84}$$

where the meaning of β is as in Proposition 2.9.

In relation to the bulk scaled linear statistic $\sum_{l=1}^N \chi_{X_l\in[0,L]}$, following the strategy outlined in [64] it has been noted in [75, §14.5.1] that for the COE and CSE (and too the CUE upon identifying $\beta = 2$)

$$\lim_{N\to\infty}\mathrm{Var}\left(\sum_{l=1}^N \chi_{X_l\in[0,L]}\right) \sim \frac{2}{\pi^2\beta}\log L + B_\beta, \tag{2.85}$$

where, with C denoting Euler's constant,

$$B_\beta = \frac{2}{\pi^2\beta}C + \frac{2}{\pi}\int_0^1 \frac{1}{y^2}\left(S_\infty^{(\beta)}(y) - \frac{y}{\pi\beta}\right)dy + \frac{2}{\pi}\int_1^\infty \frac{1}{y^2}S_\infty^{(\beta)}(y)\,dy. \tag{2.86}$$

Substituting (2.82) gives [64]

$$B_1 = \frac{2}{\pi^2}C + \frac{2}{\pi^2}\left(1 + \log 2\pi\right) - \frac{1}{4}, \tag{2.87}$$

while substituting (2.82) leads to the formula [126, Eq. (16.1.4)]

$$B_4 = \frac{1}{2\pi^2}C + \frac{1}{2\pi^2}\left(1 + \log 4\pi\right) + \frac{1}{16}. \tag{2.88}$$

Remark 2.10. A recent result [1] gives that for a one-dimensional point process, in the limit $L \to \infty$

$$\lim_{N\to\infty}\mathrm{Var}\left(\sum_{l=1}^N \chi_{X_l\in[0,L]}\right) \asymp \left(L^2\int_{|x|<c/L}S_\infty(x)\,dx + \int_{|x|>c/L}\frac{S_\infty(x)}{x^2}\,dx\right), \tag{2.89}$$

which is consistent with (2.85) and furthermore gives some insight into the structure of (2.86). In fact [1] gives an analogous asymptotic bound in the d-dimensional case for $\sum_{l=1}^N \chi_{|\mathbf{r}_l|<L}$.

2.8. *Bulk scaling of the circular β-ensemble*

Unitary random matrices with eigenvalue PDF proportional to

$$\prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta \tag{2.90}$$

are said to form the circular β-ensemble. The cases $\beta = 1, 2$ and 4 are realised by the COE, CUE and CSE respectively. For general $\beta > 0$ there is a realisation in terms of certain unitary Hessenberg random matrices [114]. As emphasised by Dyson [63], upon writing (2.90) in the form

$$e^{-\beta \sum_{1 \leq j < k \leq N} \phi(e^{i\theta_j}, e^{i\theta_k})}, \qquad \phi(z_j, z_k) = -\log |z_k - z_j|, \tag{2.91}$$

there is analogy with the equilibrium statistical mechanics of particles repelling pairwise via the logarithmic potential and confined to a circle. The interpretation of β is then as the inverse temperature.

For $\beta = p/q$ a positive rational number in reduced form, the bulk scaled structure function $S_\infty(k; \beta)$ is known explicitly [83]. This functional form shows that the quantity

$$F(k; \beta) = \frac{\pi\beta}{k} S(k; \beta) \tag{2.92}$$

in the range $0 < k < \min(2\pi, \pi\beta)$ extends to an analytic function of k about the origin with radius of convergence $\min(2\pi, \pi\beta)$. The leading terms of the corresponding power series in k — in this the coefficient of k^j is a polynomial of degree j in $(2/\beta)$ which has particular palindromic properties — are known up to an including $j = 10$ [84, 78], with the first two being

$$f(k; \beta) = 1 + \frac{1}{2\pi}(1 - 2/\beta)k + \cdots \tag{2.93}$$

Hence

$$S(k; \beta) = \frac{|k|}{\pi\beta} + \frac{1}{2\pi^2\beta}(1 - 2/\beta)k^2 + \cdots \tag{2.94}$$

Since the derivation of (2.84) is determined entirely by the leading term in this expansion, we see that this same expression, derived previously for $\beta = 1, 2$ and 4, holds for all $\beta > 0$.

The derivation of (2.45) for $\beta = 2$ and (2.85) in the cases $\beta = 1, 4$, when used in conjunction with the knowledge (2.94) affirms the formula (2.85) as valid for general $\beta > 0$. Moreover the recent work [92], using a β-generalisation of the Fisher-Hartwig conjecture from the theory of Toeplitz determinants [80] has computed for the constant B_β in (2.85)

$$B_\beta = \frac{2}{\pi^2\beta}\left(C + \log\beta + \sum_{q=1}^{\infty}\left(\frac{2}{\beta}\psi^{(1)}(2q/\beta) - \frac{1}{q}\right)\right), \tag{2.95}$$

where $\psi^{(1)}(z) := \frac{d^2}{dz^2}\log\Gamma(z)$. The work [156] shows that the constant B_β also occurs in the next order term of the global scaling of $\mathrm{Var}\,(N_L)$ for $0 < L < 2\pi$, with the leading term being proportional to $\log N$.

Remark 2.11. Setting $\beta = 2\alpha/N$, then taking the limit $N \to \infty$ corresponds to a scaled high temperature limit. This is known to give a well defined statistical state for $\alpha > -1$. In particular, we have from [103] that

$$\lim_{N \to \infty} \frac{1}{N} \text{Var}^{(2\alpha/N)} \Big(\sum_{j=1}^{N} f(x_j) \Big) = 2 \sum_{k=1}^{\infty} \frac{1}{1 + \alpha/k} |f_l|^2. \tag{2.96}$$

Note that when viewed as a large N statement, substituting for α in terms of β and N reclaims (2.84) with $f = g$.

2.9. Two-dimensional support

In the mid 1960's Ginibre introduced into random matrix theory the study of the eigenvalue statistics of, among other ensembles, $N \times N$ standard complex Gaussian random matrices [95]. For this ensemble, to be denoted GinUE, all the eigenvalues are in the complex plane. It was shown in [95] that the statistical state of the eigenvalues forms a determinantal point process, with the $N \to \infty$ bulk correlation kernel

$$K_\infty^{\text{GinUE}}(w, z) = \frac{1}{\pi} e^{-(|w|^2 + |z|^2)/2} e^{w\overline{z}} \tag{2.97}$$

(in distinction to the derivation of (2.41), no scaling of the eigenvalues is required as part of the limit). Consequently the corresponding two-point correlation function has the simple functional form

$$\rho_{(2),\infty}^{\text{GinUE}}(w, z) = \frac{1}{\pi^2} \Big(1 - \exp(-|w - z|^2) \Big). \tag{2.98}$$

This in turn implies that up to a constant the structure function, defined by the two-dimensional version of the integral in (2.42), is also a Gaussian

$$S_\infty^{\text{GinUE}}(\mathbf{k}) = \frac{1}{\pi} \Big(1 - e^{-|\mathbf{k}|^2/4} \Big). \tag{2.99}$$

The analogue of (2.44) can now readily be deduced.

Proposition 2.12. *Let $z_l = x_l + iy_l$. We have [75, Exercises 15.4]*

$$\lim_{L \to \infty} \lim_{N \to \infty} \text{Cov}^{\text{GinUE}} \Big(\sum_{l=1}^{N} f(x_l/L, y_l/L), \sum_{l=1}^{N} g(x_l/L, y_l/L) \Big)$$
$$= \frac{1}{(2\pi)^2} \frac{1}{4\pi} \int_{\mathbb{R}^2} \hat{f}(\mathbf{k}) \hat{g}(-\mathbf{k}) |\mathbf{k}|^2 \, d\mathbf{k}, \tag{2.100}$$

valid provided the integral converges.

Proof. Starting with the two-dimensional version of (2.43), the essential point in the derivation of (2.100) is the small $|\mathbf{k}|$ form of $S_\infty^{\text{GinUE}}(\mathbf{k})$. Thus, we read off from (2.99) that

$$S_\infty^{\text{GinUE}}(\mathbf{k}) \sim \frac{|\mathbf{k}|^2}{4\pi}. \tag{2.101}$$

$\square$

As is consistent with (2.11) set

$$C_\infty^{\mathrm{GinUE}}(\mathbf{r} - \mathbf{r}') := \rho_{(2),\infty}^{\mathrm{GinUE}}(\mathbf{r} - \mathbf{r}', \mathbf{0}) - \frac{1}{\pi^2} + \frac{1}{\pi}\delta(\mathbf{r} - \mathbf{r}'), \qquad (2.102)$$

so that

$$\int_{\mathbb{R}^2} C_\infty^{\mathrm{GinUE}}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{r} = S_\infty^{\mathrm{GinUE}}(\mathbf{k}). \qquad (2.103)$$

For a general region $\Lambda \in \mathbb{C}$, the two-dimensional analogue of (2.43) then gives

$$\lim_{N\to\infty} \mathrm{Var}^{\mathrm{GinUE}}\Big(\sum_{l=1}^{N}\chi_{z_l\in\Lambda}\Big) = \int_{\mathbb{R}^2} d\mathbf{r} \int_{\mathbb{R}^2} d\mathbf{r}'\, C_\infty^{\mathrm{GinUE}}(\mathbf{r} - \mathbf{r}')\chi_{\mathbf{r}\in\Lambda}\chi_{\mathbf{r}'\in\Lambda}. \quad (2.104)$$

Simple manipulation of (2.104) gives

$$\lim_{N\to\infty} \mathrm{Var}^{\mathrm{GinUE}}\Big(\sum_{l=1}^{N}\chi_{z_l\in\Lambda}\Big)$$
$$= \int_{\mathbb{R}^2} d\mathbf{r} \int_{\mathbb{R}^2} d\mathbf{r}'\, C_\infty^{\mathrm{GinUE}}(\mathbf{r})(\chi_{\mathbf{r}+\mathbf{r}'\in\Lambda} - 1)\chi_{\mathbf{r}'\in\Lambda} + |\Lambda|\int_{\mathbb{R}^2} C_\infty^{\mathrm{GinUE}}(\mathbf{r})\,d\mathbf{r}.$$

According to (2.103) the final term in this expression is equal to $S_\infty^{\mathrm{GinUE}}(\mathbf{0})$, which from (2.98) is equal to 0. On the other hand, as the region Λ is scaled to infinity in a self similar manner, $\Lambda \mapsto \lambda\Lambda$, with $\lambda \to \infty$, the quantity in the first term $\int_{\mathbb{R}^2} d\mathbf{r}'\,(\chi_{\mathbf{r}+\mathbf{r}'\in\Lambda} - 1)\chi_{\mathbf{r}'\in\Lambda}$ for fixed $\mathbf{r}$ is to leading order proportional to the surface area of $\lambda\Lambda$, $|\partial(\lambda\Lambda)|$ say [124]. Hence in this limit

$$\lim_{N\to\infty} \mathrm{Var}^{\mathrm{GinUE}}\Big(\sum_{l=1}^{N}\chi_{z_l\in\lambda\Lambda}\Big) \sim c_\Lambda|\partial(\lambda\Lambda)| \qquad (2.105)$$

for some proportionality c_Λ. An illustration for Λ a square centred at the origin and rotated at random is given in Figure 2.2. Generally in two or more dimensions point processes with the property that the variance of the number of particles in a region scales with the surface area of the region have been termed hyperuniform [161, 160, 94].

The quantity

$$\frac{1}{|\Lambda|}\int_{\mathbb{R}^2}\chi_{\mathbf{r}+\mathbf{r}'\in\Lambda}\chi_{\mathbf{r}'\in\Lambda}\,d\mathbf{r}' \qquad (2.106)$$

relevant to a direct computation of (2.104) has been evaluated in [161] for the case of $\Lambda = \Lambda_R$ a disk of radius R centred at the origin. Then (2.106) is rotationally invariant and thus a function of r/R, where $r := |\mathbf{r}|$, to be denoted $\alpha(r/R)$ say. We see from the definition that $\alpha(r/R) = 0$ for $r \geq 2R$. For $0 < r < 2R$ the result of [161] gives

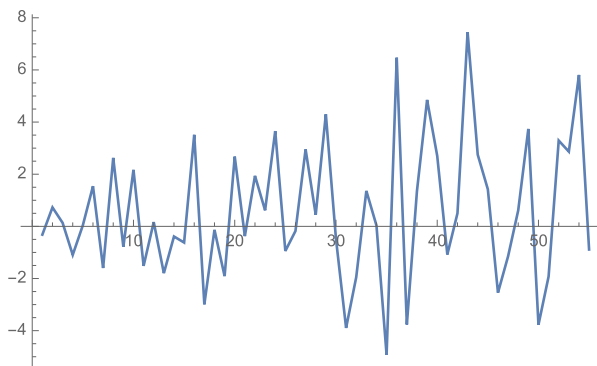$$\alpha(r/R) = \frac{2}{\pi}\Big(\mathrm{Arcos}\,x - x(1 - x^2)^{1/2}\Big), \quad x = r/2R,$$

FIG 2.2. *Plot of values of the random variable $(\sum_{j=1}^{N}(\chi_{|x_j|<L/2}\chi_{|y_j|<L/2})^{\circ} - L^2/\pi)$ for $z_j = x_j + iy_j$ the eigenvalues of a single $N = 1,600$ GinUE matrix, as a function of $L = 1, 2, \ldots, 56$; the values of the random variable have been joined for visual clarity. The symbol $\circ$ indicates that for each L the square has been rotated by some uniformly chosen angle. Note that the growth is approximately of the order of $\sqrt{L}$.*

which has the large R form

$$\alpha(r/R) = 1 - \frac{2}{\pi}\frac{r}{R} + O\left(\frac{r}{R}\right)^2.$$

This substituted in (2.104) implies that for large R

$$\lim_{N\to\infty}\text{Var}^{\text{GinUE}}\left(\sum_{l=1}^{N}\chi_{z_l \in \Lambda_R}\right) \sim -2R\int_{\mathbb{R}^2}|\mathbf{r}|C_{\infty}^{\text{GinUE}}(\mathbf{r})\,d\mathbf{r} = \frac{R}{\sqrt{\pi}}, \qquad (2.107)$$

where the equality follows upon recalling the definition (2.102) and the exact result (2.98). Note that this is consistent with (2.105).

One feature of the GinUE eigenvalues is that to leading order their density is uniform in the disk $|z| < 1$. This feature is shared by zeros of the random polynomial [101, 102]

$$p_N(z) = a_0 + a_1 z + \cdots + a_N z^N,$$

where each coefficient a_j is a zero mean complex random variable with variance $\sigma_j^2 = 1/j!$. The bulk large N limiting form of the zeros two-point correlation function is known from [102] to be given by

$$\rho_{(2)}(z_1, z_2) = \frac{1}{\pi^2}f(|z_1 - z_2|^2/2), \qquad (2.108)$$

where

$$f(x) := \frac{(\sinh^2 x + x^2)\cosh x - 2x\sinh x}{\sinh^3 x} = \frac{1}{2}\frac{d^2}{dx^2}(x^2\coth x); \qquad (2.109)$$

for the equality in (2.109) see [81]. The asymptotic relation in (2.107) applies equally as well to the two-dimensional point process for the zeros of this complex Gaussian polynomial (to be denoted cGP), and gives [81]

$$\lim_{R\to\infty} \frac{1}{|\partial\Lambda_R|} \text{Var}^{\text{cGP}}\Big(\sum_{l=1}^N \chi_{z_l\in\Lambda_R}\Big) = \frac{1}{8\pi^{3/2}}\zeta(3/2), \qquad (2.110)$$

where $\zeta(s)$ denotes the Riemann zeta function, and use has been made of (2.109) to compute the integral in (2.107).

With regards to the analogue of (2.100), we know the driving feature is the small $|\mathbf{k}|$ form of the structure function (2.101). Defining $S_\infty^{\text{cGP}}(\mathbf{k})$ by the analogue of (2.103) and expanding for small $|\mathbf{k}|$ shows

$$S_\infty^{\text{cGP}}(\mathbf{k}) = c_0 + c_2|\mathbf{k}|^2 + c_4|\mathbf{k}|^4 + \cdots, \qquad c_{2j} \propto \int_0^\infty r^{2j+1}C_\infty^{\text{cGP}}(r)\,dr.$$

It follows from (2.108) and (2.109) that $c_0 = c_2 = 0$. The first nonzero coefficient is c_4, which is readily computed giving

$$S_\infty^{\text{cGP}}(\mathbf{k}) \sim c_4|\mathbf{k}|^4, \qquad c_4 = \zeta(3)/8\pi. \qquad (2.111)$$

This implies that for $L \to \infty$ [81]

$$\lim_{N\to\infty} \text{Cov}^{\text{cGP}}\Big(\sum_{l=1}^N f(x_l/L, y_l/L), \sum_{l=1}^N g(x_l/L, y_l/L)\Big)$$
$$\sim \frac{c_4}{(2\pi)^2}\frac{1}{L^2}\int_{\mathbb{R}^2}\hat{f}(\mathbf{k})\hat{g}(-\mathbf{k})|\mathbf{k}|^4\,d\mathbf{k}, \quad (2.112)$$

assuming the integral converges, or in words the covariance goes to zero at a rate proportional to $1/L^2$.

2.10. Summarising remarks and heuristics

2.10.1. The two classes of large N limits

The results of this Section have been based on the double integral formula for the covariance (2.5). Starting from this generic formula, the aim has been to give its limiting form in two distinct large N settings. One is a global scaling limit, in which for $N \to \infty$ the eigenvalue support is a finite integral. In the analysis of this Section, which has relied on explicit Fourier analysis of the two-point correlation function, analytic results for the global scaling limit of (2.5) were obtained for Dyson's circular ensembles, a deformation of the CUE due to Gaudin, and the ensemble of real orthogonal matrices with Haar measure. The latter is distinct as translation invariance is broken. In all these cases it has been possible to reduce the double integral to a single integral with a simple integrand

involving the Fourier transform of the linear statistics. The mechanism for this is that by direct calculation the Fourier transform of the structure function for these ensembles could be shown to have a simple form. For all the ensembles analysed in this limit, it was found that for large N the covariance is of order unity for smooth linear statistics, This behaviour contrasts with that of a gas of noninteracting eigenvalues, for which the covariance is proportional to N; recall (2.28). For the linear statistic counting the number of eigenvalues in an interval, the linear statistic is a step function and so not smooth. Exact calculation leads to the conclusion that the variance, in the global scaling limit of the same ensembles analysed in the case of a smooth statistic, is then proportional to $\log N$.

The other large N setting of interest is to first compute what in statistical mechanics is referred to as the thermodynamic limit, and termed above as the bulk scaling limit. Thus the coordinates are scaled so that the mean density is of order unity and the limit $N \to \infty$ then performed. Next a length scale L is introduced into the linear statistics so that they vary on this scale, and finally the large L limit is considered. Direct analysis of this limit is simpler than for the global scaling limit. In addition to the ensembles already analysed, it is possible to study the covariance of two linear statistics for the eigenvalues of the circular β ensemble, for the real eigenvalues of the ensemble of $N \times N$ real Gaussian matrices, and the eigenvalues of complex Ginibre ensemble. In the case of real Gaussian matrices the covariance is proportional to L, which is a characteristic property of the structure function being nonzero at the origin. A feature of the eigenvalues of the complex Ginibre ensemble is that the variance of the counting function for the number of eigenvalues in a region scales with the length of the boundary of that region.

2.10.2. Consistency with log-gas predictions

The log-gas analogy for the eigenvalue PDF for the circular β ensemble (2.90) leads to predictions for both the smoothed bulk and global scaled forms of the quantity $C_{(2),N}(x, x')$, as required for the determination of the corresponding fluctuation formulas [21, 107, 73]. The fluctuation formulas obtained using this heuristic are consistent with the exact results obtained in the case of the circular β ensemble and moreover the working can be extended to apply to other random matrix ensembles. This is possible because of log-gas analogies for those random matrix ensembles too.

When the eigenvalue PDF permits a Bolzmann factor interpretation $e^{-\beta U}$, it is possible to take the viewpoint that a linear statistic $U_u := \sum_{j=1}^{N} u(x_j)$ is a perturbing external one body potential, so that the perturbed Boltzmann factor becomes $e^{-\beta(U_u+U)}$. Expanding the factor $e^{-\beta U_u}$ to first order in u, $e^{-\beta U_u} \approx 1 - \beta \sum_{j=1}^{N} u(x_j)$ we can check from the definitions that

$$q_u(x') := \langle n_{(1),N}(x') \rangle_u - \langle n_{(1),N}(x') \rangle_{u=0} = -\beta \int_I u(x) C_{(2),N}(x, x') \, dx, \ (2.113)$$

where $C_{(2),N}(x, x')$ is the quantity (2.11) computed in the absence of U_u and $n_{(1),N}(x') := \sum_{j=1}^N \delta(x' - x_j)$. The key hypothesis is that for large N, the LHS of (2.113) is determined by the macroscopic electrostatics implied by the pair potential $\phi(z, z')$ in (2.91), and thus satisfies the integral equation

$$-\int_0^{2\pi} \log |\sin(x - x')/2| q_u(x') \, dx' = u(x) + C. \tag{2.114}$$

Here the constant C is determined by the particle conservation condition

$$\int_0^{2\pi} q_u(x') \, dx' = 0. \tag{2.115}$$

The functional form of $q_u(x)$ can readily be determined [75, Prop. 14.3.4].

Proposition 2.13. *With $u(x) = \sum_{p=-\infty}^{\infty} u_p e^{ip\theta}$ specified, the solution $q_u(x)$ of the integral equation (2.114), subject to the constraint (2.115), is given by the Fourier series*

$$q_u(x) = -\frac{1}{\pi} \sum_{p=-\infty}^{\infty} |p| u_p e^{ip\theta}. \tag{2.116}$$

Proof. Substituting the Fourier series implicit in (2.24) for $\log |\sin(x - x')/2|$ in (2.114) together with the Fourier series of $u(x)$ gives

$$\sum_{p=-\infty}^{\infty} \alpha_p e^{ip\theta} \int_0^{2\pi} q_u(x') e^{-ipx'} \, dx' = \sum_{p=-\infty}^{\infty} |p| u_p e^{ip\theta} + C,$$

where α_p is defined in (2.25). Equating coefficients of $e^{ip\theta}$ and requiring (2.115) gives the value of the Fourier coefficients of $q_u(x)$ and (2.116) follows.   □

Now it follows from (2.113), the definition of $C_{(2),N}(x, x')$ (2.11) and (2.5) that

$$\text{Cov}\left(\sum_{l=1}^N f(x_l), \sum_{l=1}^N g(x_l)\right) = -\frac{1}{\beta} \int_I f(x) q_g(x) \, dx. \tag{2.117}$$

In the case of the circular-β ensemble, assuming the validity of the hypothesis that for large N, q_u is determined by (2.114) and we see by substituting (2.116) that the fluctuation formula (2.76) results, now predicted to be valid for general $\beta > 0$.

We know from Proposition 2.7 that the limiting covariance formula in the case of the independent eigenvalues for random $O^+(N)$ matrices takes the simple form (2.68) involving the cosine transform. The joint PDF for the independent eigenvalues for Haar distributed $O^+(N)$ matrices is proportional to (see e.g. [75, Eq. (2.62)])

$$\prod_{1 \le j < k \le N/2} |\cos x_k - \cos x_j|^\beta, \qquad 0 \le x_l \le \pi, \tag{2.118}$$

with $\beta = 2$. Writing this in Boltzmann factor form the corresponding pair potential is $\phi^c(x_j, x_k) = -\log |\cos x_k - \cos x_j|$ (here the superscript "c" indicates the

involvement of cosine). Hence from the viewpoint of macroscopic electrostatics the task is to solve the integral equation

$$-\int_0^\pi \log|\cos x - \cos x'|q_u^c(x')\,dx' = u(x) + C, \quad \text{subject to} \quad \int_0^\pi q_u^c(x')\,dx' = 0. \tag{2.119}$$

Knowledge of the expansion (see e.g. [75, Exercises 1.4 q.4])

$$\log(2|\cos x - \cos t|) = -\sum_{n=1}^\infty \frac{2}{n}\cos nx \cos nt$$

shows that with $u(x)$ given in terms of its cosine expansion as in (2.67), the solution of (2.119) is given by

$$q_u^c(x) = -\frac{2}{\pi}\sum_{p=1}^\infty p u_p^c \cos px. \tag{2.120}$$

Substituting this in the RHS of (2.117) with $\beta = 2$ reclaims (2.36) and moreover predicts that its generalisation for $\beta > 0$ in the sense of (2.118) is to multiply the RHS therein by $2/\beta$.

Remark 2.14. Gaudin's eigenvalue PDF (2.29) can be written in Boltzmann factor form involving a pair potential. However this pair potential, as seen in the second expression of (2.29), is not long range in an appropriate scaling limit. Due to this, it is not expected that the hypothesis of an analogue of (2.114) will be valid. Indeed, assuming it is leads to a result for the covariance which is in contradiction to the exact result (2.36).

3. Other structures leading to explicit formulas

3.1. The Gaussian β-ensemble

Under the change of variables $y_j = \cos x_j$ and with $N/2$ replaced by N (this for convenience) the PDF (2.118) becomes proportional to

$$\prod_{l=1}^N (1 - y_l^2)^{-1/2}\prod_{1 \le j < k \le N} |y_k - y_j|^\beta, \qquad |y_l| < 1. \tag{3.1}$$

This is an example of a Jacobi β-ensemble (see e.g. [75, §3.11] and Section 3.2 below). It follows from the derivation of (2.120) that the macroscopic log-gas viewpoint predicts for the corresponding fluctuation formula,

$$\lim_{N \to \infty} \mathrm{Cov}^{\mathrm{J}}\Big(\sum_{j=1}^N f(x_j), \sum_{j=1}^N g(x_j)\Big) = \frac{2}{\beta}\sum_{n=1}^\infty n f_n^c g_n^c \tag{3.2}$$

independent of the details of the one body term $\prod_{l=1}^N (1 - y_l^2)^{-1/2}$. The important point is that the eigenvalue support is the interval $(-1, 1)$, or more generally a

single interval (a, b), and that the underlying pair potential in the Boltzmann factor analogy is $-\log |y_j - y_k|$.

The leading order eigenvalue support being a single interval is shared by a number of random matrix ensembles with an eigenvalue PDF of the form

$$\prod_{l=1}^{N} e^{-\beta N V(x_l)} \prod_{1 \leq j < k \leq N} |x_k - x_j|^{\beta}, \tag{3.3}$$

and thus also having an underlying logarithmic pair potential. Our interest in this section is in the Gaussian β-ensemble, specified by setting $V(x) = x^2$ in (3.3). Even though for finite N the eigenvalues may be located anywhere on the real line, as $N \to \infty$ their support is the single interval, $(-1, 1)$ say. The log-gas argument of subsection 2.10.2 then predicts an identical expression to (3.2) for the covariance,

$$\lim_{N \to \infty} \mathrm{Cov}^{\mathrm{G}} \left(\sum_{j=1}^{N} f(x_j), \sum_{j=1}^{N} g(x_j) \right) = \frac{2}{\beta} \sum_{n=1}^{\infty} n f_n^{\mathrm{c}} g_n^{\mathrm{c}}$$

$$= \frac{2}{\beta} \frac{1}{8\pi^2} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} d\phi \, \frac{(f(\cos \theta) - f(\cos \phi))(g(\cos \theta) - g(\cos \phi))}{|e^{i\theta} - e^{i\phi}|^2}$$

$$= \frac{2}{\beta} \frac{1}{4\pi^2} \int_{-1}^{1} dx \int_{-1}^{1} dy \, \frac{(f(x) - f(y))(g(x) - g(y))}{(x - y)^2} \frac{1 - xy}{(1 - x^2)^{1/2}(1 - y^2)^{1/2}}. \tag{3.4}$$

Here the second equality can be seen to imply the first upon using the identity (2.25) and integrating by parts. In the second equality the integration domain can be reduced to $[0, \pi]^2$ by replacing the denominator by

$$\frac{1}{|e^{i\theta} - e^{i\phi}|^2} + \frac{1}{|e^{i\theta} - e^{-i\phi}|^2}$$

and multiplying by 2. A simple change of variables then gives the third equality [117]. The reference [41] gives a different perspective on the first equality starting from the third equality.

The identity

$$\frac{(1 - xy)}{(1 - x^2)^{1/2}(1 - y^2)^{1/2}(x - y)^2} = \frac{1}{(1 - x^2)^{1/2}} \frac{\partial^2}{\partial x \partial y} \left((1 - y^2)^{1/2} \log |x - y| \right) \tag{3.5}$$

allows for the rewrite of (3.2) in the case $f = g$ [111, 44]

$$\lim_{N \to \infty} \mathrm{Var}^{\mathrm{G}} \left(\sum_{j=1}^{N} f(x_j) \right) = \frac{2}{\beta} \frac{1}{\pi^2} \int_{-1}^{1} dy \, \frac{f(y)}{\sqrt{1 - y^2}} \int_{-1}^{1} dx \, \frac{f'(x)\sqrt{1 - x^2}}{x - y}. \tag{3.6}$$

One can check too that the LHS of (3.5) can be written as [13]

$$-\frac{1}{2} \frac{\partial^2}{\partial x \partial y} \log \left(\frac{1 - xy + \sqrt{(1 - x^2)(1 - y^2)}}{1 - xy - \sqrt{(1 - x^2)(1 - y^2)}} \right). \tag{3.7}$$

This substituted in (3.2) gives, upon integration by parts,

$$\lim_{N \to \infty} \mathrm{Cov}^{\mathrm{G}} \left(\sum_{j=1}^{N} f(x_j), \sum_{j=1}^{N} g(x_j) \right)$$

$$= \frac{2}{\beta} \frac{1}{4\pi^2} \int_{-1}^{1} dx\, f'(x) \int_{-1}^{1} dy\, g'(y) \log \left(\frac{1 - xy + \sqrt{(1 - x^2)(1 - y^2)}}{1 - xy - \sqrt{(1 - x^2)(1 - y^2)}} \right)$$

$$= \frac{2}{\beta} \frac{1}{4\pi^2} \oint_{\substack{|z|=1 \\ y>0}} dz \oint_{\substack{|w|=1 \\ v>0}} dw\, f'(x) g'(u) \log \left| \frac{1 - zw}{1 - z\bar{w}} \right| \frac{yv}{zw}, \tag{3.8}$$

where in the final expression $z = x + iy$, $w = u + iv$; see [142] for details relating to the second equality, which provides a link with the Gaussian free field [29]. According to (2.6) a corollary of the third form in (3.2) is that for the Gaussian (and Jacobi) β-ensembles,

$$\lim_{N \to \infty} \rho^T_{(2),N}(x,y) \doteq -\frac{1}{\beta} \frac{1}{\pi^2} \frac{(1 - xy)}{(1 - x^2)^{1/2}(1 - y^2)^{1/2}(x - y)^2}, \tag{3.9}$$

supported on $|x|, |y| < 1$. Here the symbol $\doteq$ is used to indicate the limiting functional form has been smoothed with respect to test functions; as seen in (2.22) the pointwise limit is not expected to exist.

The density of eigenvalues for the Gaussian β-ensemble in the large N limit on their support $(-1, 1)$ is the Wigner semi-circle law $\rho^{\mathrm{W}}_{(1)}(x) = \frac{2}{\pi}(1 - x^2)^{1/2}$; see e.g. [75, §1.4]. The case $\beta = 2$ — referred to as the Gaussian unitary ensemble (GUE) — is realised by the complex Hermitian random matrices $H^{\mathrm{c}} = \frac{1}{2}(X + X^\dagger)$, where X is an $N \times N$ complex standard Gaussian matrix; scaling these matrices by $1/\sqrt{2N}$ gives rise to (3.3) with $\beta = 2$. Analogously, the case $\beta = 1$ — known as the Gaussian orthogonal ensemble (GOE) — is realised by the real symmetric random matrices $H^{\mathrm{r}} = \frac{1}{2}(X + X^T)$, where X is an $N \times N$ real standard Gaussian matrix. Scaling these matrices by $1/\sqrt{2N}$ as for the GUE gives (3.3) with $\beta = 1$ as the eigenvalue PDF. A realisation in terms of Gaussian random matrices is known in the case $\beta = 4$ too (see e.g. [75, §1.3.2]), which is referred to as the Gaussian symplectic ensemble (GSE). Together the values $\beta = 1, 2$ and 4 are referred to as the classical cases, with their underlying symmetries isolated by Dyson [66]

For the GOE case $\beta = 1$ the functional form (3.9) was first derived in the 1978 work of French, Mello and Pandey [90]. This was done through an analysis of the covariance formula for the pair of linear statistics $\sum_{j=1}^{N} x_j^p = \mathrm{Tr}\, H^p$, $\sum_{j=1}^{N} x_j^q = \mathrm{Tr}\, H^q$. As such the strategy used was a generalisation of the method of moments as introduced by Wigner to study the eigenvalue density [166]. Soon after Pandey [139] realised that this polynomial covariance could usefully be encoded by considering instead

$$\mathrm{Cov} \left(\sum_{j=1}^{N} \frac{1}{x - x_j}, \sum_{j=1}^{N} \frac{1}{y - x_j} \right) = \mathrm{Cov} \left(\mathrm{Tr}(x\mathbb{I}_N - H)^{-1}, \mathrm{Tr}(y\mathbb{I}_N - H)^{-1} \right). \tag{3.10}$$

Studying the mean value of one such linear statistic, i.e.

$$\left\langle \sum_{j=1}^{N} \frac{1}{x - x_j} \right\rangle, \tag{3.11}$$

corresponds to the Steiltjes transform of the eigenvalue density, the analysis of which was introduced into random matrix theory by Pastur [143]; see also the text [144]. A number of derivations of (3.9) additional to those those of [90, 139] have been listed in the recent work [149].

Here we will focus on a method of derivation of (3.9) based on the loop equation formalism, first made use of in this context in relation to the GUE [6], and generalised to the Gaussian β-ensemble for general $\beta > 0$ in [39, 131, 167]. Denote the covariance (3.10) by $\overline{W}_2^{\mathrm{G}}(x, y; N, \kappa)$ and the mean (3.11) by $\overline{W}_1(x; N, \kappa)$, where $\kappa := \beta/2$. An integration by parts procedure gives that these quantities for the Gaussian β-ensemble are related by the first loop equation

$$\left(\frac{\kappa - 1}{2N} \frac{\partial}{\partial x_1} - 2\kappa x_1 \right) \overline{W}_1^{\mathrm{G}}(x_1; N, \kappa)$$
$$+ 2N\kappa + \frac{\kappa}{2N} \left(\overline{W}_2^{\mathrm{G}}(x_1, x_1; N, \kappa) + (\overline{W}_1^{\mathrm{G}}(x_1; N, \kappa))^2 \right) = 0. \tag{3.12}$$

To progress further, the $1/N$ expansions

$$\overline{W}_1^{\mathrm{G}}(x; N, \kappa) = N W_{1,0}^{\mathrm{G}}(x; \kappa) + W_{1,1}^{\mathrm{G}}(x; \kappa) + \frac{1}{N} W_{2,1}^{\mathrm{G}}(x; \kappa) + \cdots,$$
$$\overline{W}_2^{\mathrm{G}}(x, y; N, \kappa) = W_{2,0}^{\mathrm{G}}(x, y; \kappa) + \frac{1}{N} W_{2,1}^{\mathrm{G}}(x, y; \kappa) + \cdots, \tag{3.13}$$

rigorously justified in [34], are introduced into (3.12). Equating like powers of N gives a quadratic equation for $W_{1,0}^{\mathrm{G}}(x; \kappa)$ with solution

$$W_{1,0}^{\mathrm{G}}(x; \kappa) = 2(x - \sqrt{x^2 - 1}) \tag{3.14}$$

independent of κ. With this established, equating terms independent of N gives a linear equation for $W_{1,1}^{\mathrm{G}}(x; \kappa)$ with solution

$$W_{1,1}^{\mathrm{G}}(x; \kappa) = \frac{1}{2} \left(1 - \frac{1}{\kappa} \right) \left(\frac{1}{\sqrt{x^2 - 1}} - \frac{x}{x^2 - 1} \right). \tag{3.15}$$

However, at order $1/N$ in (3.13), the two quantities $W_{1,2}^{\mathrm{G}}(x; \kappa)$ and $W_{2,0}^{\mathrm{G}}(x, y; \kappa)$, both of which are unknown are involved. To separate these unknowns the second equation of the loop hierarchy is needed. As well as involving $\overline{W}_1^{\mathrm{G}}$ and $\overline{W}_2^{\mathrm{G}}$, this second equation involves the three point quantity $\overline{W}_3^{\mathrm{G}} = \overline{W}_3^{\mathrm{G}}(x_1, x_2, x_3; N, \kappa)$; see e.g. [167] for its precise definition. For large N, continuing the pattern from (3.13), $\overline{W}_3^{\mathrm{G}} = O(1/N)$ and so does not contribute to leading order in the second

loop equation. In fact the only unknown to leading order is $W_{2,0}^{\mathrm{G}}(x, y; \kappa)$, with the equation linear in this quantity and having solution

$$W_{2,0}^{\mathrm{G}}(x, y; \kappa) = \frac{2}{\beta} \left(\frac{xy - 1}{2(x-y)^2 \sqrt{(x^2-1)(y^2-1)}} - \frac{1}{2(x-y)^2} \right). \qquad (3.16)$$

Now generally for I an open interval, $\alpha(t)$ continuous on I and

$$f(z) = \int_I \frac{\alpha(t)}{z - t} \, dt, \quad z \notin I,$$

the inverse formula for the Stieltjes transform gives

$$\alpha(t) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \operatorname{Im} f(t - i\epsilon), \qquad t \in I.$$

Applying this with respect to both the x and y variable in (3.16) the functional form in (3.9) is obtained. By noting that for γ a simple contour enclosing $(-1, 1)$ in the complex plane, and f analytic on and within γ, we have by Cauchy's integral formula the representation

$$f(x) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - x} \, dz \qquad (3.17)$$

shows

$$\left(\frac{1}{2\pi i} \right)^2 \oint dw \, f(w) \oint dz \, g(z) \operatorname{Cov} \left(\sum_{j=1}^N \frac{1}{w - x_j}, \sum_{j=1}^N \frac{1}{z - x_j} \right)$$
$$= \operatorname{Cov} \left(\sum_{j=1}^N f(x_j), \sum_{j=1}^N g(x_j) \right). \quad (3.18)$$

This fact, together with certain large deviation bounds, enables the rigorous deduction of (3.2) for f, g analytic in a neighbourhood of $(-1, 1)$ from knowledge of (3.16); see e.g. [34]. Replacing the Cauchy integral formula by the Helffer-Sjöstrand formula [71, 138, 133, 104] allows for the conditions on f and g to be further weakened.

Remark 3.1. 1. Consider the linear statistics $\sum_{j=1}^N x_j^p$, $(p = 1, 2)$. As noted in [11], in the case of the Gaussian β-ensemble the corresponding characteristic functions (2.17) are simple to evaluate,

$$\hat{P}_{N,f=x}(t) = e^{-t^2/4\beta}, \qquad \hat{P}_{N,f=x^2}(t) = (1 - it/\beta N)^{-(1/2)(N + \beta N(N-1)/2)}. \tag{3.19}$$

Recalling (2.21), it follows

$$\lim_{N \to \infty} \operatorname{Var}^{\mathrm{G}} \left(\sum_{j=1}^N x_j \right) = \frac{1}{2\beta}, \qquad \lim_{N \to \infty} \operatorname{Var}^{\mathrm{G}} \left(\sum_{j=1}^N x_j^2 \right) = \frac{1}{4\beta}, \qquad (3.20)$$

as is consistent with (3.2). We note too that in the limit $N \to \infty$ the second characteristic function in (3.36) becomes a Gaussian like the first upon the recentring $\hat{P}_{N,f=x^2}(t) \mapsto \hat{P}_{N,f=x^2}(t)e^{-it\langle \sum_{j=1}^{N} x_j^2 \rangle}$. That the rescaled limiting distribution of the polynomial linear statistics $\sum_{j=1}^{N} x_j^k$ ($k \in \mathbb{Z}^+$) is a Gaussian with variance as implied by (3.2) was first established by Johansson [111]. In [34] this result was extended to a wider class of linear statistics using a loop equation analysis; see also [22], [118] and [35].

2. With $f = x^{k_1}$, $g = x^{k_2}$ it is a known corollary of (3.2) (see e.g. [129, 61, 37, 51]) that

$$2^{p+q} \lim_{N\to\infty} \mathrm{Cov}^{\mathrm{G}}\Big(\sum_{j=1}^{N} x_j^p, \sum_{j=1}^{N} x_j^q \Big) =$$

$$\begin{cases} \dfrac{2}{\beta} \dfrac{(2k_1)!(2k_2)!}{(k_1!)^2(k_2!)^2} \dfrac{k_1 k_2}{k_1 + k_2}, & p = 2k_1,\ q = 2k_2 \\[3mm] \dfrac{2}{\beta} \dfrac{(2k_1+1)!(2k_2+1)!}{(k_1!)^2(k_2!)^2} \dfrac{k_1 k_2}{k_1 + k_2 + 1}, & p = 2k_1 + 1,\ q = 2k_2 + 1 \end{cases} \tag{3.21}$$

It is pointed out in [130, 51] that from a particular combinatorial viewpoint, equivalent enumeration formulas are contained in the work of Tutte [164].

3. (Variance for number of particles in an interval) As in Proposition 2.2, an example of a linear statistic for which (3.2) breaks down in $f = g = \chi_{(a,b)}$, for $(a,b) \subset (-1,1)$. Let $N_{(a,b)} := \sum_{j=1}^{N} \chi_{x_j \in (a,b)}$. It is proved in [16] that for $(a,b) = (0,1)$

$$\lim_{N\to\infty} \frac{1}{\log N} \mathrm{Var}^{\mathrm{G}}(N_{(a,b)}) = \frac{1}{\pi^2 \beta} \tag{3.22}$$

(cf. (2.15)). This was conjectured to hold true in the general case, a fact which has been established in the special cases $\beta = 1, 2$ and 4 [153].

A half line scaling is possible. Suppose in (3.2) that $f(-1+x) = F(X/L)$, $g(-1+x) = G(X/L)$, where $F(X), G(X)$ are assumed to decay at infinity. In the second equality of (3.2) change variables $-1+x = X/L$, $-1+y = Y/L$. This shows

$$\lim_{L\to\infty} \lim_{N\to\infty} \mathrm{Cov}^{\mathrm{G}}\Big(\sum_{j=1}^{N} f(x_j), \sum_{j=1}^{N} g(x_j) \Big)\Big|_{\substack{f(-1+x)=F(X/L) \\ g(-1+x)=G(X/L)}}$$

$$= \frac{1}{\beta} \frac{1}{4\pi^2} \int_0^\infty dX \int_0^\infty dY \frac{(F(X)-F(Y))(G(X)-G(Y))}{(X-Y)^2} \frac{X+Y}{(XY)^{1/2}}. \tag{3.23}$$

The identity,

$$\frac{1}{2} \frac{1}{\sqrt{XY}} \frac{X+Y}{(X-Y)^2} = \frac{\partial^2}{\partial X \partial Y} \log \left| \frac{\sqrt{X} - \sqrt{Y}}{\sqrt{X} + \sqrt{Y}} \right|$$

substituted in (3.23) gives, upon integration by parts, the rewrite of the RHS

of (3.23),

$$-\frac{1}{\beta\pi^2} \int_0^\infty dX\, F'(X) \int_0^\infty dY\, G'(Y) \log\left|\frac{\sqrt{X}-\sqrt{Y}}{\sqrt{X}+\sqrt{Y}}\right|. \tag{3.24}$$

Furthermore, making use of the Fourier transform

$$-\log\left|\tanh\frac{\pi y}{4\pi}\right| = \frac{1}{2}\int_{-\infty}^\infty \frac{\tanh\pi x}{x}e^{ixy}\,dx,$$

shows that an alternative form to (3.24) is [21, 52]

$$\frac{1}{\beta\pi^2}\int_{-\infty}^\infty \hat{F}^{\mathrm{e}}(k)\hat{G}^{\mathrm{e}}(-k)k\tanh(\pi k)\,dk, \qquad \hat{a}^{\mathrm{e}}(k):=\int_{-\infty}^\infty e^{ikx}a(e^x)\,dx. \tag{3.25}$$

The change of variables $X=u^2, Y=v^2$ shows that (3.23) permits the further rewrites [20, 128]

$$\frac{1}{\beta}\frac{1}{(2\pi)^2}\int_{-\infty}^\infty du \int_{-\infty}^\infty dv\, \frac{(F(u^2)-F(v^2))(G(u^2)-G(v^2))}{(u-v)^2}$$
$$= \frac{1}{\beta}\frac{1}{(2\pi)^2}\int_{-\infty}^\infty |k|\hat{F}^{\mathrm{s}}(k)\hat{G}^{\mathrm{s}}(-k)\,dk, \tag{3.26}$$

where $\hat{h}^{\mathrm{s}}(k):=\int_{-\infty}^\infty h(x^2)e^{ikx}\,dk$.

Both the GOE and GUE allow for extensions to involve what historically has been termed an external source. With G a GOE matrix ($\beta=1$) or GUE matrix ($\beta=2$) the corresponding ensembles with an external source (to be denoted $\mathrm{G}^{\boxplus}$) are specified by the sum

$$A+G, \tag{3.27}$$

where it is assumed that A is real symmetric ($\beta=1$) or complex Hermitian ($\beta=2$). Suppose furthermore that as $N\to\infty$ the eigenvalue density of $\frac{1}{\sqrt{2N}}A$ has a compactly supported limiting density with corresponding measure $d\mu(x)$. Let $\tilde{m}(z)$ be the solution of the Pastur equation

$$\tilde{m}(z) = \int_{-\infty}^\infty \frac{1}{t-2z-\tilde{m}(z)}\,d\mu(t) \tag{3.28}$$

which has positive imaginary part for z in the upper half complex plane. The quantity $\tilde{m}(z)$ then corresponds to the Stieltjes transform of the limiting scaled eigenvalue density of (3.27); see e.g. [144]. We have from [53, 108, 123]

$$W_{2,0}^{\mathrm{G}^{\boxplus}}(x,y) = -\frac{2}{\beta}\frac{\partial^2}{\partial x\partial y}\log\left(1-\int_{-\infty}^\infty \frac{d\mu(t)}{(t-2x-\tilde{m}(z))(t-2y-\tilde{m}(z))}\right). \tag{3.29}$$

Note that in the special case that $d\mu(t) = \delta(t)dt$ corresponding to $A=0$ in (3.27), it follows from (3.28) that $1/(2z+\tilde{m}(z)) = -\tilde{m}(z)$. Using this in (3.29) in this same special case reclaims the form of $W_{2,0}^{\mathrm{G}}(x,y)$ given in (3.52)

below. A formula of a different type for the variance of a polynomial linear statistic in the case of the GUE with a source has been given in [60], which comes about through its relation to multiple orthogonal polynomials. Another point of interest is that the external source model (3.27) can be viewed in terms of Dyson Brownian motion; see e.g. [75, §11.1] for $\beta = 1, 2$ and 4, and [76] for general $\beta > 0$. A study of the corresponding fluctuation formulas for linear statistics from this perspective can be found in [23].

Linear statistics ranging over a restriction of the full set of eigenvalues are of interest [15, 99, 98]. With the eigenvalues ordered $x_1 > x_2 > \cdots > x_N$, considered is the fluctuation of $\sum_{j=1}^{K} f(x_j)$, where $K/N \to \gamma$ as $N \to \infty$. Define $c = c(\gamma)$ by

$$\gamma = \frac{2}{\pi} \int_c^1 \sqrt{1-x^2}\, dx, \tag{3.30}$$

so that the fraction of eigenvalues in the interval $(c, 1)$ of the Wigner semi-circle is γ. With $f_c(x) = (f(x) - f(c))\chi_{1>x>c}$ it is proved in [15] that for the GUE

$$\lim_{N\to\infty} \mathrm{Var}^{\mathrm{GUE}}\Big(\sum_{j=1}^{K} f(x_j)\Big) = \lim_{N\to\infty} \mathrm{Var}^{\mathrm{GUE}}\Big(\sum_{j=1}^{N} f_c(x_j)\Big), \tag{3.31}$$

with the RHS in turn being given by any of the formulas (3.2) and (3.8). A case of particular interest is $f(x) = x^2$. Use of the first formula in (3.8) and evaluating the integral via computer algebra shows

$$\lim_{N\to\infty} \mathrm{Var}^{\mathrm{GUE}}\Big(\sum_{j=1}^{K} x_j^2\Big) =$$
$$\frac{1}{\pi^2}\frac{1}{8}\Big(3c^4 - 4c^3\sqrt{1-c^2}\mathrm{Arccos}\,c + c^2(-7 + 2c\sqrt{1-c^2}\mathrm{Arccos}\,c) + 4 + (\mathrm{Arccos}\,c)^2\Big). \tag{3.32}$$

Up to a simple scaling of c, this formula was first derived in the recent paper [98, Eq. (70)], from a large deviations viewpoint. Setting $c = 0$ and using (3.30) and (3.31) implies

$$\mathrm{Var}^{\mathrm{GUE}}\Big(\sum_{j=1}^{N} x_j^2 \chi_{x_j>0}\Big) = \frac{1}{16}\Big(1 + \frac{16}{\pi^2}\Big). \tag{3.33}$$

The analogue of this result in the GOE case is known from [125, §6.2], where it was shown to have relevance to the distribution of intrinsic volumes for the cone of positive semidefinite matrices.

Also of interest are linear statistics associated with submatrices. For an $N \times N$ matrix H and $I \subset \{1, \ldots, N\}$, $|I| \leq N$, denote by $H(I)$ the $|I| \times |I|$ Hermitian matrix formed by the intersection of the rows and columns labelled by I of H. Let H be chosen from the GOE $(\beta = 1)$ or the GUE $(\beta = 2)$. Motivated by the relevance of submatrices of random Hermitian matrices to models admitting a

stepped surfaces interpretation — see the overview [30] — and moreover relating to the Gaussian free field, Borodin [29], using methods from [9, Ch. 2], took up the problem of computing the covariance for the pair of linear statistics $\text{Tr}(H(I_p))^{k_p}, \text{Tr}(H(I_q))^{k_q}$, under the assumption that

$$\lim_{N\to\infty} \frac{|I_p|}{N} =: b_p > 0, \quad \lim_{N\to\infty} \frac{|I_q|}{N} =: b_q > 0, \quad \lim_{N\to\infty} \frac{|I_p \cap I_q|}{N} =: c_{pq} > 0.$$

Let $T_n(x)$ denote the n-th Chebyshev polynomial of the first kind, $T_n(\cos\theta) = \cos n\theta$. Let $\tilde{H}(I_p)$ denote $H(I_p)$ scale so that its limiting eigenvalue support is the interval $(-1, 1)$. From [29] we have

$$\lim_{N\to\infty} \text{Cov}\Big(\text{Tr}(T_{k_p}(\tilde{H}(I_p))), \text{Tr}(T_{k_q}(\tilde{H}(I_q))) \Big) = \delta_{k_p,k_q} \frac{k_p}{2\beta} \Big(\frac{c_{pq}}{\sqrt{b_p b_q}} \Big)^{k_p}. \quad (3.34)$$

Note that this is consistent with the first line of (2.6) in the case $b_p = b_q = c_{pq} = 1$. This theme, with emphasis placed on the form of the covariance written in a form relating to the Gaussian free field — recall the final expression in (3.8) — has been followed up in [31, 62, 59, 40, 46], amongst other works.

As our final point specifically in relation to the Gaussian β-ensemble, we will review fluctuation formulas associated with a particular high temperature limit, analogous to that introduced in Remark 2.11 for the circular β-ensemble. This high temperature limit is specified by replacing $e^{-\beta N V(x_l)}$ in (3.3) by $e^{-x_l^2/2}$, setting $\beta = 2\alpha/N$ and taking $N \to \infty$. It is known that the corresponding normalised density, $\rho_{(1)}(x; \alpha)$ say, has the exact functional form [4] (see [86] for a derivation via loop equations)

$$\rho_{(1)}(x; \alpha) = \frac{1}{\sqrt{2\pi}\Gamma(1+\alpha)} \frac{1}{|D_{-\alpha}(ix)|^2},$$

where $D_\mu(z)$ denotes the parabolic cylinder function. Introduce now the orthogonal polynomials with respect to $\rho_{(1)}(x; \alpha)$. These are the so called associated Hermite polynomials $\{p_n^{\text{H}}(x; \alpha)\}$, which can be generate through the three term recurrence

$$p_{n+1}^{\text{H}}(x; \alpha) = x p_n^{\text{H}}(x; \alpha) - (n+\alpha) p_{n-1}^{\text{H}}(x; \alpha),$$

with $p_0^{\text{H}}(x; \alpha) = 1$, $p_1^{\text{H}}(x; \alpha) = x$. The case $\alpha = 0$ corresponds to the classical Hermite polynomials. In relation to a fluctuation formula, the recent work of Nakano, Trihn and Trinh [134] has shown that with $P_n^{\text{H}}(x; \alpha) := \int^x p_n^{\text{H}}(t; \alpha)\, dt$,

$$\lim_{N\to\infty} \frac{1}{N} \text{Cov}^{\text{G}}(P_m^{\text{H}}, P_n^{\text{H}})\Big|_{\substack{w(x)=e^{-x^2/2} \\ \beta=2\alpha/N}} = \delta_{m,n} \frac{(\alpha+1)\cdots(\alpha+n)}{n+1}. \quad (3.35)$$

3.2. The Laguerre and Jacobi β-ensembles

The potential $V(x) = -(\alpha \log x - x)/2$, $x \in \mathbb{R}^+$ substituted in (3.3) corresponds to the Laguerre β-ensemble. In the limit $N \to \infty$ the normalised density limits

to the Marčenko–Pastur functional form

$$\frac{\sqrt{(x-c)(d-x)}}{2\pi(1+\alpha)x},$$

where (c, d) is the interval of support with $c = (1 - \sqrt{1+\alpha})^2$, $d = (1 + \sqrt{1+\alpha})^2$. Although this is distinct from the Wigner semi-circle functional form for the normalised eigenvalue density as holds for the Gaussian β-ensemble, a loop equation analysis [88] gives for $W_{2,0}^L$ a functional form which includes the corresponding result (3.16) for the Gaussian β-ensemble. Thus it is found

$$W_{2,0}^L(x, y; \kappa = \beta/2) = \frac{2}{\beta}\left(\frac{xy - (c+d)(x+y)/2 + cd}{2(x-y)^2\sqrt{((x-c)(x-d)(y-c)(y-d))}} - \frac{1}{2(x-y)^2}\right),$$

$$(3.36)$$

supported on $x, y \in (c, d)$ (to reclaim (3.16) set $c = -1$, $d = 1$) as first identified in [6]. Consequently, by applying the inverse Stieltjes transform,

$$\lim_{N\to\infty} \rho_{(2),N}^T(x, y) \doteq -\frac{1}{\beta}\frac{1}{\pi^2}\frac{(-cd + (c+d)(x+y)/2 - xy)}{(c-x)(d-x)(c-y)(d-y)}. \tag{3.37}$$

Substituting (3.37) in (2.6) gives one particular functional form of the limiting covariance. More revealing is to change variables in the linear statistic $f(x)$ by writing $f(\alpha_1 + \alpha_2 \cos\theta)$ with $\alpha_1 = (c+d)/2$, $\alpha_2 = (d-c)/2$, and similarly for $g(x)$. In this new variable, performing a cosine expansion as on the RHS of (2.67) then gives the simplified expression

$$\lim_{N\to\infty} \mathrm{Cov}^L\Big(\sum_{j=1}^N f(x_j), \sum_{j=1}^N g(x_j)\Big) = \frac{2}{\beta}\sum_{n=1}^\infty n f_n^c g_n^c; \tag{3.38}$$

cf. the first expression in (3.3).

Remark 3.2. 1. As observed in [11], for the Laguerre β-ensemble it is simple to compute the characteristic function for the linear statistic $\sum_{j=1}^N x_j$,

$$\hat{P}_{N,f=x}^L(t) = (1 - 2it/N\beta)^{-N(1+N\beta/2)-\beta N(N-1)/2}. \tag{3.39}$$

This implies

$$\lim_{N\to\infty} \mathrm{Var}^L\Big(\sum_{j=1}^N x_j\Big) = \frac{2}{\beta}(\alpha+1), \tag{3.40}$$

which is readily checked to be consistent with (3.38). Upon the recentring of replacing $\hat{P}_{N,f=x}^L(t)$ by $\hat{P}_{N,f=x}^L(t)e^{-it\sum_{j=1}^N x_j}$, we see that the $N \to \infty$ form of (3.39) is a Gaussian. For linear statistics analytic in the neighbourhood of the eigenvalue support, the loop equation analysis of [34] gives that the limiting recentred distribution is a Gaussian with variance determined by (3.38).

2. Denote the limiting covariance for the monomial linear statistics $\sum_{j=1}^N x_j^{k_1}$, $\sum_{j=1}^N x_j^{k_2}$ in the Laguerre (Gaussian) cases by μ_{k_1,k_2}^L (μ_{k_1,k_2}^G). Thus for k_1, k_2

of the same parity $\mu^{\mathrm{G}}_{k_1,k_2}$ is given by (3.21), while for k_1, k_2 of different parity $\mu^{\mathrm{G}}_{k_1,k_2} = 0$. It is shown in [51] that the covariance formula as obtained by substituting (3.37) in (2.6) implies

$$\mu^{\mathrm{L}}_{k_1,k_2} = \alpha_1^{k_1+k_2} \sum_{p=0}^{k_1} \sum_{q=0}^{k_2} \binom{k_1}{p} \binom{k_2}{q} \left(\frac{\alpha_2}{\alpha_1}\right)^{p+q} \mu^{\mathrm{G}}_{p,q}. \tag{3.41}$$

Also, in the special case $\alpha = 0$ so that $c = 0$, $d = 4$, $\alpha_1 = \alpha_2 = 2$ there is the simplification [51]

$$\mu^{\mathrm{L}}_{k_1,k_2}\Big|_{\alpha=0} = \frac{1}{\beta} 2^{k_1+k_2+2} \frac{1}{k_1+k_2} \binom{2k_1-1}{k_1} \binom{2k_2-1}{k_2}. \tag{3.42}$$

The Laguerre β-ensemble has been specified in terms of the eigenvalue PDF (3.3) with potential $V(x) = -(\alpha \log x + x)/2$, $x > 0$. It is a standard result in random matrix theory (see e.g. [75, §3.2]) that with $\alpha N = (n - N) + 1 - 2/\beta$ this eigenvalue PDF is realised by Wishart matrices $W = \frac{1}{N} X^\dagger X$, where X is an $n \times N$ ($n \geq N$) standard real ($\beta = 1$) or complex ($\beta = 2$) Gaussian random matrix. In a statistical setting the matrix X is the centred data matrix and W is proportional to the sample covariance. More generally the Wishart class involves the centred data matrix having the form $X \Sigma^{1/2}$, where X is as above and Σ is an $N \times N$ positive definite matrix. We will use the symbol $\mathrm{L}^{\boxtimes}$ to indicate this setting. Let $1 + \alpha_\infty = \lim_{M,N\to\infty} M/N$ and suppose Σ has a limiting eigenvalue density with compact support specified by the measure $d\nu(x)$. Specify $\tilde{m}(z)$ — the Stieltjes transform of the limiting eigenvalue density of $\frac{1}{N} \Sigma^{1/2} X^\dagger X \Sigma^{1/2}$ — as the solution of

$$\alpha_\infty - z\tilde{m}(z) = (1 + \alpha_\infty) \int_{-\infty}^{\infty} \frac{1}{1 + t\tilde{m}(z)} d\nu(t),$$

which has $\mathrm{Im}\,\tilde{m}(z) > 0$ for z in the upper half complex plane; see e.g. the text [144] in relation to this result. Results of Bai and Silverstein [12], and further developed in [122, 151, 133, 123], give

$$W^{\mathrm{L}^{\boxtimes}}_{2,0} = \frac{1}{\beta} \frac{\partial^2}{\partial x \partial y} \log \left(\frac{\tilde{m}(x) - \tilde{m}(y)}{x - y}\right) \tag{3.43}$$

and, with I denoting the interval of support of the density,

$$\mathrm{Cov}^{\mathrm{L}^{\boxtimes}}(f,g) = \frac{1}{\beta \pi^2} \int_I dx \int_I dy\, f'(x) g'(y) \log \left|\frac{\tilde{m}(x) - \overline{\tilde{m}(y)}}{\tilde{m}(x) - \tilde{m}(y)}\right|. \tag{3.44}$$

As for the Gaussian β-ensemble, the Laguerre β-ensemble admits a scaled high temperature limit. This is specified by replacing $e^{-\beta N V(x_l)}$ in (3.3) by $x_l^{\alpha_1} e^{-x_l}$, setting $\beta = 2\alpha/N$ and taking $N \to \infty$. The normalised density, $\rho_{(1)}(x; \alpha_1, \alpha)$ say, is then given in terms of the Whittaker function $W_{\zeta,\mu}(z)$ by [5]

$$\rho_{(1)}(x; \alpha_1, \alpha) = \frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+\alpha_1)} \frac{1}{|W_{-\alpha-\alpha_1/2,(1+\alpha_1)/2}(-x)|^2}, \quad x > 0. \tag{3.45}$$

The PDF for the corresponding mean $x\rho_{(1)}(x;\alpha_1,\alpha)/(\alpha_1+\alpha)$ can be recognised as the weight function for the associated Laguerre polynomials $\{p_n^{\mathrm{L}}(x;\alpha_1,\alpha)\}$, defined by the three term recurrence

$$p_{n+1}^{\mathrm{L}} = xp_n^{\mathrm{L}} - (\alpha + 2\alpha_1 + 2n + 1)p_n^{\mathrm{L}} - (\alpha + \alpha_1 + n)(\alpha_1 + n)p_{n-1}^{\mathrm{L}},$$

with $p_0^{\mathrm{L}} = 1$, $p_1^{\mathrm{L}} = x - (\alpha + 2\alpha_1 + 1)$. A companion result to (3.35) obtained in [134] gives that with $P_n^{\mathrm{L}}(x;\alpha_1,\alpha) := \int^x p_n^{\mathrm{L}}(t;\alpha_1,\alpha)\,dt$,

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Cov}^{\mathrm{L}}(P_m^{\mathrm{L}}, P_n^{\mathrm{L}})\Big|_{\substack{w(x)=x^{\alpha_1}e^{-x} \\ \beta=2\alpha/N}} = \delta_{m,n}\frac{\alpha+\alpha_1}{n+1}\prod_{j=1}^n (\alpha_1+j)(\alpha+\alpha_1+j). \tag{3.46}$$

We turn now to the consideration of the Jacobi β-ensemble. An example of a Jacobi β-ensemble is seen in (3.1). In this example the support is $(-1,1)$. Note that when written in the form (3.3) this gives a $V(x)$ proportional to $1/N$, which is sub-leading. We will consider instead a version of the Jacobi β-ensemble supported on $(0,1)$ with $V(x)$ of order unity and given explicitly by $V(x) = \gamma_1 \log x + \gamma_2 \log(1-x)$. The corresponding $N\to\infty$ normalised density is known to be given by a functional form first identified by Wachter [165],

$$(\gamma_1 + \gamma_2 + 2)\frac{\sqrt{(x-c^{\mathrm{J}})(d^{\mathrm{J}}-x)}}{2\pi x(1-x)} \tag{3.47}$$

supported on $(c^{\mathrm{J}}, d^{\mathrm{J}})$ with these endpoints specified by

$$\left(\sqrt{\frac{\gamma_1+1}{\gamma_1+\gamma_2+2}\Big(1 - \frac{1}{\gamma_1+\gamma_2+2}\Big)} \pm \sqrt{\frac{1}{\gamma_1+\gamma_2+2}\Big(1 - \frac{\gamma_1+1}{\gamma_1+\gamma_2+2}\Big)}\right)^2. \tag{3.48}$$

In the present context the main point is that a loop equation analysis [88] confirms that the limiting functional form (3.36) for the two-point quantity $W_{2,0}$ again holds true, as expected for all β-ensembles with density supported on a single interval. Hence the covariance is given by (3.38). In the case of monomial linear statistics, the formula (3.41) again holds, with $\alpha_1 = (c^{\mathrm{J}}+d^{\mathrm{J}})/2$, $\alpha_2 = (d^{\mathrm{J}} - c^{\mathrm{J}})/2$. In the special case $\gamma_1 = \gamma_2 = 0$ the support is $(c^{\mathrm{J}}, d^{\mathrm{J}}) = (0,1)$ and the covariance formula for monomial linear statistics simplifies to be related to (3.42) [51],

$$\mu_{k_1,k_2}^{\mathrm{J}}\Big|_{\gamma_1=\gamma_2=0} = 2^{-k_1-k_2}\mu_{k_1,k_2}^{\mathrm{L}}\Big|_{\alpha=0}. \tag{3.49}$$

Remark 3.3. 1. For the Jacobi β-ensemble specified by (3.1) with $\prod_{l=1}^N(1 - y_l^2)^{-1/2}$ replaced by $\prod_{l=1}^N(1-y_l)^{\lambda_1}(1+y_l)^{\lambda_2}$, where $\lambda_1,\lambda_2 > -1$ and fixed, the analogue of (3.22) has been proved by Killip [113], provided the interval (a,b) has $a = -1$ or $b = 1$.

2. A loop equation analysis has been applied to various discretisation of the classical β-ensembles [32, 58]. In the so-called one cut regime, the universal form of $W_{2,0}$ as given by the RHS of (3.36) is recovered.

3. All convex potentials $V(x)$ in (3.3) are known to lead to a one cut regime for the corresponding eigenvalue density. However, without this assumption, the eigenvalue density may consist of several intervals and the fluctuation formula for a linear statistic typically involves quasi-periodic terms [152].

4. The scaled high temperature limit of the Jacobi β-ensemble is specified by replacing $e^{-\beta NV(x_l)}$ in (3.3) by $x_l^{\alpha_1}(1-x_l)^{\alpha_2}$, setting $\beta = 2\alpha/N$ and taking $N \to \infty$. The normalised density is known in terms of Gauss hypergeometric functions [163, 86]. Define polynomials $\{p_j^{\mathrm{J}}(x; \alpha, \alpha_1, \alpha_2)\}_{j=0,1,\dots}$ orthonormal with respect to $\frac{1}{C}x(1-x)$ times the density, where C is a normalisation. It is shown in the recent work [135] that with P_j^{J} defined as the anti-derivative of p_j^{J}, the scaled high temperature limit of $\frac{1}{N}\mathrm{Cov}^{\mathrm{J}}(P_m^{\mathrm{J}}, P_n^{\mathrm{J}})$ is diagonal and of order unity, and an explicit formula is obtained for the diagonal entries.

3.3. Wigner matrices

In the paragraph below (3.3) the GOE and GUE were defined in terms of real symmetric and complex Hermitian matrices, with elements on, and elements above, the diagonal independently and identically distributed as particular zero mean Gaussians. If the requirement of a Gaussian distribution is weakened to some other zero mean, finite variance distributions, the GOE and GUE generalise to what is termed the real symmetric and complex Hermitian Wigner matrices. Specifically, following [13] it is assumed the variances are such $\langle |x_{ij}|^2 \rangle = 1$ $(i < j)$ and $\langle x_{ii}^2 \rangle = \sigma^2$. In this setting, the celebrated Wigner semi-circle law (see e.g. [144]) is equivalent to the result that, after scaling the matrices by $1/\sqrt{2N}$, the limit of (3.11) which we denote by $W_{1,0}^{\mathrm{W}}(x)$ is again given by (3.14) and thus

$$W_{1,0}^{\mathrm{W}}(x) = 2(x - \sqrt{x^2 - 1}). \tag{3.50}$$

Note the independence on σ^2 and higher moments of the distribution of the entries. Generalising results obtained earlier by D'Anna and Zee [54] and Khorunzhy et al. [112], Bai and Yao [13] have computed the scaled limit of the two-point quantity (3.10) for Wigner matrices. As is consistent with the usage in (3.13), we denote this limiting quantity by $W_{2,0}^{\mathrm{W}}(x, y; \kappa)$, where $\kappa = 1/2$ (real case), $\kappa = 1$ (complex case).

Proposition 3.4. *Let* $\langle |x_{ij}|^4 \rangle = 1$ $(i < j)$ *be finite and independent of* (i, j). *In the complex case, with* $x_{ij} = x_{ij}^{\mathrm{r}} + ix_{ij}^{\mathrm{i}}$ *require that* $\langle |x_{ij}^{\mathrm{r}}|^2 \rangle = \langle |x_{ij}^{\mathrm{i}}|^2 \rangle$. *Subject only to a further technical condition on the decay of the tails of the distribution, one has*

$$W_{2,0}^{\mathrm{W}}(x, y; \kappa) = \left(\frac{d}{dx}W_{1,0}^{\mathrm{W}}(x)\right)\left(\frac{d}{dy}W_{1,0}^{\mathrm{W}}(y)\right)$$
$$\times \left(\sigma^2 - 1/\kappa + 2\tilde{\beta}W_{1,0}^{\mathrm{W}}(x)W_{1,0}^{\mathrm{W}}(y) + \frac{(1/\kappa)}{(1 - W_{1,0}^{\mathrm{W}}(x)W_{1,0}^{\mathrm{W}}(y))^2}\right), \tag{3.51}$$

where $\tilde{\beta} = \langle (|x_{12}|^2 - 1)^2 \rangle - 1/\kappa$.

Remark 3.5. 1. In the special case of the GOE we have $1/\kappa = \sigma^2 = 2$ and $\tilde{\beta} = 0$, while for the GUE we have $1/\kappa = \sigma^2 = 1$ and $\tilde{\beta} = 0$. This implies that as an alternative to (3.16), for $W_{2,0}^G(x, y; \kappa)$ we have the functional forms

$$W_{2,0}^G(x, y; \kappa) = \frac{2}{\beta}\Big(\frac{d}{dx}W_{1,0}^W(x; \kappa)\Big)\Big(\frac{d}{dy}W_{1,0}^W(y; \kappa)\Big)\frac{1}{(1 - W_{1,0}^W(x; \kappa)W_{1,0}^W(y; \kappa))^2}$$

$$= -\frac{2}{\beta}\frac{\partial^2}{\partial x \partial y}\log(1 - W_{1,0}^W(x; \kappa)W_{1,0}^W(y; \kappa)). \tag{3.52}$$

The second of these is known earlier from the work of Brézin et al. [38].

2. The rewrite of one of the terms in (3.51) implied by (3.52) can be extended to the remaining terms

$$\Big(\frac{d}{dx}W_{1,0}^W(x)\Big)\Big(\frac{d}{dy}W_{1,0}^W(y)\Big)\Big(\sigma^2 - 1/\kappa + 2\tilde{\beta}W_{1,0}^W(x)W_{1,0}^W(y)\Big)$$

$$= \frac{\partial^2}{\partial x \partial y}\Big((\sigma^2 - 1/\kappa)W_{1,0}^W(x)W_{1,0}^W(y) + \frac{\tilde{\beta}}{2}\Big(1 - 2xW_{1,0}^W(x)\Big)\Big(1 - 2yW_{1,0}^W(y)\Big)\Big), \tag{3.53}$$

where use has been made of the quadratic equation satisfied by $W_{1,0}^W$. An extension of the functional form (3.7), obtained by taking the inverse Stieltjes transform of $W_{2,0}^W(x, y; \kappa)$ so rewritten with respect to both x and y, is the result

$$\lim_{N \to \infty} \rho_{(2)}^{T,W}(x, y) \doteq \frac{1}{4\pi^2}\frac{\partial^2}{\partial x \partial y}\Big(\frac{2}{\beta}\log\Big(\frac{1 - xy + \sqrt{(1 - x^2)(1 - y^2)}}{1 - xy - \sqrt{(1 - x^2)(1 - y^2)}}\Big)$$

$$+ (\sigma^2 - (2/\beta) + 2\tilde{\beta}xy)4(1 - x^2)^{1/2}(1 - y^2)^{1/2}\Big). \tag{3.54}$$

Notice the separation of variables in the terms which differ from the Gaussian result. This substituted in the limiting form of (2.6) implies [13]

$$\lim_{L \to \infty}\lim_{N \to \infty}\text{Cov}^W\Big(\sum_{j=1}^N f(x_j), \sum_{j=1}^N g(x_j)\Big) = \frac{1}{4\pi^2}\int_{-1}^1 dx\, f'(x)\int_{-1}^1 dy\, g'(y)V(x, y), \tag{3.55}$$

where $V(x, y)$ is the functional form in (3.54) after the partial derivatives. Moreover, with f_n^c, g_n^c as in (3.2) it is shown in [13] that the RHS of (3.55) can be rewritten in the form

$$(\sigma^2 - (2/\beta))f_1^c g_1^c + 2\tilde{\beta}f_2^c g_2^c + \frac{2}{\beta}\sum_{l=1}^N l f_l^c g_l^c. \tag{3.56}$$

We note too the work of Shcherbina [151] for an independent evaluation of the limit in (3.55) in the real case for $f = g$, which gives a functional form generalising the final equality in (3.4). A comprehensive study of conditions on f

for which this formula is valid has recently been given by Landon and Sosoe [119]; this work also reviews earlier work along these lines as part of the Introduction section. In the complex case Bao and Xie [17] remove the requirement of [13] that $\langle |x_{ij}^{\mathrm{r}}|^2 \rangle = \langle |x_{ij}^{\mathrm{i}}|^2 \rangle$; the covariance formula now depends on the parameter $\Phi := \langle |x_{ij}^{\mathrm{r}}|^2 \rangle - \langle |x_{ij}^{\mathrm{i}}|^2 \rangle$,

$$\sigma^2 f_1^{\mathrm{c}} g_1^{\mathrm{c}} + 2(\langle |x_{ij}|^4 \rangle - \Phi^2 - 2) f_2^{\mathrm{c}} g_2^{\mathrm{c}} + \sum_{l=2}^{N} l(1 + \Phi^l) f_l^{\mathrm{c}} g_l^{\mathrm{c}}. \tag{3.57}$$

Note that when $\Phi = 1$, which corresponds to real Wigner matrices, this is consistent with (3.56) for $\beta = 1$.

3. In relation to the proof of the central limit theorem associated with a linear statistic for Wigner matrices, for which (3.55) implies the variance, the recent work [14] highlights the strategy introduced in [112] as being particularly influential. Denote $G = (x\mathbb{I} - H)^{-1}$ as the resolvent of the Wigner matrix H, so that $\operatorname{Tr} G$ is equal to (3.11). The corresponding matrix elements then satisfy the simple identity

$$G_{jm} = -x^{-1} \delta_{j,m} + x^{-1} \sum_{k=1}^{N} G_{jk} H_{km}.$$

To average over the distribution of the entries of H, in the Gaussian case use can be made of the identity

$$\langle G_{jk} H_{km} \rangle = \langle H_{km}^2 \rangle \Big\langle \frac{\partial}{\partial H_{km}} G_{jk} \Big\rangle$$

as follows from

$$\langle \xi f(\xi) \rangle = \langle \xi^2 \rangle \langle f'(\xi) \rangle. \tag{3.58}$$

As a replacement to (3.58) in the case of distributions outside the Gaussian class, it is proposed in [112] to make use of the particular cumulant expansion

$$\langle \xi f(\xi) \rangle = \sum_{l=0}^{p} \frac{\kappa_{l+1}}{l!} \langle f^{(p)}(\xi) \rangle + R_{p+1}. \tag{3.59}$$

Here $\{\kappa_{l+1}\}_{l=0,1,\dots}$ refers to the cumulants of the distribution of ξ, $f^{(l)}$ denotes the l-th derivative of f, and R_{p+1} is a remainder term which can be bounded in terms of $f^{(l+1)}$. In [14] (3.59) is attributed to Barbour [18].

4. The results for the covariance of linear statistics for the Gaussian external source model of Remark 3.1.5, and the Gaussian sample covariance matrices of Remark 3.2.3, have been generalised to Wigner matrices. In fact the references cited to arrive at (3.29) and (3.43) are formulated in this more general setting.

5. Let the independent upper triangular diagonals of an Hermitian matrix be labelled $d = 1$ (main diagonal), $d = 2$ (first diagonal above the main diagonal), etc. A band Hermitian matrix has all such independent diagonals $d > d^*$ for

some d^* with all entries equal to zero. In the case that the entries in the diagonals $d = 1, \ldots, d^*$ are random and as for Wigner matrices, and d^* is dependent on N such that $d^* \to \infty$, $d^*/N \to 0$ as $N \to \infty$ a generalisation of the covariance formula (3.55) has been derived in [154, 106].

3.4. Singular values of random matrix products

The singular values of an $n \times N$ $(n \geq N)$ matrix X are the eigenvalues of $XX^\dagger$. In the case $X = G_1 G_2$ with each G_i an independent GinUE matrix, a loop equation analysis of the square singular values has been carried out in [55]. The already known result [146] that the limiting resolvent satisfies the cubic equation

$$x^2 (W_{1,0}^{G^2}(x))^3 - x W_{1,0}^{G^2}(x) + 1 = 0 \qquad (3.60)$$

was recovered, and the limiting second order resolvent $W_{2,0}^{G^2}$ was expressed in terms of $W_{1,0}^{G^2}$. Analogous to the results (3.21) and (3.42) it was found from this that the corresponding limiting covariance of the monomial statistics $\sum_{j=1}^{N} x_j^{k_1}$, $\sum_{j=1}^{N} x_j^{k_2}$, to be denoted $\mu_{k_1,k_2}^{G^2}$ say, has the explicit evaluation

$$\mu_{k_1,k_2}^{G^2} = \frac{2k_1 k_2}{3(k_1 + k_2)} \binom{3k_1}{k_1} \binom{3k_2}{k_2}. \qquad (3.61)$$

In the case of a product of M independent GinUE matrices, Gorin and Sun [97] have given a double integral formula for $\mathrm{Cov}(p_j(x), p_k(x))$ which however appears to be difficult to evaluate. This is similarly true of the formula for the variance of a more general, not necessarily polynomial, linear statistics in the case of the product of two real Wigner matrices given in [96].

In the case of product of complex, rectangular Ginibre matrices the work of Lambert [117] does provide an easy to evaluate single contour integral formula for the variance of a polynomial linear statistics. This work is based on special properties of the biorthogonal functions underpinning integrability of the singular values of the products [3]. To state the result, let G_j be an $N_j \times N_{j-1}$ rectangular GinUE matrix and consider the squared singular values of the product $W_N = G_M G_{M-1} \cdots G_1$ where $N_0 = 1$ and $N_j = N + \eta_j$ with $\eta_j \geq 0$. Divide the squared singular values by $\prod_{j=1}^{M} N_j$. Reading off from [117, Th. 4.2], the following fluctuation formula holds true.

Proposition 3.6. *In terms of the above notation, suppose $N/N_j \to \gamma_j \in [0, 1]$ and $N \to \infty$. Then the variance of the polynomial linear statistic of the squared singular values $\sum_{j=1}^{N} p(x_j)$ is given by*

$$\sum_{k=1}^{\infty} k C_k C_{-k}, \qquad (3.62)$$

where

$$C_k = \frac{1}{2\pi i} \oint p\Big(z^{-M} \prod_{l=0}^{M} (z + \gamma_l)\Big) z^{-k} \frac{dz}{z}. \qquad (3.63)$$

Moreover, the limiting distribution of this linear statistic is a Gaussian.

Specifically, in the case of a product of two square GinUE matrices, and with $p(x) = x^l$ we have that (3.62) reduces to

$$\sum_{k=1}^{l} k \binom{3l}{2l+k} \binom{3l}{2l-k} = \frac{l}{3} \binom{3l}{l}^2, \qquad (3.64)$$

where the value of the sum has been obtained using computer algebra. This is in agreement with the case $k_1 = k_2$ of (3.61).

Closely related to the squared singular values of random complex GinUE matrices is the Laguerre Muttalib–Borodin model eigenvalue PDF in the variables $x_l \mapsto x_l^{1/M}$, proportional to

$$\prod_{l=1}^{N} x_l^c e^{-x_l^{1/M}} \prod_{1 \le j < k \le N} (x_k - x_j)(x_k^{1/M} - x_j^{1/M}), \qquad x_l > 0. \qquad (3.65)$$

For example, when $M = 2$ the scaled limiting resolvent satisfies (3.60) [89]. After scaling $x_l \mapsto x_l/N$, it has been shown in [117] that the formula of Proposition 3.6 remains true with $\gamma_0 = \gamma_1 = \cdots = \gamma_M = 1$.

3.5. Global scaling of Ginibre matrices and generalisations

The eigenvalue PDF for GinUE is proportional to

$$\prod_{l=1}^{N} e^{-\beta |z_l|^2/2} \prod_{1 \le j < k \le N} |z_k - z_j|^\beta \qquad (3.66)$$

with $\beta = 2$; see e.g. [75, §15.1.1]. As in §2.8 the parameter β has the interpretation of inverse temperature in a particular equilibrium statistical mechanics analogy. The latter relates to a system of N particles in two-dimensions with potential energy

$$U = \frac{1}{2} \sum_{l=1}^{N} |z_l|^2/2 - \sum_{1 \le j < k \le N} \log |z_k - z_j|,$$

which up to an additive constant is realised by the two-dimensional one-component plasma model (2dOCP) of N log-potential unit charges in the presence of a disk of radius $\sqrt{N}$ containing a uniform smeared out neutralising background.

The global scaling limit corresponds to the replacement $z_l \mapsto \sqrt{N} z_l$. Then, for all $\beta > 0$, the density is the uniform distribution on the unit disk, as can be established by potential theoretic considerations [150, 43]. From a random matrix viewpoint, this latter feature is an example of the circular law [27], which tells us that for non-Hermitian random matrices with identically distributed,

zero mean and finite variance entries, the eigenvalue density in the global scaling limit is uniform inside a disk.

In relation to fluctuation formulas, with

$$C_N^{\text{2dOCP}}(\mathbf{r} - \mathbf{r}') = \rho_{(2),N}^{\text{2dOCP}}(\mathbf{r} - \mathbf{r}', \mathbf{0}) - \frac{N^2}{\pi^2} + \frac{N}{\pi}\delta(\mathbf{r} - \mathbf{r}') \qquad (3.67)$$

(cf. (2.102); here N/π corresponds to the density inside the unit disk, assuming global scaling) write

$$S_\infty^{\text{2dOCP}}(\mathbf{k}) = \int_{\mathbb{R}^2} \Big(\lim_{N\to\infty} C_N^{\text{2dOCP}}(\mathbf{r}, \mathbf{0}) \Big) e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}. \qquad (3.68)$$

Linear response arguments [107] predict that

$$S_\infty^{\text{2dOCP}}(\mathbf{k}) = \frac{|\mathbf{k}|^2}{2\pi\beta} \qquad (3.69)$$

(note the consistency with (2.101) in the case $\beta = 2$), or equivalently

$$\lim_{N\to\infty} C_N^{\text{2dOCP}}(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi\beta}\nabla^2\delta(\mathbf{r} - \mathbf{r}'). \qquad (3.70)$$

The validity of (3.70) is restricted to $\mathbf{r}, \mathbf{r}'$ strictly inside of the unit disk. On the boundary, different linear response arguments predict [107], [75, §15.4.3]

$$\lim_{N\to\infty} C_N^{\text{2dOCP}}(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi^2\beta}\Big(\frac{\partial^2}{\partial\theta_1\partial\theta_2} \log\Big| \sin(\theta_1 - \theta_2)/2 \Big| \Big)\delta(r_1 - 1)\delta(r_2 - 1). \qquad (3.71)$$

The two results (3.70) and (3.71) together, substituted in (2.5), predict [74]

$$\lim_{N\to\infty} \text{Cov}^{\text{2dOCP}}\Big(\sum_{j=1}^N f(\mathbf{r}_j), \sum_{j=1}^N g(\mathbf{r}_j) \Big)$$
$$= \frac{1}{2\pi\beta} \int_{|\mathbf{r}|<1} \nabla f \cdot \nabla g\, dx dy + \frac{1}{\beta} \sum_{n=-\infty}^\infty |n| f_n g_{-n}, \quad (3.72)$$

where f_n, g_n are the angular Fourier components of $f(\mathbf{r})|_{|\mathbf{r}|=1}, g(\mathbf{r})|_{|\mathbf{r}|=1}$ (cf. (2.13)).

In the case that $f(\mathbf{r}) = f(|\mathbf{r}|)$ or $g(\mathbf{r}) = g(|\mathbf{r}|)$ the second term in (3.72) vanishes. Further setting $f = g$ in the GinUE case $\beta = 2$ it is simple to compute the limiting characteristic function for the linear statistic $\sum_{j=1}^N f(|\mathbf{r}_j|)$ using the Vandermonde determinant form of $\prod_{1\le j<k\le N}(z_k - z_j)$; see e.g. [75, Eq. (1.173)]. This calculation shows [74]

$$\lim_{N\to\infty} \text{Var}^{\text{GinUE}}\Big(\sum_{j=1}^N f(|\mathbf{r}_j|) \Big) = \frac{1}{2} \int_0^1 r(f'(|\mathbf{r}|))^2\, dr, \qquad (3.73)$$

which is consistent with the appropriate specialisation of (3.72). Proofs of (3.72), and the underlying Gaussian fluctuation formula, have been given in [148, 8] for $\beta = 2$ and in [120] for general $\beta > 0$.

A generalisation of (3.66) is the PDF proportional to

$$\exp\Big(-\frac{\beta}{2}\sum_{j=1}^{N}\Big(\frac{x_j^2}{1+\tau}+\frac{y_j^2}{1-\tau}\Big)\Big)\prod_{1\le j<k\le N}|z_k-z_j|^{\beta},\quad 0\le\tau<1. \qquad (3.74)$$

After scaling $z_j\mapsto\sqrt{N}z_j$ the leading order support is an ellipse with semi-axes $A=1+\tau$, $B=1-\tau$ [57, 82]. In the case $\beta=2$ (3.74) corresponds to the eigenvalue PDF of the complex nonsymmetric random matrices $J=H+ivA$ [91]. Both H and A are Gaussian Hermitian random matrices, and with $X=H,A$ and $\tau=(1-v^2)/(1+v^2)$, have joint PDFs for their elements proportional to $\exp\Big(-\frac{1}{1+\tau}\mathrm{Tr}X^2\Big)$. Parametrising the boundary of the ellipse by

$$x+iy=\cosh(\xi_b+i\eta),\qquad 0\le\eta<2\pi,\ \tanh\xi_b=(1-\tau)/(1+\tau) \qquad (3.75)$$

the only modification of (3.72) required, beyond altering the integration domain in the first term, is to replace f_n,g_n therein by the angular Fourier components in η of $f(\mathbf{r}),g(\mathbf{r})$ on the boundary (3.75) [74]. For the particular linear statistic $f(\mathbf{r})=c_{10}x+c_{01}y$, completing the square gives for the characteristic function

$$\hat{P}_{N,f}(t)=e^{-t^2(c_{10}^2(1+\tau)+c_{01}^2(1-\tau))/(2\beta)}. \qquad (3.76)$$

Hence the variance is equal to $(c_{10}^2(1+\tau)+c_{01}^2(1-\tau))/(2\beta)$, which indeed is consistent with the specified modification of (3.72). In the case $\beta=2$ a derivation of the elliptic analogue of (3.71) is possible [82, 7, 2]. For general $\beta>0$ the validity of (3.72), with the RHS modified as noted, is covered by the results of [120] which proves too that the linear statistic has Gaussian fluctuations.

Generalising GinUE matrices to $N\times N$ complex matrices having general i.i.d. complex entries z_{ij} with $\langle z_{ij}\rangle=\langle z_{ij}^2\rangle=0$, $\langle|z_{ij}|^2\rangle=1$ gives the analogue of the Wigner class of complex Hermitian matrices. It is shown in the work of Cipolloni, Erdös and Schröder [47] that the covariance formula (3.72) with $\beta=2$ is then extended to include the additional term

$$\kappa_4\Big(\frac{1}{\pi}\int_{|z|<1}f(\mathbf{r})\,dxdy-\frac{1}{2\pi}\int_0^{2\pi}f(\theta)\,d\theta\Big)\Big(\frac{1}{\pi}\int_{|z|<1}g(\mathbf{r})\,dxdy-\frac{1}{2\pi}\int_0^{2\pi}g(\theta)\,d\theta\Big), \qquad (3.77)$$

where $\kappa_4:=\langle|z_{ij}|^4\rangle-2$ is the fourth cumulant of the distribution of the entries. Moreoever, the work [47] places particular emphasis on the class of test functions for which the covariance formula has a rigorous proof, and provides an extensive list, and discussion, of previous literature. The technical ideas underlying the results of [47] have been reviewed in [45].

Modifying GinUE to have standard real rather than standard complex entries gives what we will refer to as the Ginibre orthogonal ensemble (GinOE). Unlike the circumstance for the GOE and GUE, where (3.3) with $V(x)=x^2$ is the functional form for the eigenvalue PDF of both, depending on the value of β, this is not the case for GinOE in relation to (3.66). In fact the eigenvalue PDF for GinOE is not absolutely continuous, and is naturally broken into sectors,

depending on the number of real eigenvalues [69]. Furthermore, the complex eigenvalues for GinOE must come in complex conjugate pairs. Despite these differences, as first found in [115, 137], and earlier in the case of polynomial test functions in [136], the fluctuation formula (3.72) with $\beta = 1$ does give the correct form of the covariance, where it is being assumed that both f and g are symmetric about the real axis. In the real analogue of the more general setting discussed in the previous paragraph the same term (3.77) is to be added [48].

Remark 3.7. 1. The fluctuation formula (3.72) with $\beta = 2$ has been shown to remain valid in the case of the eigenvalues of products of i.i.d. complex random matrices in [116], albeit for a class of test function contained strictly inside the eigenvalue support; see too [50].

2. The recent work [42] gives a generalisation of (3.73) to the case that f is discontinuous.

Acknowledgments

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