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Infinite convolutions of probability measures on Polish semigroups*

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Abstract: This expository paper is intended for a short self-contained introduction to the theory of infinite convolutions of probability measures on Polish semigroups. We give the proofs of the Rees decomposition theorem of completely simple semigroups, the Ellis–Żelazko theorem, the convolution factorization theorem of convolution idempotents, and the convolution factorization theorem of cluster points of infinite convolutions.

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1. Introduction

As a natural generalization of random walks on an integer lattice, the theory of infinite convolutions of probability measures on topological semigroups has been extensively studied and widely applied to various problems. For this theory, there are celebrated textbooks Rosenblatt [42], Mukherjea–Tserpes [33] and Högnäs–Mukherjea [16], which include a lot of applications of the theory; see also Mukherjea's lecture notes [30] for applications to random matrices, and Ito–Sera–Yano [17] for applications to the problem of resolution of σ-fields.

The aim of this paper is to help the reader to gain the basic knowledge of this thoery conveniently. We mainly follow [16] and we make some modifications on the proofs. For a potential application, we develop the theory for topological semigroups equipped with a Polish topology, while the textbooks [42, 33, 16] deal with semigroups equipped with a locally compact Hausdorff second countable topology.

The goal of this paper is the convolution factorization theorem of cluster points of infinite convolutions, which will be stated as Theorem 4.9. The key to the proof is the convolution factorization theorem of convolution idempotents, which will be stated as Theorem 4.6, and the study of probability measures with convolution invariance, which will be stated as Proposition 4.5. Theorems 4.6 and 4.9 are based on the product decomposition theorem for completely simple semigroups, which will be called the *Rees decomposition* and stated as Theorem 2.10. To show that the algebraic decomposition is compatible with a Polish topology, we need the Ellis–Żelazko theorem, which will be stated as Theorem 3.2.

The Ellis theorem [11](1957) asserts that an algebraic group where the product mapping is separately continuous is a topological group, where the topology is locally compact Hausdorff second countable. It was extended to completely metrizable topologies by Żelazko [52](1960).

The study of infinite convolutions on compact groups was initiated by Kawada–Itô [19](1940), It was investigated further by Urbanik [50](1957), Kloss [23](1959), and Stromberg [46](1960), and extended to the context of locally compact groups by Tortrat [49](1964) and Csiszár [7](1966). The convolution invariance Proposition 4.5 is due to Mukherjea [27](1972), which originates from the Choquet–Deny equation [2](1960); for later studies, see [51, 39, 9, 8, 38, 24, 48]. Theorem 4.6 for convolution idempotents is due to Mukherjea–Tserpes [32] (1971); for ealier studies, see Collins [6](1962), Pym [37](1962), Heble–Rosenblatt [14](1963), Schwarz [45](1964), Choy [3](1970), Duncan [10](1970), and Sun–Tserpes [47](1970); see also [12]. Theorem 4.9 for cluster points of infinite convolutions is due to Rosenblatt [40](1960) in the compact case and to Mukherjea [29](1979) in the locally compact case; for studies earlier than [29], see Glicksberg [13](1959), Collins [5](1962), Schwarz [44](1964), Rosenblatt [41](1965), Lin [25](1966), Mukherjea [28](1977), and Mukherjea–Sun [31](1978); for related papers, see [34, 43, 26, 1].

This paper is organized as follows. In Section 2 we review the theory of algebraic semigroups. In Section 3 we study the theory of Polish semigroups, where the Ellis–Żelazko theorem is proved and utilized. Section 4 is devoted to the convolution factorization theorems of convolution idempotents and of cluster points of infinite convolutions. In Section 5, we give two examples for the theorem of infinite convolutions.

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2. Algebraic semigroup

We say that a non-empty set S is a *semigroup* if it is endowed with multiplication

$$S \times S \ni (a, b) \mapsto ab \in S \tag{2.1}$$

which is associative, i.e.,

$$(ab)c = a(bc), \quad a, b, c \in S. \tag{2.2}$$

For two subsets A and B of S, we denote their product by

$$AB = \{ab : a \in A,\ b \in B\}. \tag{2.3}$$

We write $A^1 = A$ and $A^n = A^{n-1}A$ for $n \geq 2$. We sometimes identify an element $a \in S$ with the singleton $\{a\}$; for instance, $aS = \{a\}S = \{ab : b \in S\}$. An element $e \in S$ is called *identity* if

$$xe = ex = x, \quad x \in S. \tag{2.4}$$

It is obvious that the identity is unique if it exists. For a semigroup S with identity e, we say that $y \in S$ is the *inverse* of $x \in S$ if $xy = yx = e$. It is obvious that the inverse of an element $x \in S$ is unique if it exists. A *group* is a semigroup S with identity such that every element has an inverse.

2.1. Left and right simplicity

Let S be a semigroup. A non-empty subset I is called a *left ideal* [*right ideal*] (of S) if $SI \subset I$ [$IS \subset I$]. If S contains no proper left ideal [right ideal], then it is called *left simple* [*right simple*]. A non-empty subset I is called an *ideal* if it is both a left and a right ideal, i.e., $SI \cup IS \subset I$. If S contains no proper ideal, then it is called *simple*. Note that being left or right simple implies being simple, but the converse statement is not true. We say that I is a *minimal left ideal* of S if I is a left ideal of S and does not contain a proper left ideal of S. We also define a *minimal right ideal* and a *minimal ideal* similarly.

Example 2.1. Let $V = \{1, 2\}$ and let S denote the set of mappings from V into itself. We equip S with the semigroup structure with respect to composition: $(fg)(v) = f(g(v))$ for $f, g \in S$ and $v \in V$. We write ι_1, ι_2 for the constant mappings: $\iota_1(v) = 1$ and $\iota_2(v) = 2$ for all $v \in V$. Then the following claims are obvious:

(i) The sets $\{\iota_1\}$ and $\{\iota_2\}$ are both minimal right ideals of S, but are not left ideals.
(ii) The set $\{\iota_1, \iota_2\}$ is a minimal left ideal of S and is a right ideal, but is not a minimal right ideal.

Lemma 2.2. *For a subsemigroup S of a semigroup S_0, the following are equivalent:*

(i) S is a minimal left ideal of S_0.
(ii) $S = S_0 a$ for all $a \in S$.

Proof. Suppose S is a minimal left ideal. Since $S_0 a$ for $a \in S$ is a left ideal of S_0 contained in S, we have $S = S_0 a$ by minimality.

Suppose $S = S_0 a$ for all $a \in S$. Let I be a left ideal of S_0 such that $I \subset S$. For any $a \in I$, we have $S = S_0 a \subset S_0 I \subset I$, which shows that S is a minimal left ideal of S_0. □

Lemma 2.3. *For a subsemigroup S of a semigroup S_0, the following are equivalent:*

(i) S is a minimal ideal of S_0.
(ii) $S = S_0 a S_0$ for all $a \in S$.

The proof of Lemma 2.3 is almost the same as that of Lemma 2.2, and so we omit it.

Lemma 2.4. *For a semigroup S, the following are equivalent:*

(i) For any semigroup S_0 of which S is a left ideal, S is a minimal left ideal of S_0.
(ii) S is left simple, or in other words, S is a minimal left ideal of S itself (if and only if $S = Sa$ for all $a \in S$ by Lemma 2.2).
(iii) There exists a semigroup S_0 such that S is a minimal left ideal of S_0.
(iv) For any $a, b \in S$, the equation $xa = b$ has at least one solution $x \in S$.

Proof. [(i) ⇒ (ii) ⇒ (iii)] These are obvious.

[(iii) ⇒ (ii)] Suppose that S is a minimal left ideal of S_0 and let I be a left ideal of S. Since $S_0 SI \subset SI \subset I \subset S$, we see that SI is a left ideal of S_0 with $SI \subset S$. Hence $SI = S$ by minimality. Since $I \subset S = SI \subset I$, we have $I = S$, which implies that S is a minimal left ideal of S.

[(ii) ⇒ (i)] Suppose that S is a left ideal of a semigroup S_0 and let I be a left iedal of S_0 such that $I \subset S$. Then $SI \subset S_0 I \subset I$, and so I is a left ideal of S. By the minimality assumption, we have $I = S$, which shows that S is a minimal left ideal of S_0.

[(ii) ⇒ (iv)] This is obvious by $S \subset Sa$.

[(iv) $\Rightarrow$ (ii)] Let $a \in S$. Then we have $S \subset Sa$ by the assumption. Since S is a semigroup, we have $Sa \subset S$. Hence we have $S = Sa$. $\square$

The next lemma treats simplicity. The proof is similar and is omitted.

Lemma 2.5. *For a semigroup S, the following are equivalent:*

(i) For any semigroup S_0 of which S is an ideal, S is a minimal ideal of S_0.
(ii) S is simple, or in other words, S is a minimal ideal of S itself (if and only if $S = SaS$ for all $a \in S$ by Lemma 2.3).
(iii) There exists a semigroup S_0 such that S is a minimal ideal of S_0.
(iv) For any $a, b \in S$, the equation $xay = b$ has at least one solution $(x, y) \in S \times S$.

Proposition 2.6. *A semigroup S which is both left and right simple is a group.*

Proof. Let $a \in S$. By Lemma 2.4, we have $ea = a$ for some $e \in S$. For any $x \in S$, we have $x = ay$ for some $y \in S$, and so we have $ex = eay = ay = x$. Similarly, there exists $e' \in S$ such that $xe' = x$ for all $x \in S$. Then we obtain $e' = ee' = e$, and thus e is identity of S.

Let $x \in S$. By Lemma 2.4, we have $xy = e$ and $y'x = e$ for some $y, y' \in S$. Since $y' = y'e = y'xy = ey = y$, we see that y is the inverse of x. $\square$

2.2. Left and right groups

Let S be a semigroup. An element $e \in S$ is called an *idempotent* if $e^2 = e$. We denote the set of all idempotents of S by

$$E(S) = \{e \in S : e^2 = e\}. \tag{2.5}$$

Note that, if e is an idempotent, then any element of Se is invariant under right multiplication by e, i.e., $x \in Se$ implies $xe = x$. A semigroup S is called a *left group* [*right group*] if S is left simple [*right simple*] and contains at least one idempotent.

Example 2.7. Let us keep the notation of Example 2.1. Then $\{\iota_1, \iota_2\}$ is a left group. In fact, both ι_1 and ι_2 are idempotents, and $\{\iota_1, \iota_2\}$ is left simple by Lemma 2.4, because $\{\iota_1, \iota_2\}$ is a minimal left ideal of S.

A semigroup S is called *left cancellative* [*right cancellative*] if, for any $a, x, y \in S$ with $ax = ay$ [$xa = ya$], we have $x = y$. An element $e \in S$ is called a *left identity* [*right identity*] if $ex = x$ [$xe = x$] for all $x \in S$.

Lemma 2.8. *Let S be a semigroup. If S is either right cancellative or left simple, then any idempotent of S is a right identity.*

Proof. Suppose S is right cancellative and let $e \in E(S)$. Then $xee = xe$ implies $xe = x$.

Suppose S is left simple and let $e \in E(S)$. By Lemma 2.4, we have $S = Se$, which yields that $xe = x$ for all $x \in S$. $\square$

Proposition 2.9. *For a semigroup S, the following are equivalent:*

(i) *S is a left group.*
(ii) *S is left simple and right cancellative.*
(iii) *For any $a, b \in S$, the equation $xa = b$ has a unique solution $x \in S$.*

Proof. [(i) $\Rightarrow$ (ii)] Let $e \in E(S)$ be fixed. By Lemma 2.8, we see that e is a right identity.

Suppose $xa = ya$. By Lemma 2.4, we have $ba = e$ for some $b \in S$. We then have $abab = aeb = ab$, so that $ab \in E(S)$ and ab is a right identity. We then obtain $x = xab = yab = y$.

[(ii) $\Rightarrow$ (iii)] Existence follows from left simplicity and Lemma 2.4. Uniqueness follows from right cancellativity.

[(iii) $\Rightarrow$ (i)] By (iii), we have $S = Sa$ for all $a \in S$, which shows by Lemma 2.4 that S is left simple.

Let $a \in S$ and take $e \in S$ such that $ea = a$ by (iii). Then we have $e^2a = ea = a$, which leads to $e^2 = e$ by right cancellativity. $\square$

2.3. Rees decomposition

Let S be a semigroup. An idempotent $e \in E(S)$ is called *primitive* if

$$ex = xe = x \in E(S) \text{ implies } x = e. \tag{2.6}$$

We say that S is *completely simple* if S is simple and contains a primitive idempotent.

Theorem 2.10 (Rees decomposition)**.** *Let S be a completely simple semigroup and let e be a primitive idempotent of S. Set*

$$L := E(Se), \quad G := eSe, \quad R := E(eS). \tag{2.7}$$

Then the following assertions hold:

(i) *$LG = Se$ is a left group and $GR = eS$ is a right group.*
(ii) *$RL \subset G$ and $eL = Re = \{e\}$.*
(iii) *$G = Se \cap eS$ is a group where e is its identity.*
(iv) *$S = LGR$ (This factorization will be called the* Rees decomposition *of S at e, and G will be called the* group factor *at e).*
(v) *The product mapping*

$$\psi : L \times G \times R \ni (x, g, y) \mapsto (xgy) \in LGR \tag{2.8}$$

is bijective with its inverse given as

$$\psi^{-1} : LGR \ni z \mapsto (ze(eze)^{-1}, eze, (eze)^{-1}ez) \in L \times G \times R. \tag{2.9}$$

Proof. (i) It is obvious that Se is a left ideal of S. Let I be a left ideal of S such that $I \subset Se$. Let $a \in I$. Note that $ae = a$ since $a \in Se$. By simplicity of S and

Lemma 2.5, we have $uav = e$ for some $u, v \in S$. Set $r = eu$ and $s = eve$. We then have

$$ras = eu(ae)ve = euave = e, \quad er = r, \quad es = se = s. \tag{2.10}$$

If we set $t = sra$, then $et = te = t$ and

$$t^2 = s(ras)ra = sera = sra = t, \tag{2.11}$$

which yields $t = e$ by primitivity. Since $e = t = sra \in srI \subset I$, we have $Se \subset SI \subset I$, which shows $I = Se$ and that Se is a minimal left ideal of S. By Lemma 2.4, we see that Se is left simple. Since Se contains an idempotent e, we see that Se is a left group. By a similar argument we see that eS is a right group.

Let us show $LG = Se$. It is obvious that $LG \subset Se$. Let $a \in Se$. Set $g := ea \in eSe = G$ and set $b = ag^{-1} \in Se$. Since $g^{-1} = g^{-1}e$, we have

$$b^2 = ag^{-1}ag^{-1} = ag^{-1}(ea)g^{-1} = ag^{-1} = b. \tag{2.12}$$

Hence we have $b \in E(Se) = L$ and $a = ae = ag^{-1}g = bg \in LG$. We now have $LG = Se$. We also have $GR = eS$ similarly.

(ii) $RL \subset (eS)(Se) \subset eSe = G$.

Let $x \in L = E(Se)$. Since $(ex)^2 = e(xe)x = exx = ex$ and $e(ex) = (ex)e = ex$, we have $ex = e$ by primitivity. We thus see that $eL = \{e\}$. We have $Re = \{e\}$ similarly.

(iii) It is obvious that $G = eSe = eS \cap Se$, since $x \in eS \cap Se$ implies $x = ex = xe = exe$. It is also obvious that e is identity of G. Let $g \in G$. Since $G \subset eSe$, we have $g = ea$ for some $a \in Se$. By the left simplicity of Se and by Lemma 2.4, we have $ba = e$ for some $b \in Se$. Since $(ab)^2 = a(ba)b = aeb = ab$, we see by Lemma 2.8 that ab is right identity. Hence $ab = abe = e$, which shows that b is the inverse of a.

(iv) $LGR = LGGR = SeeS = SeS = S$ by Lemma 2.5.

(v) Let $z = xgy$ with $(x, g, y) \in L \times G \times R$. Since $x = xx = xex$ and since $exgye \in eSe = G$, we have

$$x = xe = x(exgye)(exgye)^{-1} = ze(eze)^{-1}. \tag{2.13}$$

We have $y = (eze)^{-1}ez$ similarly. Since $ex = ye = e$ by (ii), we obtain

$$g = ege = (ex)g(ye) = eze. \tag{2.14}$$

The proof is now complete. □

Corollary 2.11. *Under the same assumptions and notation as Theorem 2.10, it holds that $\{Sy = LGy : y \in R\}$ is the family of all minimal left ideals of S.*

Proof. Any minimal left ideal of S is of the form Sz for some $z \in S$. We represent $z = xgy$ and then we obtain $Sz = LG(Rx)gy = LGy$, since $RL \subset G$.

Conversely, for any $z \in LGy$, we have $z = xgy$ for some $(x, g) \in L \times G$, so that we have $LGyz = LG(yx)gy = LGy$, which shows by Lemma 2.2 that LGy is a minimal left ideal. □

Corollary 2.12. *Under the same assumptions and notation as Theorem 2.10, the following assertions hold:*

(i) For $z = xgy$ with $(x, g, y) \in L \times G \times R$, z is idempotent if and only if $g = (yx)^{-1}$.
(ii) All idempotents of S are primitive.
(iii) Let e' be another idempotent of S and represent it as $e' = a(ba)^{-1}b$ for $(a, b) \in L \times R$. Let $S = L'G'R'$ denote the Rees decomposition of S at e'. Then

$$L'G' = LGb, \quad G' = aGb, \quad G'R' = aGR. \tag{2.15}$$

Proof. (i) Suppose $z^2 = z$. Then $xgyxgy = xgy$. Since $eL = Re = \{e\}$, we have $gyxg = g$, which shows $g = (yx)^{-1}$.

Conversely, suppose $g = (yx)^{-1}$. Then $z^2 = x(gyxg)y = xgy = z$.

(ii) Let $e_1, e_2 \in S$ be two idempotents of S and represent them as $e_i = a_i(b_ia_i)^{-1}b_i$ for $(a_i, b_i) \in L \times R$, $i = 1, 2$. Suppose $e_1e_2 = e_2e_1 = e_2$. Then $a_1((b_1a_1)^{-1}(b_1a_2)(b_2a_2)^{-1})b_2 = a_2((b_2a_2)^{-1}(b_2a_1)(b_1a_1)^{-1})b_1 = a_2(b_2a_2)^{-1}b_2$, which shows $a_1 = a_2$ and $b_1 = b_2$ by the injectivity of the product mapping ψ. Hence we have $e_1 = e_2$, which shows that e_1 is a primitive idempotent.

(iii) We have $L'G' = Se' = LG(Ra)(ba)^{-1}b = LGb$ and $G'R' = aGR$ similarly. We also have $G' = e'Se' = a(ba)^{-1}(bL)G(Ra)(ba)^{-1}b = aGb$. □

Corollary 2.13. *A left group S is completely simple. The Rees decomposition of S at $e \in E(S)$ is given as $S = LG$ with $R = \{e\}$.*

Proof. Suppose $ex = xe = x \in E(S)$. By Lemma 2.4, we have $yx = e$ for some $y \in S$. Hence $x = ex = yxx = yx = e$, which shows that e is an primitive idempotent. Hence S is completely simple. Let $S = LGR$ denote the Rees decomposition of S at e. Since $S = Se$ by Lemma 2.4 and since $Re = \{e\}$, we obtain $S = Se = LGRe = LG$. □

For later use we prove the following proposition.

Proposition 2.14. *Suppose that a semigroup S contains a minimal left ideal A and a minimal right ideal B as well. Then BA is a group and its identity is a primitive idempotent of S. If, in addition, S is simple, then S is completely simple.*

Proof. Since $(BA)(BA) = (BAB)A \subset BA$, we see that BA is a subsemigroup of S. To prove right simplicity of BA, let I be a right ideal of BA. Since IB is a right ideal of S and $IB \subset BAB \subset B$, we see that $IB = B$ by minimality. Hence $BA = IBA \subset I$, which shows right simplicity of BA. By a similar argument we obtain left simplicity of BA. We thus conclude by Proposition 2.6 that BA is a group.

Let e be the identity of BA and suppose $ex = xe = x \in E(S)$. Then $x = xx = exxe \in (BAS)(SBA) \subset BA$. Since BA is a group and since $x^2 = x$, we have $x = xx^{-1} = e$, which shows that e is a primitive idempotent of S. □

2.4. Kernel

A minimal ideal of a semigroup S will be called a *kernel* of S.

Theorem 2.15. *Let S be a semigroup. Then the following assertions hold:*

(i) If S contains a minimal left ideal, then S contains a unique kernel K, and $SzS = K$ for all $z \in K$.
(ii) If S contains a minimal left ideal and a minimal right ideal as well, then the unique kernel of S is completely simple.
(iii) If S contains a completely simple kernel K, then it is the unique kernel of S. Let $K = LGR$ denote the Rees decomposition at e. Then $Sz = Kz = LGz$ for all $z \in K$.

Proof. (i) Let $\mathcal{A}$ denote the family of all minimal left ideals of S and suppose $\mathcal{A}$ is not empty. We shall prove that $K := \bigcup \mathcal{A}$ is a unique kernel of S.

Let $z \in K$ and take $A \in \mathcal{A}$ such that $z \in A$. Then $Sz = A$ by Lemma 2.2. For $x \in S$, we see that $Ax \in \mathcal{A}$; in fact, for any left ideal I of S such that $I \subset Ax$, we see that $J = \{a \in A : ax \in I\} \subset A$ is a left ideal of S, so that $J = A$ by minimality and thus $I = Ax$. Hence $SzS = AS = \bigcup_{x \in S} Ax \subset \bigcup \mathcal{A} = K$, which shows by Lemma 2.3 that K is a kernel of S.

Let K' be another kernel of S. Since $K \cap K'$ contains KK' which is not empty, we see that $K \cap K'$ is an ideal contained both in K and in K'. Thus $K \cap K' = K = K'$ by minimality.

(ii) By (i) and Lemma 2.3, we see that the unique kernel K of S is both a minimal left ideal of K and a minimal right ideal of K. By Proposition 2.14, we see that K is completely simple.

(iii) Suppose K is a completely simple kernel of S with a primitive idempotent e. By Theorem 2.10, K contains a left group Ke. By Lemma 2.4, we see that Ke is a minimal left idal of S. Hence by (i) the kernel of S is unique.

For $z \in K$, we represent $z = xgy \in LGR$. Then by (i) Sz is a minimal left ideal of K containing y. By Corollary 2.11, we see that $Sz = LGy = LGz = Kz$. $\square$

3. Topological semigroup

A semigroup S is called *topological* if S is endowed with a topology such that the product mapping $S \times S \ni (x, y) \mapsto xy \in S$ is jointly continuous. A semigroup S is called *Polish* if S is a topological semigroup with respect to a Polish topology, i.e. a separable and completely metrizable topology.

For a topological space, it is well-known (see, e.g. [20, Theorem 1.5.3]) that being locally compact Polish is equivalent to being locally compact Hausdorff with a countable base. It is elementary that being compact Polish is equivalent to being compact metrizable.

For $a \in S$ and $A \subset S$, we write

$$a^{-1}A = \{x \in S : ax \in A\}, \quad Aa^{-1} = \{x \in S : xa \in A\}. \tag{3.1}$$

If S contains identity e and $a \in S$ has its inverse $a^{-1} \in S$, then $(a^{-1})A = a^{-1}A$; in fact,

$$(a^{-1})A = \{a^{-1}x \in S : x \in A\} = \{y \in S : ay \in A\} = a^{-1}A. \tag{3.2}$$

Lemma 3.1. *Let S be a Polish semigroup. Then the following assertions hold:*

- *(i) For $a \in S$ and for a closed [open, Borel] subset A, both $a^{-1}A$ and Aa^{-1} are also closed [open, Borel].*
- *(ii) If A is a subsemigroup of S, then so is its closure $\overline{A}$.*
- *(iii) Let A be a closed subsemigroup of S. Then $E(A)$, eA, Ae and eAe are closed for all $e \in E(A)$.*
- *(iv) For two compact subsets K and K', the product KK' is also compact.*

Proof. (i) If we write $\psi_a : S \to S$ for the translation $\psi_a(x) = ax$, then $a^{-1}A = \psi_a^{-1}(A)$. Since ψ_a is continuous, we obtain the desired results.

(ii) Let $a, b \in \overline{A}$ and take $\{a_n\}, \{b_n\} \subset A$ such that $a_n \to a$ and $b_n \to b$. Then we have $ab = \lim a_n b_n \in \overline{A}$.

(iii) Let $\{e_n\} \subset E(A)$ such that $e_n \to e \in S$. Since A is closed, we have $e \in A$. Since $e_n^2 = e_n$ for all n, we have $e^2 = e$, which shows $e \in E(A)$.

Let $\{x_n\} \subset eA$ such that $x_n \to x \in S$. Since $eA \subset A$ and since A is closed, we have $x \in A$. Then $ex = \lim ex_n = \lim x_n = x$, which shows $x = ex \in eA$.

(iv) Let $\psi : S \times S \to S$ denote the jointly continuous product mapping: $\psi(x, y) = xy$. Since $KK' = \psi(K \times K')$ and $K \times K'$ is compact, we see that KK' is compact. □

3.1. Topological group

A group S is called *topological* if G is a topological semigroup and the inverse mapping $G \ni g \mapsto g^{-1} \in G$ is continuous.

Theorem 3.2 (Ellis [11] and Żelazko [52])**.** *If a group G is a topological semigroup with respect to a completely metrizable topology, then it is a topological group.*

Proof. We borrow the proof from Pfister [36]. Let e denote the identity of G and let d be a complete metric of G.

Let U_0 be a open neighborhood of e. By the joint continuity of the product mapping, we can construct a sequence $\{U_n\}_{n=1}^\infty$ of open balls of e such that the radius of U_n decreases to 0 and $\overline{U}_n\overline{U}_n \subset U_{n-1}$ for $n = 1, 2, \ldots$, where $\overline{U}_n$ stands for the closure of U_n.

Let $\{x_n\}_{n=1}^\infty$ be a subsequence of an arbitrary sequence of G which converges to e. It then suffices to construct a subsequence $\{n(k)\}_{k=1}^\infty$ of $\{1, 2, \ldots\}$ such that $x_{n(k)}^{-1} \to e$. We write $y_k := x_{n(1)} \cdots x_{n(k)}$.

Set $n(0) = 0$ and $y_0 = x_0 = e$. If we have $n(0), n(1), \ldots, n(k-1)$, then we can take $n(k) > n(k-1)$ such that $x_{n(k)} \in U_k$ and $d(y_k, y_{k-1}) < 2^{-k}$, since $y_{k-1}x_n \to y_{k-1}$ as $n \to \infty$. By completeness of d, we see that y_k converges to a limit $y \in G$. Let n be fixed for a while. Since yU_{n+1} is a neighborhood of y, we see that $y_{k-1} \in yU_{n+1}$ for large k. For $j > k$, we have $U_{j-1}U_j \subset U_jU_j \subset U_{j-1}$, and hence

$$y_k^{-1}y_j = x_{n(k+1)} \cdots x_{n(j-1)}x_{n(j)} \in U_{k+1} \cdots U_j \subset U_k, \tag{3.3}$$

which implies $y_k^{-1}y \in \overline{U}_k \subset U_{k-1}$. We now obtain

$$x_{n(k)}^{-1} = (y_{k-1}^{-1}y_k)^{-1} = y_k^{-1}y_{k-1} \in y_k^{-1}yU_{n+1} \subset U_{k-1}U_{n+1} \subset U_{n+1}U_{n+1} \subset U_n \tag{3.4}$$

for large k. Thus we obtain $x_{n(k)}^{-1} \to e$. □

Corollary 3.3. *Suppose that a Polish semigroup S contains a completely simple kernel K. Let $K = LGR$ denote the Rees decomposition of K at $e \in E(K)$. Then it holds that L, G, R and K are closed subsets, and that the product mapping*

$$\psi : L \times G \times R \ni (x, g, y) \mapsto xgy \in LGR \tag{3.5}$$

is a homeomorphism.

Proof. By Corollary 2.15, we have $Ke = Se$, $eK = eS$ and $eKe = eSe$. By Lemma 3.1, we see that $L = E(Ke)$, $G = eKe$ and $R = E(eK)$ are all closed. By Theorem 3.2, we see that G is a Polish group. We now see that the inverse

$$\psi^{-1} : LGR \ni z \mapsto (ze(eze)^{-1}, eze, (eze)^{-1}ez) \in L \times G \times R \tag{3.6}$$

is continuous. Consequently, we see that K is closed. □

3.2. Compact semigroup

Theorem 3.4. *A compact Polish semigroup S contains a compact completely simple kernel.*

Proof. Let $\mathcal{I}$ denote the family of all closed left ideals of S. The family $\mathcal{I}$ contains S and is endowed with a partial order by the usual inclusion. For any linearly ordered subfamily $\mathcal{J}$ of $\mathcal{I}$ has a lower bound in $\mathcal{I}$; in fact, the intersection $\bigcap \mathcal{J}$ is not empty by compactness of S and is a closed left ideal of S such that $\bigcap \mathcal{J} \subset J$ for all $J \in \mathcal{J}$. Hence, by Zorn's lemma, we see that $\mathcal{I}$ contains a minimal element, say A.

Let us prove that A is a minimal left ideal of S. Let I be a left ideal of S such that $I \subset A$. For $a \in I$, we have $Sa \in \mathcal{I}$ and $Sa \subset SI \subset I \subset A$, which yields $Sa = I = A$ by the minimality of A in $\mathcal{I}$. This shows that A is a minimal left ideal of S.

Similarly we see that S contains a minimal right ideal. By Theorem 2.15, we see that S contains a completely simple kernel K. By Corollary 3.3, we see that K is a closed subset of S, and hence K is compact. □

Proposition 3.5. *Let S be a Polish semigroup and let $a \in S$. Suppose that any subsequence of $\{a^n\}_{n=1}^\infty$ has a convergent further subsequence. Then the set C of all cluster points of $\{a^n\}_{n=1}^\infty$ is a compact abelian group. If we denote the identity of C by e, then $C = \overline{\{e, ae, a^2e, \dots\}}$.*

Proof. Let C denote the set of all cluster points of $\{a^n\}_{n=1}^\infty$. By the assumption, we see that C is a compact abelian semigroup. By Theorem 3.4, we see that C contains a compact completely simple kernel K. Since the Rees decomposition of K is $LGR = GRL = G$ by commutativity, we see that K is a compact abelian group. Let e denote the identity of K. Then, for any $x \in C$, we can find a subsequence $\{n(k)\}$ of $\{1, 2, \dots\}$ such that $x = e \lim_{k\to\infty} a^{n(k)} \in eC \subset KC \subset K \subset C$, which shows $K = eC = C$. It is now easy to see that $C = \overline{\{e, ae, a^2e, \dots\}}$. □

Remark 3.6. In the settings of Proposition 3.5, suppose that the sequence $\{a^n\}_{n=1}^\infty$ has multiple points. Let p and q be the smallest positive integers such that $a^{q+p} = a^q$. Then we have $\{a^n : n = 1, 2, \dots\} = \{a, a^2, \dots, a^{q+p-1}\}$ and

$$K = \{a^q, a^{q+1}, \dots, a^{q+p-1}\} = \{e, ae, \dots, a^{p-1}e\} \tag{3.7}$$

with $e = a^{rp}$, where r is the unique integer such that $q \le rp \le q + p - 1$.

4. Convolutions of probability measures on Polish semigroups

4.1. Convolutions

Let S be a Polish semigroup. Let $\mathcal{B}(S)$ denote the family of all Borel sets of S and $\mathcal{P}(S)$ the family of all probability measures on $(S, \mathcal{B}(S))$.

For $\mu, \nu \in \mathcal{P}(S)$, we define the *convolution* $\mu * \nu \in \mathcal{P}(S)$ of μ and ν by

$$\mu * \nu(B) = \iint 1_B(xy)\mu(\mathrm{d}x)\nu(\mathrm{d}y), \quad B \in \mathcal{B}(S). \tag{4.1}$$

Since $1_B(xy) = 1_{By^{-1}}(x) = 1_{x^{-1}B}(y)$, we have

$$\mu * \nu(B) = \int \mu(By^{-1})\nu(\mathrm{d}y) = \int \nu(x^{-1}B)\mu(\mathrm{d}x), \quad B \in \mathcal{B}(S). \tag{4.2}$$

For $a \in S$, we write δ_a for the Dirac mass at a: $\delta_a(B) = 1_B(a)$. It is obvious that

$$\mu * \delta_x(B) = \mu(Bx^{-1}), \quad \delta_x * \mu(B) = \mu(x^{-1}B), \quad B \in \mathcal{B}(S), \tag{4.3}$$

which will be called *translations* of μ.

For $\mu \in \mathcal{P}(S)$, we denote its *topological support* by

$$\mathcal{S}(\mu) = \{x \in S : \mu(U) > 0 \text{ for all open neighborhood } U \text{ of } x\}. \tag{4.4}$$

It is obvious that $\mathcal{S}(\mu)$ is closed and $\mu(\mathcal{S}(\mu)^c) = 0$.

Lemma 4.1. *For $\mu, \nu \in \mathcal{P}(S)$, it holds that*

$$\mathcal{S}(\mu * \nu) = \overline{\mathcal{S}(\mu)\mathcal{S}(\nu)}. \tag{4.5}$$

Proof. Let $a \in \mathcal{S}(\mu)$ and $b \in \mathcal{S}(\nu)$. For any open neighborhood U of ab, the joint continuity of the product mapping allows us to take open neighborhoods U_1 of a and U_2 of b such that $U_1 U_2 \subset U$, so that

$$\mu * \nu(U) \geq \iint 1_{U_1 U_2}(xy)\mu(\mathrm{d}x)\nu(\mathrm{d}y) \geq \mu(U_1)\nu(U_2) > 0, \tag{4.6}$$

which yields $ab \in \mathcal{S}(\mu * \nu)$ and hence $\overline{\mathcal{S}(\mu)\mathcal{S}(\nu)} \subset \mathcal{S}(\mu * \nu)$.

Let $a \in \overline{\mathcal{S}(\mu)\mathcal{S}(\nu)}^c$. Then we can take an open neighborhood U of a such that $U \subset \{\mathcal{S}(\mu)\mathcal{S}(\nu)\}^c$, so that

$$\mu * \nu(U) \leq \iint 1_{\{\mathcal{S}(\mu)\mathcal{S}(\nu)\}^c}(xy)\mu(\mathrm{d}x)\nu(\mathrm{d}y) \leq \mu(\mathcal{S}(\mu)^c) + \nu(\mathcal{S}(\nu)^c) = 0, \tag{4.7}$$

which shows $a \in \mathcal{S}(\mu * \nu)^c$ and hence $\mathcal{S}(\mu * \nu) \subset \overline{\mathcal{S}(\mu)\mathcal{S}(\nu)}$. □

Proposition 4.2. *Let S be a completely simple Polish semigroup. Let $S = LGR$ denote the Rees decomposotion at $e \in E(S)$. For the inverse of the product mapping $\psi : L \times G \times R \to LGR$, we denote*

$$(z^L, z^G, z^R) := \psi^{-1}(z) = (ze(eze)^{-1}, eze, (eze)^{-1}ez) \in L \times G \times R, \quad z \in LGR. \tag{4.8}$$

For $\mu \in \mathcal{P}(S)$, we define

$$\mu^L(B) = \mu(z : z^L \in B), \quad \mu^G(B) = \mu(z : z^G \in B), \quad \mu^R(B) = \mu(z : z^R \in B) \tag{4.9}$$

for $B \in \mathcal{B}(S)$. Then, for $\mu, \nu \in \mathcal{P}(S)$, it holds that

$$(\mu * \nu)^L = \mu^L, \quad (\mu * \nu)^R = \nu^R. \tag{4.10}$$

Proof. This is obvious by noting that $(z_1 z_2)^L = z_1^L$ and $(z_1 z_2)^R = z_2^R$. □

We equip $\mathcal{P}(S)$ with the topology of weak convergence: $\mu_n \to \mu$ if and only if $\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$ for all $f \in C_b(S)$, the class of all bounded continuous functions on S. It is well-known (see, e.g. [35, Theorems 6.2 and 6.5 of Chapter 2]) that $\mathcal{P}(S)$ is a Polish space.

Proposition 4.3. *Let S be a Polish semigroup. Then the convolution mapping $\mathcal{P}(S) \times \mathcal{P}(S) \ni (\mu, \nu) \mapsto \mu * \nu \in \mathcal{P}(S)$ is jointly continuous. Consequently, $\mathcal{P}(S)$ is a Polish semigroup.*

Proof. Note that, if we take independent random variables X and Y taking values in S such that $X \overset{\mathrm{d}}{=} \mu$ and $Y \overset{\mathrm{d}}{=} \nu$, then $\mu * \nu$ coincides with the law of the product XY. The desired result now follows from the Skorokhod coupling thoerem (see, e.g. [18, Theorem 4.30]), which asserts that $\mu_n \to \mu$ implies that we can take random variables $\{X_n\}, X$ taking values in S such that $X_n \overset{\mathrm{d}}{=} \mu_n$, $X \overset{\mathrm{d}}{=} \mu$ and $X_n \to X$ a.s. □

4.2. Translation invariance

Let S be a Polish semigroup. A probability measure $\mu \in \mathcal{P}(S)$ is called *ℓ^*-invariant* [*r^*-invariant*] if $\delta_x * \mu = \mu$ [$\mu * \delta_x = \mu$] for all $x \in S$.

Theorem 4.4. *Let S be a Polish semigroup and let $\mu \in \mathcal{P}(S)$. Suppose that μ is both ℓ^*-invariant and r^*-invariant. Then $\mathcal{S}(\mu)$ is a compact Polish group, and μ coincides with the normalized unimodular Haar measure on $\mathcal{S}(\mu)$ (see e.g. [4, Chapter 9] for the Haar measure).*

Proof. Note that

$$\mathcal{S}(\mu) = \mathcal{S}(\delta_x * \mu) = \overline{x\mathcal{S}(\mu)}, \quad x \in S, \tag{4.11}$$

which implies that $\mathcal{S}(\mu)$ is a left ideal of S. Similarly $\mathcal{S}(\mu)$ is a right ideal of S, and hence $\mathcal{S}(\mu)$ is an ideal of S.

Let us prove that, for any $x \in \mathcal{S}(\mu)$, the subsemigroup xS is left-cancellative. Let $y, a, b \in S$ be such that $(xy)(xa) = (xy)(xb)$. Since $\mathcal{S}(\mu) = \mathcal{S}(\mu * \delta_{xyx}) = \overline{\mathcal{S}(\mu)xyx}$, we can take $\{z_n\} \subset \mathcal{S}(\mu)$ such that $z_n xyx \to x$, and hence

$$xa = \lim z_n xyxa = \lim z_n xyxb = xb, \tag{4.12}$$

which shows that xS is left-cancellative. Similarly Sx is right-cancellative.

Let $a, b \in \mathcal{S}(\mu)$ be fixed. We shall prove that the subsemigroup $D := a\mathcal{S}(\mu)b$ contains an idempotent. Note that

$$\mu(D) = (\delta_a * \mu * \delta_b)(D) = \mu(a^{-1}Db^{-1}) \geq \mu(\mathcal{S}(\mu)) = \mu(S) = 1, \tag{4.13}$$

which shows $\mu(D) = 1$. For $x \in D$, we have

$$\mu(D) \leq \mu(x^{-1}(xD)) = (\mu * \delta_x)(xD) = \mu(xD) \leq \mu(D), \tag{4.14}$$

which shows $\mu(xD) = \mu(D) = 1$. We define two mappings $\theta, \beta : S \times S \to S \times S$ by

$$\theta(x, y) = (x, xy), \quad \beta(x, y) = (y, x). \tag{4.15}$$

Since $(x, y) \in \theta(D \times D)$ if and only if $x \in D$ and $y \in xD$, we have

$$(\mu \otimes \mu)(\beta \circ \theta(D \times D)) = (\mu \otimes \mu)(\theta(D \times D)) = \int_D \mu(xD)\mu(\mathrm{d}x) = \mu(D)^2 = 1. \tag{4.16}$$

This shows that $\beta \circ \theta(D \times D) \cap \theta(D \times D)$ is not empty, so that $(vw, v) = (x, xy)$ for some $v, w, x, y \in D$. We now have $x(yw) = vwyw = x(yw)^2$, which implies $yw = (yw)^2$ by left-cancellativity of D.

Let $e := yw \in E(D) = E(a\mathcal{S}(\mu)b)$. By the left- and right-cancellativity of $a\mathcal{S}(\mu)b$ and by Lemma 2.8, we see that e is identity of $a\mathcal{S}(\mu)b$. By Lemma 3.1, we see that

$$\mathcal{S}(\mu) = \overline{ea\mathcal{S}(\mu)b} = e\,(\overline{a\mathcal{S}(\mu)b}) = e\mathcal{S}(\mu) \subset a\mathcal{S}(\mu)b\mathcal{S}(\mu) \subset a\mathcal{S}(\mu) \subset \mathcal{S}(\mu), \tag{4.17}$$

which shows $a\mathcal{S}(\mu) = \mathcal{S}(\mu)$. Similarly we have $\mathcal{S}(\mu)b = \mathcal{S}(\mu)$. By Lemma 2.4, Proposition 2.6 and Theorem 3.2, we see that $\mathcal{S}(\mu)$ is a Polish group.

By the ℓ^*-invariance, we have $\mu * \mu = \mu$. We now apply [35, Theorem 3.1 of Chapter 3] to obtain the desired result. □

4.3. Convolution invariance

Proposition 4.5 (Mukherjea [27]). *Let S be a Polish semigroup and let $\mu, \nu \in \mathcal{P}(S)$. Suppose*

$$\nu = \mu * \nu = \nu * \mu. \tag{4.18}$$

Then, for any $x \in \mathcal{S}(\mu)$ and any $a \in \mathcal{S}(\nu)$, it holds that

$$\nu * \delta_{xa} = \nu * \delta_a, \quad \delta_{ax} * \nu = \delta_a * \nu. \tag{4.19}$$

Proof. Let $a \in \mathcal{S}(\nu)$, $f \in C_b(S)$ and $\varepsilon > 0$ be fixed for a while, and set

$$g(x) = \max\left\{ \int f \mathrm{d}(\nu * \delta_x) - \int f \mathrm{d}(\nu * \delta_a) - \varepsilon, 0 \right\}, \quad x \in S. \tag{4.20}$$

It is obvious that $g \in C_b(S)$, g is non-negative and $g(a) = 0$. By $\nu = \nu * \mu$, we have

$$\int f \mathrm{d}(\nu * \delta_x) - \int f \mathrm{d}(\nu * \delta_a) - \varepsilon \tag{4.21}$$

$$= \int \left\{ \int f \mathrm{d}(\nu * \delta_{yx}) - \int f \mathrm{d}(\nu * \delta_a) - \varepsilon \right\} \mu(\mathrm{d}y) \leq \int g(yx)\mu(\mathrm{d}y), \tag{4.22}$$

so that we have

$$g(x) \leq \int g(yx)\mu(\mathrm{d}y), \quad x \in S. \tag{4.23}$$

In addition, by $\nu = \mu * \nu$, we have

$$\int \left\{ g(x) - \int g(yx)\mu(\mathrm{d}y) \right\} \nu(\mathrm{d}x) = \int g \mathrm{d}\nu - \int g \mathrm{d}(\mu * \nu) = 0, \tag{4.24}$$

which shows that the equality in (4.23) holds for ν-a.e. $x \in S$. Since g is continuous, we see that the equality in (4.23) holds for all $x \in \mathcal{S}(\nu)$. Since $a \in \mathcal{S}(\nu)$ and $g(a) = 0$, we see, again by continuity of g, that

$$g(ya) = 0, \quad y \in \mathcal{S}(\mu). \tag{4.25}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\int f \mathrm{d}(\nu * \delta_{ya}) \leq \int f \mathrm{d}(\nu * \delta_a), \quad a \in \mathcal{S}(\nu),\ y \in \mathcal{S}(\mu). \tag{4.26}$$

Since $\nu = \nu * \mu$, we have $\int \left\{ \int f \mathrm{d}(\nu * \delta_{ya}) - \int f \mathrm{d}(\nu * \delta_a) \right\} \mu(\mathrm{d}y) = 0$, which implies

$$\int f \mathrm{d}(\nu * \delta_{ya}) = \int f \mathrm{d}(\nu * \delta_a), \quad a \in \mathcal{S}(\nu),\ y \in \mathcal{S}(\mu),\ f \in C_b(S). \tag{4.27}$$

Since $f \in C_b(S)$ is arbitrary, we obtain $\nu * \delta_{ya} = \nu * \delta_a$ for all $a \in \mathcal{S}(\nu)$ and $y \in \mathcal{S}(\mu)$. We obtain $\delta_{ay} * \nu = \delta_a * \nu$ similarly. □

4.4. Convolution idempotent

We denote the n-fold convolution by μ^n, i.e. $\mu^1 = \mu$ and $\mu^n = \mu^{n-1} * \mu$ for $n = 2, 3, \ldots$.

Theorem 4.6 (Mukherjea–Tserpes [32]). *Let S be a Polish semigroup and let $\mu \in \mathcal{P}(S)$. Suppose that $\mu^2 = \mu$. Then $\mathcal{S}(\mu)$ is completely simple and its group factor is compact. Let $\mathcal{S}(\mu) = LGR$ denote the Rees decomposition at $e \in E(\mathcal{S}(\mu))$. Then μ admits the convolution factorization*

$$\mu = \mu^L * \omega_G * \mu^R, \tag{4.28}$$

where μ^L and μ^R have been introduced in (4.9) and ω_G stands for the normalized unimodular Haar measure on the compact Polish group G.

Remark 4.7. The convolution factorization (4.28) is equivalent to the following assertion: If we let Z be a random variable whose law is μ, then

$$Z^L,\ Z^G \text{ and } Z^R \text{ are independent and the law of } Z^G \text{ is } \omega_G. \tag{4.29}$$

Here $(Z^L, Z^G, Z^R) = \psi^{-1}(Z)$ with $\psi : L \times G \times R \to LGR$ denoting the product mapping; see Proposition 4.2.

Proof of Theorem 4.6. Since $\mathcal{S}(\mu) = \overline{\mathcal{S}(\mu)\mathcal{S}(\mu)}$, we see that $\mathcal{S}(\mu)$ is a closed subsemigroup of S. By Proposition 4.5, we see that, for any $a \in \mathcal{S}(\mu)$,

$$\mu * \delta_{xa} = \mu * \delta_a, \quad \delta_{ax} * \mu = \delta_a * \mu, \quad x \in \mathcal{S}(\mu). \tag{4.30}$$

Then, for $a \in \mathcal{S}(\mu)$, we have

$$\mu * \delta_{ay} = \mu * \delta_a \ (y \in \mathcal{S}(\mu * \delta_a)).,\quad \delta_{za} * \mu = \delta_a * \mu \ (z \in \mathcal{S}(\delta_a * \mu)) \tag{4.31}$$

In fact, for $y \in \mathcal{S}(\mu * \delta_a) = \overline{\mathcal{S}(\mu)a}$, we may take $\{x_n\} \subset \mathcal{S}(\mu)$ such that $x_n a \to y$, so that $\mu * \delta_a = \mu * \delta_{ax_na} \to \mu * \delta_{ay}$.

Let $a \in \mathcal{S}(\mu)$ be fixed and set $\nu = \delta_a * \mu * \delta_a$. Then $\mathcal{S}(\nu) = \overline{a\mathcal{S}(\mu)a}$ is a closed subsemigroup of S. For any $y \in \mathcal{S}(\nu) = \overline{a\mathcal{S}(\mu)a}$, we may take $\{x_n\} \subset \mathcal{S}(\mu)$ such that $ax_na \to y$, so that, using (4.30), we have

$$\nu = \delta_a * \mu * \delta_a = \delta_{ax_na^2} * \mu * \delta_a = \delta_{ax_na} * \nu \to \delta_y * \nu, \tag{4.32}$$

which shows that $\nu|_{\mathcal{S}(\nu)}$ is ℓ^*-invariant. We see similarly that $\nu|_{\mathcal{S}(\nu)}$ is r^*-invariant. We may now apply Theorem 4.4 to see that $\mathcal{S}(\nu) = \overline{a\mathcal{S}(\mu)a}$ is a compact Polish group. Its identity is an idempotent of $\mathcal{S}(\mu)$.

Let $e \in E(\mathcal{S}(\mu))$. By the above argument with $a = e$, we see that $G := e\mathcal{S}(\mu)e$ is a compact Polish group (note that $e\mathcal{S}(\mu)e$ is closed by Lemma 3.1). Set $A := \mathcal{S}(\mu)e$. For $y \in A$, using (4.31), we have

$$\overline{Ay} = \overline{\mathcal{S}(\mu)ey} = \mathcal{S}(\mu * \delta_{ey}) = \mathcal{S}(\mu * \delta_e) = \mathcal{S}(\mu)e = A. \tag{4.33}$$

Since $Ay \cap e\mathcal{S}(\mu)e$ is a left ideal of the group $e\mathcal{S}(\mu)e$, we see that $Ay \cap e\mathcal{S}(\mu)e = e\mathcal{S}(\mu)e$, i.e. $e\mathcal{S}(\mu)e \subset Ay$, which shows $e \in Ay$. Hence

$$A = Ae \subset AAy \subset Ay \subset \overline{Ay} = A, \tag{4.34}$$

which yields $Ay = A$ for all $y \in A$. By Lemma 2.4, we see that A is a left group. We see similarly that $B := e\mathcal{S}(\mu)$ is a right group. By Theorem 2.15, we see that $\mathcal{S}(\mu)$ contains a completely simple kernel K, which is closed by Corollary 3.3.

By (4.30), we have

$$\mu * \delta_e * \mu = \int (\mu * \delta_e * \delta_a)\mu(\mathrm{d}a) = \int (\mu * \delta_a)\mu(\mathrm{d}a) = \mu * \mu = \mu. \tag{4.35}$$

By Lemma 2.5, we have $K = \mathcal{S}(\mu)e\mathcal{S}(\mu)$, and hence we obtain

$$K = \overline{K} = \overline{\mathcal{S}(\mu)e\mathcal{S}(\mu)} = \mathcal{S}(\mu * \delta_e * \mu) = \mathcal{S}(\mu), \tag{4.36}$$

which shows that $\mathcal{S}(\mu)$ is completely simple.

By (4.31), we see that $\mu * \delta_e$ is r^*-invariant on $A = \mathcal{S}(\mu)e = LG$, so that $\mu * \delta_e = \mu * \delta_e * \omega_G$. Hence, for any $B \in \mathcal{B}(\mathcal{S}(\mu))$,

$$\mu(B) =(\mu * \delta_e * \mu)(B) = (\mu * \delta_e * \omega_G * \mu)(B) \tag{4.37}$$

$$= \int \mu(\mathrm{d}z_1) \int \mu(\mathrm{d}z_2) \int \omega_G(\mathrm{d}g)1_B(z_1egz_2) \tag{4.38}$$

$$= \int \mu(\mathrm{d}z_1) \int \mu(\mathrm{d}z_2) \int \omega_G(\mathrm{d}g)1_B(z_1^L g z_2^R) = (\mu^L * \omega_G * \mu^R)(B), \tag{4.39}$$

which completes the proof. □

The following proposition is a converse to Theorem 4.6.

Proposition 4.8. *Let S be a Polish semigroup and let $\mu_1, \mu_2 \in \mathcal{P}(S)$. Let G be a compact Polish subgroup of S and suppose that $\mathcal{S}(\mu_2 * \mu_1) \subset G$. Then $\mu := \mu_1 * \omega_G * \mu_2$ satisfies $\mu^2 = \mu$.*

Proof. For any $B \in \mathcal{B}(S)$, we have

$$\mu^2(B) = (\mu_1 * \omega_G * \mu_2 * \mu_1 * \omega_G * \mu_2)(B) \tag{4.40}$$

$$= \int \mu_1(\mathrm{d}z_1) \int \omega_G(\mathrm{d}g_1) \int (\mu_2 * \mu_1)(\mathrm{d}g_2) \int \omega_G(\mathrm{d}g_3) \int \mu_2(\mathrm{d}z_2) 1_B(z_1 g_1 g_2 g_3 z_2) \tag{4.41}$$

$$= \int \mu_1(\mathrm{d}z_1) \int \omega_G(\mathrm{d}g_1) \int \mu_2(\mathrm{d}z_2) 1_B(z_1 g_1 z_2) = (\mu_1 * \omega_G * \mu_2)(B) = \mu(B), \tag{4.42}$$

which completes the proof. □

4.5. Infinite convolutions

Theorem 4.9 (Rosenblatt [40] and Mukherjea [29]). *Let S_0 be a Polish semigroup and let $\mu \in \mathcal{P}(S_0)$. Suppose that the sequence $\{\mu^n\}_{n=1}^{\infty}$ is tight. Let S denote the closure of the semigroup generated by $\mathcal{S}(\mu)$, i.e.*

$$S := \overline{\bigcup_{n=1}^{\infty} \mathcal{S}(\mu)^n}. \tag{4.43}$$

Then the following assertions hold:

(i) There exists $\nu \in \mathcal{P}(S)$ such that $\nu^2 = \nu$, $\mu * \nu = \nu * \mu = \nu$ and

$$\mu_n := \frac{1}{n} \sum_{k=1}^{n} \mu^k \underset{n\to\infty}{\longrightarrow} \nu. \tag{4.44}$$

(ii) The family $\mathcal{K}$ of cluster points of $\{\mu^n : n = 1, 2, \ldots\}$ is a compact abelian group such that

$$\mathcal{S}(\nu) = \overline{\bigcup_{\lambda \in \mathcal{K}} \mathcal{S}(\lambda)}. \tag{4.45}$$

(iii) Let η denote the identity of $\mathcal{K}$. Then $\mathcal{S}(\eta)$ is a completely simple semigroup. Let $\mathcal{S}(\eta) = LHR$ denote the Rees decomposition at $e \in E(\mathcal{S}(\eta))$. Then H is a compact group and η admits the convolution factorization

$$\eta = \eta^L * \omega_H * \eta^R. \tag{4.46}$$

(iv) $\mathcal{S}(\nu)$ is a completely simple kernel of S containing the idempotent e. The Rees decomposition of $\mathcal{S}(\nu)$ at e is of the form $\mathcal{S}(\nu) = LGR$, where G is a compact group containing H, and ν admits the convolution factorization

$$\nu = \eta^L * \omega_G * \eta^R. \tag{4.47}$$

(v) For $g \in G$, we write $\omega_{gH} := \delta_g * \omega_H$. It holds that H is a closed normal subgroup of G and that there exists a Polish group isomorphism $F : \mathcal{K} \to G/H$ such that

$$\lambda = \eta^L * \omega_{F(\lambda)} * \eta^R, \tag{4.48}$$

Consequently, there exists $\gamma \in G$ such that $\mu^k * \eta$ admits the convolution factorization

$$\mu^k * \eta = \eta^L * \omega_{\gamma^k H} * \eta^R, \quad k = 1, 2, \ldots, \tag{4.49}$$

and furthermore, $\mathcal{K}$ and G/H may be represented as

$$\mathcal{K} = \overline{\{\eta, \mu * \eta, \mu^2 * \eta, \ldots\}}, \quad G/H = \overline{\{H, \gamma H, \gamma^2 H, \ldots\}}. \tag{4.50}$$

Remark 4.10. Note that the factors L and R in the Rees decompositions at e of $\mathcal{S}(\eta)$ and $\mathcal{S}(\nu)$ are common. By (i) of Corollary 2.12, we see that

$$E(\mathcal{S}(\nu)) = \{x(yx)^{-1}y : x \in L, \ y \in R\} = E(\mathcal{S}(\eta)). \tag{4.51}$$

Remark 4.11. If the order of the group $\mathcal{K}$ or G/H is finite, say p, then

$$\mathcal{K} = \{\eta, \mu * \eta, \ldots, \mu^{p-1} * \eta\}, \quad G/H = \{H, \gamma H, \ldots, \gamma^{p-1} H\} \tag{4.52}$$

with $\gamma^p \in H$. It is now obvious that $\lim_{n\to\infty} \mu^n$ converges if and only if $p = 1$.

Proof of Theorem 4.9. (i) Let $\|\cdot\|$ denote the total variation norm. For $j = 1, 2, \ldots$, we have

$$\left\|\mu_n - \mu^j * \mu_n\right\| \le \frac{1}{n}\left\|\sum_{k=1}^{n} \mu^k - \sum_{k=1}^{n} \mu^{k+j}\right\| = \frac{1}{n}\left\|\sum_{k=1}^{j} \mu^k - \sum_{k=n+1}^{n+j} \mu^k\right\| \le \frac{2j}{n} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.53}$$

Since $\{\mu^n\}$ is tight, we see that $\{\mu_n\}$ is also tight. Let ν_1, ν_2 be cluster points of $\{\mu_n\}$. For $i = 1, 2$, we see by (4.53) that $\mu^j * \nu_i = \nu_i * \mu^j = \nu_i$ for $j = 1, 2, \ldots$, so that $\mu_n * \nu_i = \nu_i * \mu_n = \nu_i$ for $n = 1, 2, \ldots$, which implies $\nu_1 = \nu_1 * \nu_2 = \nu_2 * \nu_1 = \nu_2$. Hence we see that $\{\mu_n\}$ converges to some $\nu \in \mathcal{P}(S_0)$ and we have $\nu^2 = \nu$ and $\mu * \nu = \nu * \mu = \nu$. We may apply Theorem 4.6 to see that $\mathcal{S}(\nu)$ is a completely simple semigroup and its group factor is compact.

(ii) Let us prove that $\mathcal{S}(\nu)$ and $\mathcal{S}(\mathcal{K}) := \overline{\bigcup_{\lambda \in \mathcal{K}} \mathcal{S}(\lambda)}$ are ideals of S. Let $a \in S$, $x \in \mathcal{S}(\nu)$ and $y \in \mathcal{S}(\mathcal{K})$. Then we may take $\{a_n\} \subset \mathcal{S}(\mu)^{m(n)} \subset \mathcal{S}(\mu^{m(n)})$ and $\{y_n\} \subset \mathcal{S}(\lambda_n)$ such that $a_n \to a$ and $y_n \to y$. Since

$$a_n x \in \mathcal{S}(\mu^{m(n)})\mathcal{S}(\nu) \subset \mathcal{S}(\mu^{m(n)} * \nu) = \mathcal{S}(\nu), \tag{4.54}$$

$$a_n y_n \in \mathcal{S}(\mu^{m(n)})\mathcal{S}(\lambda_n) \subset \mathcal{S}(\mu^{m(n)} * \lambda_n) \subset \mathcal{S}(\mathcal{K}), \tag{4.55}$$

we obtain $ax = \lim a_n x \in \mathcal{S}(\nu)$ and $ay = \lim a_n y_n \in \mathcal{S}(\mathcal{K})$, which shows that $\mathcal{S}(\nu)$ and $\mathcal{S}(\mathcal{K})$ are both left ideals of S. Similarly we see that they are also right ideals of S.

Let U be an open subset containing $\mathcal{S}(\nu)$. We shall prove that $\mu^n(U) \to 1$. Let $\varepsilon > 0$. By tightness, we may take a compact subset K_1 such that $\inf_n \mu^n(K_1) > 1 - \varepsilon$. We may take a compact subset $K_2 \subset \mathcal{S}(\nu)$ such that $\nu(K_2) > 1 - \varepsilon$. Since $K_1 K_2 \subset S\mathcal{S}(\nu) \subset \mathcal{S}(\nu) \subset U$, we have $K_1 \times K_2 \subset \widetilde{U} := \{(x, y) \in S_0 \times S_0 : xy \in$

$U\}$. By the Wallace theorem (see, e.g., [21, Theorem 12 of Chapter 5]), we may take open subsets V_1 and V_2 such that $K_1 \subset V_1$, $K_2 \subset V_2$ and $V_1 \times V_2 \subset \widetilde{U}$, which implies $V_1 V_2 \subset U$. Since $\mu_n \to \nu$, we have $\liminf_n \mu_n(V_2) \geq \nu(V_2) \geq \nu(K_2) > 1-\varepsilon$. We may then take some n_0 such that $\mu^{n_0}(V) > 1-\varepsilon$. We now have

$$\mu^{n+n_0}(U) = \iint 1_U(xy)\mu^n(\mathrm{d}x)\mu^{n_0}(\mathrm{d}y) \geq \mu^n(V_1)\mu^{n_0}(V_2) > (1-\varepsilon)^2, \quad (4.56)$$

which leads to $\mu^n(U) \to 1$.

By the tightness assumption, we may apply Proposition 3.5 to see that $\mathcal{K}$ is a compact abelian group. Let $\lambda \in \mathcal{K}$ and let $x \in S(\lambda)$. Suppose that $x \notin \mathcal{S}(\nu)$. We could then take disjoint open sets U and V such that $\mathcal{S}(\nu) \subset U$ and $x \in V$. If we let $\delta := \lambda(V)/2 > 0$, then $\mu^n(V) > \delta$ for infinitely many n, and then $\liminf_n \mu^n(U) \leq \liminf_n \mu^n(V^c) \leq 1-\delta$, which would contradict $\mu^n(U) \to 1$. Hence we obtain $\mathcal{S}(\mathcal{K}) \subset \mathcal{S}(\nu)$. Since $\mathcal{S}(\nu)$ is a minimal ideal of S by Lemma 2.5 and since $\mathcal{S}(\mathcal{K})$ is an ideal of S, we see that $\mathcal{S}(\mathcal{K}) = \mathcal{S}(\nu)$.

(iii) By Theorem 4.6, we see that $\mathcal{S}(\eta)$ is a completely simple semigroup. Let $\mathcal{S}(\eta) = LHR$ denote the Rees decomposition at $e \in E(\mathcal{S}(\eta))$ (hence $RL \subset H$). Then the group factor H is compact and η admits the convolution factorization (4.46).

(iv) We have already seen in (i) that $\mathcal{S}(\nu)$ is a completely simple kernel of S. Since $\mathcal{S}(\eta) \subset \mathcal{S}(\mathcal{K}) = \mathcal{S}(\nu)$, we have $e \in E(\mathcal{S}(\nu))$. Let $\mathcal{S}(\nu) = L'GR'$ denote the Rees decomposition at e. As a consequence of Theorem 4.6, we see that ν admits the convolution factorization $\nu = \eta^{L'} * \omega_G * \eta^{R'}$. Since $\mathcal{S}(\eta) \subset \mathcal{S}(\nu)$ and $L = E(\mathcal{S}(\eta)e))$ etc., we see that $L \subset L'$, $H \subset G$ and $R \subset R'$.

Let us prove that $L' = L$ and $R' = R$. Let $z = xgy \in L'GR'$. Since $\mathcal{S}(\nu) = \mathcal{S}(\mathcal{K})$, we may take $z_n \in \mathcal{S}(\lambda_n)$ such that $z_n \to z$. Since $\mathcal{K}$ is abelian, we have $\lambda_n * \lambda_n^{-1} = \lambda_n^{-1} * \lambda_n = \eta$, and by Proposition 4.2 we have $\lambda_n^{L'} = \eta^{L'} = \eta^L$ and $\lambda_n^{R'} = \eta^{R'} = \eta^R$. Hence we obtain $x_n := z_n^{L'} \in \mathcal{S}(\lambda_n^{L'}) = \mathcal{S}(\eta^L) = L$ and $y_n := z_n^{R'} \in \mathcal{S}(\lambda_n^{R'}) = \mathcal{S}(\eta^R) = R$, and thus $x = \lim x_n \in L$ and $y = \lim y_n \in R$, which shows $L' = L$ and $R' = R$.

(v) Let $\lambda \in \mathcal{K}$. For $z = xgy \in \mathcal{S}(\lambda) \subset \mathcal{S}(\nu) = LGR$, since $RL \subset H$, we have

$$xgy \in LgHR \subset LHRxgyLHR \subset \mathcal{S}(\eta)\mathcal{S}(\lambda)\mathcal{S}(\eta) \subset \mathcal{S}(\eta * \lambda * \eta) = \mathcal{S}(\lambda). \quad (4.57)$$

Hence we have $\mathcal{S}(\lambda) = LG_\lambda R$ for $G_\lambda := \bigcup\{gH : z = xgy \in \mathcal{S}(\lambda)\} \subset G$, and we also have $G_\lambda = \bigcup\{Hg : z = xgy \in \mathcal{S}(\lambda)\}$ similarly. Note that $G_\lambda H = HG_\lambda = G_\lambda$. Take $g_\lambda \in G$ such that $Hg_\lambda^{-1} \subset G_{\lambda^{-1}}$. Then we obtain

$$LHg_\lambda^{-1}G_\lambda R \subset LG_{\lambda^{-1}}RLG_\lambda R \subset \mathcal{S}(\lambda^{-1})\mathcal{S}(\lambda) \subset \mathcal{S}(\lambda^{-1} * \lambda) \subset \mathcal{S}(\eta) = LHR, \quad (4.58)$$

which yields that $Hg_\lambda^{-1}G_\lambda \subset H$ and hence $G_\lambda = g_\lambda H$. Similarly, we obtain $G_\lambda = Hg_\lambda$.

For any $h \in H$ and $g \in G \subset \mathcal{S}(\nu) = \mathcal{S}(\mathcal{K})$, we may take $z_n = x_n g_n y_n \in \mathcal{S}(\lambda_n)$ such that $z_n \to g$ and consequently $g_n \to g$. In a similar way to (4.58), we have

$$g_n h g_n^{-1} \in (g_n H)(Hg_n^{-1}) = G_{\lambda_n} G_{\lambda_n^{-1}} \subset \mathcal{S}(\eta) = LHR, \quad (4.59)$$

which shows $g_n h g_n^{-1} \in eLHRe = H$. Letting $n \to \infty$, we obtain $ghg^{-1} \in H$, which shows that H is a normal subgroup of G. Since G and H are both compact, we see by [15, Theorem 5.22] that the quotient group $G/H = \{gH : g \in G\}$ is also compact. Let $\pi : G \to G/H$ denote the natural projection.

Since

$$\mathcal{S}(\eta^R * \lambda * \eta^L) = \overline{\mathcal{S}(\eta^R)\mathcal{S}(\lambda)\mathcal{S}(\eta^L)} = \overline{RLG_\lambda RL} \subset \overline{Hg_\lambda HH} = g_\lambda H, \tag{4.60}$$

we obtain the convolution factorization

$$\lambda = \eta\lambda\eta = \eta^L * \omega_H * (\eta^R * \lambda * \eta^L) * \omega_H * \eta^R = \eta^L * \omega_{g_\lambda H} * \eta^R. \tag{4.61}$$

We now define the mapping $F : \mathcal{K} \to G/H$ by $F(\lambda) := g_\lambda H$. For $\lambda_1, \lambda_2 \in \mathcal{K}$, then

$$\lambda_1 * \lambda_2 = \eta^L * \omega_{g_{\lambda_1} H} * (\eta^R * \eta^L) * \omega_{g_{\lambda_2} H} * \eta^R = \eta^L * \omega_{(g_{\lambda_1} g_{\lambda_2} H)} * \eta^R, \tag{4.62}$$

since $RL \subset H$, which shows that F is a group homomorphism. Injectivity of F is obvious by (4.61). Let $g \in G$. As we have seen it above, we may take $z_n = x_n g_n y_n \in \mathcal{S}(\lambda_n)$ such that $g_n \to g$ and $g_n H = g_{\lambda_n} H$. Then, by (4.61), we have

$$\lambda_n = \eta^L * \omega_{g_{\lambda_n} H} * \eta^R \to \eta^L * \omega_{gH} * \eta^R =: \lambda. \tag{4.63}$$

This shows that $\lambda \in \mathcal{K}$ and $F(\lambda) = gH$, which yields surjectivity of F. Suppose $\mathcal{K} \ni \lambda_n \to \lambda \in \mathcal{K}$. By (4.61), we have

$$\omega_{F(\lambda_n)} = \delta_e * \lambda_n * \delta_e \to \delta_e * \lambda * \delta_e = \omega_{F(\lambda)} \quad \text{in } \mathcal{P}(G), \tag{4.64}$$

which shows by the continuity of the natural projection π that

$$\delta_{F(\lambda_n)} = \omega_{F(\lambda_n)} \circ \pi^{-1} \to \omega_{F(\lambda)} \circ \pi^{-1} = \delta_{F(\lambda)} \quad \text{in } \mathcal{P}(G/H), \tag{4.65}$$

which implies $F(\lambda_n) \to F(\lambda)$ and we have seen continuity of F. Since $\mathcal{K}$ is compact and G/H is Hausdorff, we see by [21, Theorem 9 of Chapter 5] that F is a homeomorphism. Since $F(\mu * \eta) \in G/H$, we may take $\gamma \in G$ such that $F(\mu * \eta) = \gamma H$, and then we obtain (4.49) since $(\mu * \eta)^k = \mu^k = \mu^k * \eta$ and F is a group homomorphism.

Finally, let us prove the representations (4.50). Since any $\lambda \in \mathcal{K}$ can be represented as $\lambda = \lambda * \eta = \lim \mu^{n(k)} * \eta$, we see that $\mathcal{K} = \overline{\{\eta, \mu * \eta, \mu^2 * \eta, \dots\}}$. Since for any $g \in G$ we have $F(\lambda) = gH$ for some $\lambda = \lim \mu^{n(k)} * \eta \in \mathcal{K}$, we obtain $gH = F(\lambda) = \lim F(\mu^{n(k)} * \eta) = \lim \gamma^{n(k)} H$ in G/H, which yields $G/H = \overline{\{H, \gamma H, \gamma^2 H, \dots\}}$. □

5. Two examples

5.1. First example

Let $V = \{1, 2\}$ and $B = \{-1, 0, 1\}$. Let S_0 denote the composition semigroup of mappings from $V \times B$ into itself. We define $e, f, g, h \in S_0$ as

$$e((v,b)) = \begin{cases} (1,1) & (b = 0, 1), \\ (1,-1) & (b = -1), \end{cases} \qquad f((v,b)) = \begin{cases} (2,1) & (b = 0, 1), \\ (2,-1) & (b = -1), \end{cases} \tag{5.1}$$

$$g((v,b)) = \begin{cases} (1,-1) & (b=0,1), \\ (1,1) & (b=-1), \end{cases} \qquad h((v,b)) = \begin{cases} (1,1) & (b=1), \\ (1,-1) & (b=0,-1). \end{cases} \tag{5.2}$$

Let $p,q,r \in (0,1)$ such that $p+q+r<1$. We define $\mu \in \mathcal{P}(S_0)$ as

$$\mu = p\delta_e + q\delta_f + r\delta_g + (1-p-q-r)\delta_h. \tag{5.3}$$

Since S_0 is a finite semigroup, we see that $\mathcal{P}(S_0)$ is compact, so that $\{\mu^n\}_{n=1}^\infty$ is tight. We may now apply Theorem 4.9 to investigate the cluster points of $\{\mu^n : n=1,2,\dots\}$.

Proposition 5.1. *The following assertions hold:*

(i) The Rees decomposition at e of $\mathcal{S}(\nu)$ is given as

$$L = \{e,f\}, \quad G = \{e,g\}, \quad R = \{e,h\}. \tag{5.4}$$

(ii) $\eta^L = (1-q)\delta_e + q\delta_f$.
(iii) $\eta^R = (1-r)\delta_e + r\delta_h$.
(iv) $H = G$.

Proof. (i) Note that $\mathcal{S}(\mu) = \{e,f,g,h\}$. We set

$$\widetilde{L} = \{e,f\}, \quad \widetilde{G} = \{e,g\}, \quad \widetilde{R} = \{e,h\} \tag{5.5}$$

and we shall prove that $e \in E(\mathcal{S}(\eta))$ and $L = \widetilde{L}$, $G = \widetilde{G}$ and $R = \widetilde{R}$. We have the following multiplication table (the table of ab for a and b varying over $\{e,f,g,h\}$):

$$\begin{array}{c|cccc} a\backslash b & e & f & g & h \\ \hline e & e & e & g & h \\ f & f & f & fg & fh \\ g & g & g & e & gh \\ h & e & e & g & h \end{array} \tag{5.6}$$

It follows from this table that $S := \overline{\bigcup_n \mathcal{S}(\mu)^n} = \{e,f,g,h,fg,fh,gh,fgh\}$, and that $SeS = \widetilde{L}\widetilde{G}\widetilde{R} = S$. Since we have

$$\begin{cases} ef = e \\ fe = f \end{cases} \qquad \begin{cases} g^2 = e \\ eg = ge = g \end{cases} \qquad \begin{cases} he = e \\ eh = h \end{cases} \tag{5.7}$$

we have $SaS = SeS = S$ for all $a \in S$. By Lemma 2.3, we see that S is a kernel of S itself. By Theorem 2.15, the kernel of S is unique, so that we obtain $\mathcal{S}(\nu) = S$. Note that $e \in E(\mathcal{S}(\nu)) = E(\mathcal{S}(\eta)) = E(S)$ by Remark 4.10. We thus obtain

$$L = E(\mathcal{S}(\nu)e) = E(Se) = E(\widetilde{L}\widetilde{G}) = \widetilde{L}, \tag{5.8}$$

$$G = e\mathcal{S}(\nu)e = eSe = \widetilde{G}, \tag{5.9}$$

$$R = E(e\mathcal{S}(\nu)) = E(eS) = E(\widetilde{G}\widetilde{R}) = \widetilde{R}. \tag{5.10}$$

(ii) By $L = \{e, f\}$ and by the multiplication table, we have

$$\mu * \eta^L = (1-q-r)\delta_e + q\delta_f + r\delta_g. \tag{5.11}$$

Since $G = \{e, g\}$, we have

$$\mu * \eta^L * \omega_G = ((1-q)\delta_e + q\delta_f) * \omega_G. \tag{5.12}$$

Since $\nu = \eta^L * \omega_G * \eta^R$, we have

$$\eta^L * \omega_G * \eta^R = \nu = \mu * \nu = \mu * \eta^L * \omega_G * \eta^R = ((1-q)\delta_e + q\delta_f) * \omega_G * \eta^R. \tag{5.13}$$

By the bijectivity of the product mapping, we obtain $\eta^L = (1-q)\delta_e + q\delta_f$.

(iii) The proof is similar to (ii), and so we omit it.

(iv) Since H is a subgroup of $G = \{e, g\}$, we have either $H = \{e\}$ or $H = G$. Suppose $H = \{e\}$. Then $\gamma = g$. By (5.11), we have

$$\eta^L * \omega_{\gamma H} * \eta^R = \mu * \eta^L * \omega_H * \eta^R = ((1-q-r)\delta_e + q\delta_f + r\delta_g) * \omega_H * \eta^R. \tag{5.14}$$

By (ii) and by the bijectivity of the product mapping, we have

$$((1-q)\delta_e + q\delta_f) * \omega_{\gamma H} = ((1-q-r)\delta_e + q\delta_f + r\delta_g) * \omega_H. \tag{5.15}$$

Since $\omega_H = \delta_e$ and $\omega_{\gamma H} = \delta_g$, we have

$$(1-q)\delta_g + q\delta_{fg} = (1-q-r)\delta_e + q\delta_f + r\delta_g, \tag{5.16}$$

which leads to a contradiction. Therefore we obtain $H = G$. $\square$

5.2. Second example

Let $V = \{1, 2, 3\}$ and consider the set of sequences of V:

$$V^\infty = \{v = (v(1), v(2), \dots) : v(i) \in V,\ i = 1, 2, \dots\}. \tag{5.17}$$

For $a = 1, 2, 3$, we define $\phi_a : V^\infty \to V^\infty$ as

$$\phi_a((v(1), v(2), \dots)) = (a, v(1), v(2), \dots). \tag{5.18}$$

Note that the set V^∞ realizes the *Sierpiński gasket* so that $\{\phi_a : a = 1, 2, 3\}$ can be regarded as the generating system of contraction mappings; see, e.g., [22, Section 1.2]. Let $\sigma : V^\infty \to V^\infty$ denote the shift mapping:

$$\sigma((v(1), v(2), \dots)) = (v(2), v(3), \dots). \tag{5.19}$$

Let $B = \{-1, 0, 1\}$ and $C = \{e^{i\theta} : \theta \in \mathbb{R}\}$ and set

$$W = V^{\infty} \times B \times C. \tag{5.20}$$

We define $\chi_{\pm 1} : B \to B$ as

$$\chi_{+1}(b) = \begin{cases} 1 & (b = 0, 1), \\ -1 & (b = -1), \end{cases} \quad \chi_{-1}(b) = \begin{cases} 1 & (b = 1), \\ -1 & (b = -1, 0). \end{cases} \tag{5.21}$$

For $a = 1, 2, 3$ and $\rho \in C$, we define $\phi_a^{\rho} : W \to W$ and $\sigma^{\rho} : W \to W$ as

$$\phi_a^{\rho}((v, b, c)) = (\phi_a(v), \chi_{+1}(b), \rho c), \quad \sigma^{\rho}((v, b, c)) = (\sigma(v), \chi_{+1}(b), \rho c) \tag{5.22}$$

and define $\tau^{\rho} : W \to W$ as

$$\tau^{\rho}((v, b, c)) = (v, -\chi_{-1}(b), \rho c). \tag{5.23}$$

Note that V^{∞} is a compact Polish space with respect to the product topology of the discrete space V, and hence W is also a compact Polish space. Let S_0 denote the composition semigroup of mappings from W into itself. Then S_0 is a Polish semigroup with respect to the topology of uniform convergence (see, e.g., [20, Theorem 4.19]).

Let $p, q \in (0, 1)$ be such that $p + q < 1$. Let $\rho_0 \in C$ be a fixed element such that $\rho_0 = e^{2\pi i t_0}$ for some irrational real t_0. We define $\mu \in \mathcal{P}(S_0)$ as

$$\mu = \frac{p}{3} \sum_{a=1,2,3} \delta_{\phi_a^{\rho_0}} + q\delta_{\sigma^{\rho_0}} + (1 - p - q)\delta_{\tau^{\rho_0}}. \tag{5.24}$$

We want to investigate the cluster points of $\{\mu^n : n = 1, 2, \ldots\}$.

Proposition 5.2. *Suppose $p > q$. Then the sequence $\{\mu^n\}_{n=1}^{\infty}$ is tight.*

Proof. Let $\widetilde{W} := V^{\infty} \times B$ and let $\widetilde{S}_0$ denote the composition semigroup of mappings from $\widetilde{W}$ into itself. We define $\widetilde{\phi}_a, \widetilde{\sigma}, \widetilde{\tau} : \widetilde{W} \to \widetilde{W}$ as

$$\widetilde{\phi}_a((v, b)) = (\phi_a(v), \chi_{+1}(b)), \quad \widetilde{\sigma}((v, b)) = (\sigma(v), \chi_{+1}(b)) \tag{5.25}$$

and

$$\widetilde{\tau}((v, b)) = (v, -\chi_{-1}(b)). \tag{5.26}$$

We define $\widetilde{\mu} \in \mathcal{P}(\widetilde{S}_0)$ as

$$\widetilde{\mu} = \frac{p}{3} \sum_{a=1,2,3} \delta_{\widetilde{\phi}_a} + q\delta_{\widetilde{\sigma}} + (1 - p - q)\delta_{\widetilde{\tau}}. \tag{5.27}$$

We notice that, if Z is a random variable whose law is $\widetilde{\mu}^n$, then the law of the random map $(v, b, c) \mapsto (Z(v, b), \rho_0^n c)$ is μ^n. Since C is compact, the sequence $\{\rho_0^n\}$ is trivially relatively compact. Consequently, for tightness of the sequence $\{\mu^n\}_{n=1}^{\infty}$, it suffices to prove tightness of the sequence $\{\widetilde{\mu}^n\}_{n=1}^{\infty}$.

Let $0 < \kappa < 1$, whose value will be specified later, and set

$$d((v,b),(v',b')) = \kappa^{1+\sup\{i\geq 1:\ v(i)=v'(i)\}} + 1_{\{b\neq b'\}} \tag{5.28}$$

for $(v,b),(v',b') \in \widetilde{W}$, where we understand that $\sup\emptyset = 0$. It is easy to see that the metric d is compatible with the topology of $\widetilde{W}$. We write

$$\Delta(f) = \sup\left\{ \frac{d(f(w),f(w'))}{d(w,w')} : w,w' \in \widetilde{W},\ 0 < d(w,w') < 1 \right\} \quad (f \in \widetilde{S}_0). \tag{5.29}$$

Note that

$$d((v,b),(v',b')) < 1 \text{ if and only if } b = b'. \tag{5.30}$$

By this fact, we easily see that

$$\Delta(\widetilde{\phi}_a) = \kappa, \quad \Delta(\widetilde{\sigma}) = \frac{1}{\kappa}, \quad \Delta(\widetilde{\tau}) = 1. \tag{5.31}$$

For $0 < \varepsilon < 1$ and $f \in \widetilde{S}_0$, we set

$$o_f(\varepsilon) = \sup\{d(f(w),f(w')) : w,w' \in \widetilde{W},\ d(w,w') \leq \varepsilon\}. \tag{5.32}$$

Note that, if $d(w,w') < 1$, then

$$d(f(w),f(w')) < 1 \text{ and } d(f(w),f(w')) \leq \Delta(f)\, d(w,w') \text{ for all } f \in \widetilde{S}_0. \tag{5.33}$$

This yields

$$\int o_f(\varepsilon)\, \widetilde{\mu}^n(\mathrm{d}f) = \int \cdots \int o_{f_1\cdots f_n}(\varepsilon)\, \widetilde{\mu}(\mathrm{d}f_1)\cdots\widetilde{\mu}(\mathrm{d}f_n) \tag{5.34}$$

$$\leq \int \cdots \int \Delta(f_1)\cdots\Delta(f_n)\, \varepsilon\, \widetilde{\mu}(\mathrm{d}f_1)\cdots\widetilde{\mu}(\mathrm{d}f_n) \tag{5.35}$$

$$= \varepsilon \left(\int \Delta(f)\, \widetilde{\mu}(\mathrm{d}f) \right)^n. \tag{5.36}$$

By (5.31), we have

$$\int \Delta(f)\, \widetilde{\mu}(\mathrm{d}f) = p\kappa + \frac{q}{\kappa} + 1 - p - q. \tag{5.37}$$

Since $p > q$, we may and do choose $0 < \kappa < 1$ so that (5.37) is less than 1, say $\kappa = (p+q)/(2p)$. Hence we obtain, for any $\delta > 0$,

$$\widetilde{\mu}^n(f \in S : o_f(\varepsilon) > \delta) \leq \frac{1}{\delta} \int o_f(\varepsilon)\, \widetilde{\mu}^n(\mathrm{d}f) \leq \frac{\varepsilon}{\delta}, \tag{5.38}$$

which implies

$$\lim_{\varepsilon\downarrow 0} \sup_n \widetilde{\mu}^n(f \in S : o_f(\varepsilon) > \delta) = 0. \tag{5.39}$$

With a slight modification thanks to compactness of $\widetilde{W}$, we can apply Theorem VII.2.2 of [35] and obtain the tightness of $\{\widetilde{\mu}^n\}_{n=1}^{\infty}$. □

We may now apply Theorem 4.9. For $v_0 \in V^\infty$, we define $\iota_{v_0} : W \to W$ as

$$\iota_{v_0}((v,b,c)) = (v_0, \chi_{+1}(b), c). \tag{5.40}$$

We write $\mathbf{1} = (1,1,\dots) \in V^\infty$ and define $h, r : W \to W$ as

$$h(v,b,c) = (\mathbf{1}, -\chi_{+1}(b), c), \quad r(v,b,c) = (\mathbf{1}, \chi_{-1}(b), c). \tag{5.41}$$

For $\rho \in C$, we define $k^\rho : W \to W$ as

$$k^\rho(v,b,c) = (\mathbf{1}, \chi_{+1}(b), \rho c). \tag{5.42}$$

Proposition 5.3. *Suppose $p > q$. Then the following assertions hold:*

(i) We may take $e = \iota_{\mathbf{1}}$. The Rees decomposition at e of $\mathcal{S}(\nu)$ is given as

$$L = \{\iota_{v_0} : v_0 \in V^\infty\}, \quad G = \{k^\rho, k^\rho h : \rho \in C\}, \quad R = \{e, r\}. \tag{5.43}$$

(ii) $\eta^R = (p+q)\delta_e + (1-p-q)\delta_r$.
(iii) η^L is the law of $\iota_{(U_1,U_2,\dots)}$, where $\{U_1, U_2, \dots\}$ is a sequence of independent random variables which are uniformly distributed on $V = \{1,2,3\}$.
(iv) $H = \{e, h\}$ and we may take $\gamma = k^{\rho_0}$.

Proof. (i) Note that $\mathcal{S}(\mu) = \{\phi_1^{\rho_0}, \phi_2^{\rho_0}, \phi_3^{\rho_0}, \sigma^{\rho_0}, \tau^{\rho_0}\}$. Let us prove that

$$\iota_{v_0}, h, r, k^\rho \in S := \overline{\textstyle\bigcup_n \mathcal{S}(\mu)^n}. \tag{5.44}$$

Note that

$$\begin{aligned} &(\tau^{\rho_0})^{2n}((v,b,c)) = (v, \chi_{-1}(b), \rho_0^{2n} c), \\ &(\tau^{\rho_0})^{2n-2} \sigma^{\rho_0} \phi_1^{\rho_0}((v,b,c)) = (v, \chi_{+1}(b), \rho_0^{2n} c). \end{aligned} \tag{5.45}$$

Since ρ_0 is an irrational rotation of the circle C, we see that for any $\rho \in C$ we can find a subsequence $\{n(k)\}$ of positive integers such that $\rho_0^{2n(k)} \to \rho$. This shows that the mappings $\chi_{\pm 1}^\rho : W \to W$ defined as

$$\chi_{\pm 1}^\rho((v,b,c)) = (v, \chi_{\pm 1}(b), \rho c) \tag{5.46}$$

both belong to S. We now obtain

$$\iota_{v_0} = \lim_{n\to\infty} \phi_{v_0(1)}^{\rho_0} \phi_{v_0(2)}^{\rho_0} \cdots \phi_{v_0(n)}^{\rho_0} \chi_{+1}^{\rho_0^{-n}} \in S \tag{5.47}$$

and

$$h = \iota_{\mathbf{1}} \tau^{\rho_0} \chi_{+1}^{\rho_0^{-1}} \in S, \quad r = \iota_{\mathbf{1}} \tau^{\rho_0} \chi_{-1}^{\rho_0^{-1}} \in S, \quad k^\rho = \iota_{\mathbf{1}} \chi_{+1}^\rho \in S. \tag{5.48}$$

We set

$$\widetilde{L} = \{\iota_{v_0} : v_0 \in V^\infty\}, \quad \widetilde{G} = \{k^\rho, k^\rho h : \rho \in C\}, \quad \widetilde{R} = \{\iota_{\mathbf{1}}, r\}. \tag{5.49}$$

We then have $S\iota_{\mathbf{1}}S = K := \widetilde{L}\widetilde{G}\widetilde{R}$; In fact, we have $S\iota_{\mathbf{1}}S \subset K$ by checking $\mathcal{S}(\mu)\iota_{\mathbf{1}}\mathcal{S}(\mu) \subset K$, and we have $K \subset S\iota_{\mathbf{1}}S$ by $k^\rho = \iota_{\mathbf{1}}\chi^\rho_{+1}$. Since we have

$$\begin{cases} \iota_{\mathbf{1}}\iota_{v_0} = \iota_{\mathbf{1}} \\ \iota_{v_0}\iota_{\mathbf{1}} = \iota_{v_0} \end{cases} \qquad \begin{cases} k^\rho k^{\rho^{-1}} = k^{\rho^{-1}}k^\rho = \iota_{\mathbf{1}} \\ k^\rho\iota_{\mathbf{1}} = \iota_{\mathbf{1}}k^\rho = k^\rho \end{cases} \qquad \begin{cases} h^2 = \iota_{\mathbf{1}} \\ h\iota_{\mathbf{1}} = \iota_{\mathbf{1}}h = h \end{cases} \qquad \begin{cases} r\iota_{\mathbf{1}} = \iota_{\mathbf{1}} \\ \iota_{\mathbf{1}}r = r \end{cases} \tag{5.50}$$

we have $SfS = S\iota_{\mathbf{1}}S = K$ for all $f \in K$. By Lemma 2.3, we see that K is a kernel of S. By Theorem 2.15, the kernel of S is unique, so that we obtain $K = \mathcal{S}(\nu)$.

We now take $e = \iota_{\mathbf{1}} \in E(K) = E(\mathcal{S}(\nu)) = E(\mathcal{S}(\eta))$ by Remark 4.10. We thus obtain

$$L = E(\mathcal{S}(\nu)e) = E(K\iota_{\mathbf{1}}) = \widetilde{L}, \tag{5.51}$$

$$G = e\mathcal{S}(\nu)e = \iota_{\mathbf{1}}K\iota_{\mathbf{1}} = \widetilde{G}, \tag{5.52}$$

$$R = E(e\mathcal{S}(\nu)) = E(\iota_{\mathbf{1}}K) = \widetilde{R}. \tag{5.53}$$

(ii) Since $e\phi^{\rho_0}_a = r\phi^{\rho_0}_a = e\sigma^{\rho_0} = r\sigma^{\rho_0} = k^{\rho_0}$ and $e\tau^{\rho_0} = r\tau^{\rho_0} = k^{\rho_0}hr$, we have

$$\eta^R * \mu = (p+q)\delta_{k^{\rho_0}} + (1-p-q)\delta_{k^{\rho_0}hr}. \tag{5.54}$$

Since $G = \{k^\rho, k^\rho h : \rho \in C\}$, we have

$$\omega_G * \eta^R * \mu = \omega_G * ((p+q)\delta_e + (1-p-q)\delta_r). \tag{5.55}$$

Since $\nu = \eta^L * \omega_G * \eta^R$, we have

$$\eta^L * \omega_G * \eta^R = \nu = \nu * \mu = \eta^L * \omega_G * \eta^R * \mu \tag{5.56}$$

$$= \eta^L * \omega_G * ((p+q)\delta_e + (1-p-q)\delta_r). \tag{5.57}$$

By the bijectivity of the product mapping, we obtain $\eta^R = (p+q)\delta_e + (1-p-q)\delta_r$.

(iii) Note that

$$\xi := \frac{1}{3}\sum_{a=1,2,3}\delta_{\phi^1_a} = E\delta_{\phi^1_U} \tag{5.58}$$

for an $\widetilde{S}$-valued random variable U which is uniformly distributed on V. For a sequence $\{U_1, U_2, \dots\}$ of independent random variables which are uniformly distributed on $V = \{1,2,3\}$, we have $\phi^1_{U_1}\cdots\phi^1_{U_n} \to \iota_{(U_1,U_2,\dots)}$ a.s. and $\delta_{\phi^1_{U_1}\cdots\phi^1_{U_n}} \to \delta_{\iota_{(U_1,U_2,\dots)}}$ a.s., which shows

$$\xi^n = E\left[\delta_{\phi^1_{U_1}\cdots\phi^1_{U_n}}\right] \underset{n\to\infty}{\longrightarrow} E[\delta_{\iota_{(U_1,U_2,\dots)}}]. \tag{5.59}$$

Note that $\phi^{\rho_0}_a\iota_{v_0} = \phi^1_a\iota_{v_0}k^{\rho_0}$, that $\sigma^{\rho_0}\iota_{v_0} = \sigma^1\iota_{v_0}k^{\rho_0}$ and that $\tau^{\rho_0}\iota_{v_0} = \iota_{v_0}k^{\rho_0}hr$. We then have

$$\mu * \eta^L = p\xi * \eta^L * \delta_{k^{\rho_0}} + q\delta_{\sigma^1} * \eta^L * \delta_{k^{\rho_0}} + (1-p-q)\eta^L * \delta_{k^{\rho_0}hr}. \tag{5.60}$$

Since $\mu * \nu = \nu$ and $\nu = \eta^L * \omega_G * \eta^R$ and by the bijectivity of the product mapping, we have

$$\eta^L = \left(\frac{p}{p+q}\xi + \frac{q}{p+q}\delta_{\sigma^1} \right) * \eta^L \tag{5.61}$$

Let $\{X_n\}_{n=1}^{\infty}$ be an asymmetric random walk independent of $\{U_n\}_{n=1}^{\infty}$ such that

$$X_0 = 0, \quad P(X_n - X_{n-1} = 1) = \frac{p}{p+q}, \quad P(X_n - X_{n-1} = -1) = \frac{q}{p+q}. \tag{5.62}$$

Set $\overline{X}_n = \max\{X_0, X_1, \dots, X_n\}$. Since $(\delta_{\sigma^1} * \xi)(v,b,c) = (v, \chi_{+1}(b), c)$, we have

$$\left(\frac{p}{p+q}\xi + \frac{q}{p+q}\delta_{\sigma^1} \right)^n = E\left[\xi^{\overline{X}_n} * (\delta_{\sigma^1})^{\overline{X}_n - X_n} \right]. \tag{5.63}$$

By the assumption $p > q$, we have $\overline{X}_n \to \infty$ a.s. By (5.59), we see that the right-hand side of (5.63) converges to $E[\delta_{\iota(U_1,U_2,\dots)}]$. By (5.61), we obtain

$$\eta^L = \left(\frac{p}{p+q}\xi + \frac{q}{p+q}\delta_{\sigma^1} \right)^n * \eta^L \underset{n\to\infty}{\longrightarrow} E[\delta_{\iota(U_1,U_2,\dots)}] * \eta^L = E[\delta_{\iota(U_1,U_2,\dots)}]. \tag{5.64}$$

(iv) Let $\alpha = p+q$ and $\beta = 1-p-q$, so that $\alpha + \beta = 1$ and $\alpha - \beta = 2(p+q) - 1 \in (-1,1)$. Note that

$$\delta_e * \mu^n * \delta_e = \delta_{k^{\rho_0^n}} * (\alpha\delta_e + \beta\delta_h)^n. \tag{5.65}$$

Since $h^2 = e$ and $he = eh = h$, we have

$$(\alpha\delta_e + \beta\delta_h)^n = \sum_{\substack{j=0,\dots,n \\ j:\ \text{even}}} \binom{n}{j} \alpha^{n-j}\beta^j\, \delta_e + \sum_{\substack{j=0,\dots,n \\ j:\ \text{odd}}} \binom{n}{j} \alpha^{n-j}\beta^j\, \delta_h \tag{5.66}$$

$$= \frac{(\alpha+\beta)^n + (\alpha-\beta)^n}{2}\delta_e + \frac{(\alpha+\beta)^n - (\alpha-\beta)^n}{2}\delta_h \underset{n\to\infty}{\longrightarrow} \frac{1}{2}\delta_e + \frac{1}{2}\delta_h = \omega_{\{e,h\}}. \tag{5.67}$$

Let $\lambda \in \mathcal{K}$, so that $\lambda = \lim \mu^{n(m)}$ for some subsequence $\{n(m)\}$. Since ρ_0 is an irrational rotation, we may find a further subsequence $\{n'(m)\}$ such that $\rho_0^{n'(m)}$ converges to some $\rho \in C$. This shows that

$$\delta_e * \lambda * \delta_e = \lim \delta_e * \mu^{n'(m)} * \delta_e = \delta_{k^\rho} * \omega_{\{e,h\}}. \tag{5.68}$$

This shows that $H = \{e,h\}$ and we may take $\gamma = k^{\rho_0}$. $\square$

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