Probability Surveys
Vol. 19 (2022) 404–449
ISSN: 1549-5787
https://doi.org/10.1214/22-PS10

A survey on the effects of free and boolean convolutions on Cauchy-Stieltjes Kernel families

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Abstract: In the setting of noncommutative probability theory and in analogy with the theory of natural exponential families (NEFs), a theory of Cauchy-Stieltjes Kernel (CSK) families has been recently introduced. It is based on the Cauchy-Stieltjes kernel $(1-\theta x)^{-1}$. In this paper, after presenting some basic concepts on NEFs and CSK families and pointing out some similarities and differences between the two families, we review the present state and developments regarding the effects of free and boolean convolutions powers on CSK families.

MSC2020 subject classifications: 60E10, 46L54.
Keywords and phrases: Variance function, Stieltjes transform, free convolution, boolean convolution.

Received September 2021.

Contents

1. Introduction

The theory of exponential families has received a great deal of attention in the classical probability and statistical literature and it remains a very interesting topic. This is in particular due to the fact that the most common distributions belong either to natural exponential families (NEFs) or to general exponential families. The notion of variance function is a fundamental concept in the theory of NEFs and many classifications of NEFs by the form of their variance function has been realized. The most important classes on $\mathbb{R}$ are the quadratic class of NEFs such that the variance function is a polynomial of degree less than or equal to two characterized by Morris [28] and the cubic class of NEFs such that the variance function is a polynomial of degree less than or equal to three characterized by Letac and Mora [25]. The multivariate version of the quadratic and cubic NEF's have been respectively described by Casalis [12] and Hassairi [23].

It is well known that the definition of a real NEF is based on the kernel $(\theta, x) \longmapsto \exp(\theta x)$. Wesolowski [36] has defined a notion of family generated by a measure ν for any kernel $k(x, \theta)$ such that

$$L(\theta) = \int k(x, \theta)\nu(dx)$$

converge in a open set Θ. It is the set of distributions

$$\{(k(x, \theta)/L(\theta))\nu(dx) \ : \ \theta \in \Theta\}.$$

Besides the exponential kernel, the most interesting example of kernels is the Cauchy-Stieltjes one $(\theta, x) \longmapsto 1/(1 - \theta x)$. In fact, the authors in [9] have introduced the definition of q-exponential families, where they identified all the q-exponential families when $|q| < 1$. In particular, they studied the case where $q = 0$, which was related to the free probability theory by using the Cauchy—Stieltjes kernel $1/(1 - \theta x)$. When $q = 1$, we get the exponential families. Bryc [6] continued the study of Cauchy-Stieltjes Kernel (CSK) families for compactly supported probability measures ν. It was in particular shown that such families can be parameterized by the mean m. With this parametrization, denoting $V(m)$ as the variance of the element with mean m, the function $m \mapsto V(m)$ called the variance function and the mean m_0 of the generating measure ν uniquely determines the family and ν. The class of quadratic CSK families is described in [6]. This class consists of the free Meixner distributions. In [10], Bryc and Hassairi have extended the results established in [6] to allow probability measures with unbounded support. They have provided a method to determine the domain of means and introduced a notion of pseudo-variance function. They have also characterized a class of cubic CSK families with support bounded from one side. A general description of polynomial variance function with arbitrary degree is given in [8]. In particular, a complete description of the cubic compactly supported CSK families is given.

On the other hand, in the setting of non-commutative probability theory, Voiculescu introduce the notion of free independence. Moreover, if X and Y are free independent random variables with laws respectively denoted by μ and ν, then $\mu \boxplus \nu$ is the law of the sum of X and Y, where the operation $\boxplus$ is the free additive convolution which is defined using the $\mathcal{R}$-transform. A multiplicative counterpart of free additive convolution, denoted by $\boxtimes$, was introduced in [5] for probability measures on the positive real line, and $\mu \boxtimes \nu$ is the law of the product of X and Y. Speicher and Woroudi [32] have introduced a new kind of convolution between probability measures in the context of non-commutative probability theory with boolean independence: the boolean additive convolution $\uplus$. Moreover, if X and Y are boolean independent random variables with laws respectively denoted by μ and ν, then $\mu \uplus \nu$ is the law of the sum of X and Y. A multiplicative counterpart of boolean additive convolution, denoted by $\rtimes$, was introduced by Bercovici [4], who showed how to calculate it using moment generating series.

In this paper we review some facts concerning the effects of free and boolean convolutions powers on CSK families. We present in section 2 some basic concepts about NEFs and CSK families. We provide some similarities and differences between the two families. In particular and in contrast to NEFs, a typical member of a given CSK family generates a different CSK family, so one can construct new CSK families by the iteration process. We relate the pseudo-variance function for the iterated family to the original pseudo-variance function, and we determine the domain of means. Section 3 is devoted to the study of free additive convolution from the perspective of CSK families. We present further similarities with NEFs and reproductive exponential models. We also explore a property of CSK families that have no counterpart in NEFs: We investigate when the domain of means can be extended beyond the natural domain. In section 4, we deal with boolean additive convolution from a point of view related to CSK families. We determine the formula for variance function under boolean additive convolution power. This formula is used to identify the relation between variance functions under boolean Bercovici-Pata bijection. We also give the connection between boolean cumulants and variance function and we relate boolean cumulants of some probability measures to Catalan numbers and Fuss Catalan numbers. In section 5, we focus on free multiplicative convolution. We determine the effect of the free multiplicative convolution on the pseudo-variance function of a CSK family. We then use the machinery of variance functions to establish some limit theorems related to this type of convolution and involving the free additive convolution and the boolean additive convolution. An explicit expression of the free multiplicative law of large numbers is also given. We are interested in section 6 on the boolean multiplicative convolution. We determine the effect of the boolean multiplicative convolution on the pseudo-variance function of a CSK family. We also identify the relation between variance functions under Belinschi-Nica type semigroup for multiplicative convolutions.

2. Cauchy-Stieltjes Kernel families

In the setting of non-commutative probability theory and in analogy with the theory of NEFs, a theory of CSK families has been recently introduced based on the Cauchy-Stieltjes kernel. In this paragraph we present some basic elements of CSK families. One start by presenting some basic concepts of NEFs. Then, we point out some similarities and differences between the two families.

2.1. About NEFs

If μ is a positive measure on the real line, we denote by

$$L_\mu(\theta) = \int_{\mathbb{R}} \exp(\theta x)\mu(dx), \tag{2.1}$$

its Laplace transform, and we denote $\Theta(\mu) = \text{interior}\{\theta \in \mathbb{R};\ L_\mu(\theta) < \infty\}$. $\mathcal{M}(\mathbb{R})$ will denote the set of measures μ such that $\Theta(\mu)$ is not empty and μ is not concentrated on one point. If μ is in $\mathcal{M}(\mathbb{R})$, we also denote

$$\kappa_\mu(\theta) = \log(L_\mu(\theta)), \qquad \theta \in \Theta(\mu), \tag{2.2}$$

the cumulate function of μ.

To each μ in $\mathcal{M}(\mathbb{R})$ and θ in $\Theta(\mu)$, we associate the following probability distribution:

$$P(\theta,\mu)(dx) = \exp(\theta x - \kappa_\mu(\theta))\mu(dx). \tag{2.3}$$

The set

$$F = F(\mu) = \{P(\theta,\mu),\ \theta \in \Theta(\mu)\} \tag{2.4}$$

is called the natural exponential family (NEF) generated by μ.

The measure μ is said to be a basis of $F(\mu)$. It is worth mentioning that a basis of F is by no means unique: If μ and μ' are in $\mathcal{M}(\mathbb{R})$, then $F(\mu) = F(\mu')$ if and only if there exists $(a,b) \in \mathbb{R}^2$ such that

$$\mu'(dx) = \exp(ax+b)\mu(dx). \tag{2.5}$$

Therefore, all measure of the form (2.5) generate the family F, in particular the elements of F. In what follows, we will see that this property fails for CSK families. In fact a typical member in a CSK family generate something different than the original family, then the construction can be iterated.

The map $\theta \longmapsto \kappa'_\mu(\theta)$ is a bijection between $\Theta(\mu)$ and its image M_F which is called the domain of means of the family F. Denote by $\phi_\mu : M_F \longrightarrow \Theta(\mu)$ the inverse of κ'_μ. We are thus led to the parametrization of F by the mean m.

For each $\mu \in \mathcal{M}(\mathbb{R})$ and $m \in M_F$, let us denote $P(m, F) = P(\phi_\mu(m), \mu)$ and rewrite $F = \{P(m, F);\ m \in M_F\}$.

The variance of $P(m, F)$ is denoted $V_F(m)$. The map $m \longmapsto V_F(m)$ is called the variance function of the NEF F and is defined for all $m \in M_F$ by

$$V_F(m) = \kappa''_\mu(\phi_\mu(m)) = (\phi'_\mu(m))^{-1}.$$

The important feature of $V_F(.)$ is that it characterizes the NEF F in the following sense: If F_1 is another NEF such that $M_F \cap M_{F_1}$ contains a non-empty open interval $\mathcal{O}$ and $V_F(m) = V_{F_1}(m)$ for $m \in \mathcal{O}$, then $F = F_1$. Thus $(M_F, V_F(m))$ completely characterizes F.

For μ in $\mathcal{M}(\mathbb{R})$ the Jørgensen set of $F(\mu)$ is defined by

$$\Lambda(\mu) = \{\lambda > 0\ :\ \exists\ \mu_\lambda\ : L_{\mu_\lambda}(\theta) = (L_\mu(\theta))^\lambda \text{ and } \Theta(\mu_\lambda) = \Theta(\mu)\}.$$

$\Lambda(\mu)$ is stable under addition which means that for $\lambda,\ \lambda' \in \Lambda(\mu)$, we have $\lambda + \lambda' \in \Lambda(\mu_1)$ and $\mu_{\lambda+\lambda'} = \mu_\lambda * \mu_{\lambda'}$. The link between μ_λ and $F_\lambda = F(\mu_\lambda)$ is revealed by the following; For $\lambda \in \Lambda(\mu)$, $M_{F_\lambda} = \lambda M_F$, and for $m \in M_{F_\lambda}$,

$$V_{F_\lambda}(m) = \lambda V_F\left(\frac{m}{\lambda}\right).$$

2.2. About CSK families

Our notations are the ones used in [16]. Let ν be a non-degenerate probability measure with support bounded from above. Then

$$M_\nu(\theta) = \int \frac{1}{1-\theta x}\nu(dx) \tag{2.6}$$

is defined for all $\theta \in [0, \theta_+)$ with $1/\theta_+ = \max\{0, \sup \operatorname{supp}(\nu)\}$.

For $\theta \in [0, \theta_+)$, we set

$$P_{(\theta,\nu)}(dx) = \frac{1}{M_\nu(\theta)(1-\theta x)}\nu(dx).$$

The set

$$\mathcal{K}_+(\nu) = \{P_{(\theta,\nu)}(dx); \theta \in (0, \theta_+)\}$$

is called the one-sided CSK family generated by ν.

Let $k_\nu(\theta) = \int x P_{(\theta,\nu)}(dx)$ denote the mean of $P_{(\theta,\nu)}$. According to [10, page 579–580] the map $\theta \mapsto k_\nu(\theta)$ is strictly increasing on $(0, \theta_+)$, it is given by the formula

$$k_\nu(\theta) = \frac{M_\nu(\theta) - 1}{\theta M_\nu(\theta)}. \tag{2.7}$$

The image of $(0, \theta_+)$ by k_ν is called the (one sided) domain of means of the family $\mathcal{K}_+(\nu)$, it is denoted $(m_0(\nu), m_+(\nu))$. This leads to a parametrization of the family $\mathcal{K}_+(\nu)$ by the mean. In fact, denoting by ψ_ν the reciprocal of k_ν, and writing for $m \in (m_0(\nu), m_+(\nu))$, $Q_{(m,\nu)}(dx) = P_{(\psi_\nu(m),\nu)}(dx)$, we have that

$$\mathcal{K}_+(\nu) = \{Q_{(m,\nu)}(dx); m \in (m_0(\nu), m_+(\nu))\}. \tag{2.8}$$

Now let

$$B = B(\nu) = \max\{0, \sup \mathrm{supp}(\nu)\} = 1/\theta_+ \in [0, \infty). \tag{2.9}$$

It is shown in [10] that the bounds $m_0(\nu)$ and $m_+(\nu)$ of the one-sided domain of means $(m_0(\nu), m_+(\nu))$ are given by

$$m_0(\nu) = \lim_{\theta \to 0^+} k_\nu(\theta) \qquad \text{and} \qquad m_+(\nu) = B - \lim_{z \to B^+} \frac{1}{G_\nu(z)}, \tag{2.10}$$

where $G_\nu(.)$ is the Cauchy transform of ν which is defined by

$$G_\nu(z) = \int \frac{1}{z - x} \nu(dx), \tag{2.11}$$

for $z \in \mathbb{C}^+ = \{x + iy \in \mathbb{C};\ y > 0\}$.

It is clear that $m_+(\nu) \leq \sup supp(\nu)$ and

$$m_0(\nu) = \lim_{\theta \searrow 0^+} k_\nu(\theta) = \int x\nu(dx) \geq -\infty.$$

It is worth mentioning here that one may define the one-sided CSK family for a measure ν with support bounded from below. This family is usually denoted $\mathcal{K}_-(\nu)$ and parameterized by θ such that $\theta_- < \theta < 0$, where θ_- is either $1/b(\nu)$ or $-\infty$ with $b = b(\nu) = \min\{0, \inf supp(\nu)\}$. The domain of means for $\mathcal{K}_-(\nu)$ is the interval $(m_-(\nu), m_0(\nu))$ with $m_-(\nu) = b - 1/G_\nu(b)$.

If ν has compact support, the natural domain for the parameter θ of the two-sided CSK family $\mathcal{K}(\nu) = \mathcal{K}_+(\nu) \cup \mathcal{K}_-(\nu) \cup \{\nu\}$ is $\theta_- < \theta < \theta_+$.

The variance function given by

$$m \mapsto V_\nu(m) = \int (x - m)^2 Q_{(m,\nu)}(dx), \tag{2.12}$$

is a fundamental concept in the theory of CSK families as presented in [6]. Unfortunately, if ν hasn't a first moment which is for example the case for free 1/2-stable law, all the distributions in the CSK family generated by ν have infinite variance. This fact has led the authors in [10] to introduce a notion of pseudo-variance function defined by

$$\mathbb{V}_\nu(m) = m\left(\frac{1}{\psi_\nu(m)} - m\right), \tag{2.13}$$

If $m_0(\nu) = \int x d\nu$ is finite, then (see [10]) the pseudo-variance function is related to the variance function by

$$\mathbb{V}_\nu(m) = \frac{m}{m - m_0} V_\nu(m). \tag{2.14}$$

In particular, $\mathbb{V}_\nu = V_\nu$ when $m_0(\nu) = 0$.

The generating measure ν is uniquely determined by the pseudo-variance function $\mathbb{V}_\nu$. In fact, if we set

$$z = z(m) = m + \frac{\mathbb{V}_\nu(m)}{m}, \tag{2.15}$$

then the Cauchy transform (2.11) satisfies

$$G_\nu(z) = \frac{m}{\mathbb{V}_\nu(m)}, \tag{2.16}$$

Also the distribution $Q_{(m,\nu)}(dx)$ may be written as $Q_{(m,\nu)}(dx) = f_\nu(x, m)\nu(dx)$ with

$$f_\nu(x,m) := \begin{cases} \frac{\mathbb{V}_\nu(m)}{\mathbb{V}_\nu(m)+m(m-x)}, & m \neq 0 \quad ; \\ 1, & m = 0, \quad \mathbb{V}_\nu(0) \neq 0 \quad ; \\ \frac{\mathbb{V}'_\nu(0)}{\mathbb{V}'_\nu(0)-x}, & m = 0, \quad \mathbb{V}_\nu(0) = 0 \quad . \end{cases} \tag{2.17}$$

Now, we recall the effect on a CSK family of applying an affine transformation to the generating measure. Consider the affine transformation $\varphi : x \longmapsto (x - \lambda)/\beta$ where $\beta \neq 0$ and $\lambda \in \mathbb{R}$ and let $\varphi(\nu)$ be the image of ν by φ. In other words, if X is a random variable with law ν, then $\varphi(\nu)$ is the law of $(X - \lambda)/\beta$, or $\varphi(\nu) = D_{1/\beta}(\nu \boxplus \delta_{-\lambda})$, where $D_r(\mu)$ denotes the dilation of measure μ by a number $r \neq 0$, that is $D_r(\mu)(U) = \mu(U/r)$. The point m_0 is transformed to $(m_0 - \lambda)/\beta$. In particular, if $\beta < 0$ the support of the measure $\varphi(\nu)$ is bounded from below so that it generates the left-sided family $\mathcal{K}_-(\varphi(\nu))$. For m close enough to $(m_0 - \lambda)/\beta$, the pseudo-variance function is

$$\mathbb{V}_{\varphi(\nu)}(m) = \frac{m}{\beta(m\beta + \lambda)}\mathbb{V}_\nu(\beta m + \lambda). \tag{2.18}$$

In particular, if the variance function exists, then $V_{\varphi(\nu)}(m) = \frac{1}{\beta^2}V_\nu(\beta m + \lambda)$.

Note that using the special case where φ is the reflection $\varphi(x) = -x$, one can transform a right-sided CSK family to a left-sided family. If ν has support bounded from above and its right-sided CSK family $\mathcal{K}_+(\nu)$ has domain of means (m_0, m_+) and pseudo-variance function $\mathbb{V}_\nu(m)$, then $\varphi(\nu)$ generates the left-sided CSK family $\mathcal{K}_-(\varphi(\nu))$ with domain of means $(-m_+, -m_0)$ and pseudo-variance function $\mathbb{V}_{\varphi(\nu)}(m) = \mathbb{V}_\nu(-m)$.

Remark 2.1. There are numerous similarities between the NEFs and the CSK families: both are parameterized by the mean, both are uniquely determined by the variance function and the so called "domain of means", and the variance function of the CSK family generated by the free additive convolution of generating measure ν has the same form as the variance function of the exponential family of the classical convolution (as we will see in the next section).

There also some differences due to the fact that the exponential kernel $\exp(\theta x)$ is always positive while the Cauchy kernel $1/(1 - \theta x)$ might be negative, and due to the fact that the variance of a CSK family might not exist. This fact has led the authors in [10] to introduce the "pseudo-variance" function that has no direct probabilistic interpretation but has similar properties to the variance function and is equal to the variance function of the CSK family generated by a measure ν of mean zero.

2.3. Iterated CSK families

One difference between the exponential and CSK families is that one can build nontrivial iterated CSK families. That is, each member of an exponential family generates the same exponential family so it does not matter which of them we use for the generating measure. But this is not so for CSK families: each member of a CSK family generates something different than the original family, so the construction can be iterated.

Suppose $Q_{(m,\nu)}$ is in the CSK family generated by a probability measure ν with support bounded from above. Consider a new CSK family generated by $Q_{(m,\nu)}$. Then, as long as $m \neq m_0$, the variance function of this new family necessarily exists. Our goal is to relate the variance function of this new family to the pseudo-variance function of the initial family. We relate the domains of means and the pseudo-variance functions of the original family $\mathcal{K}_+(\nu)$ and the new family $\mathcal{K}_+(Q_{(m_1,\nu)})$. Fix $m_1 \in (m_0(\nu), m_+(\nu))$, and consider $Q_{(m_1,\nu)} = P_{(\theta_1,\nu)} \in \mathcal{K}_+(\nu)$, with $\theta_1 \in (0, \theta_+(\nu))$. Define

$$M_{P_{(\theta_1,\nu)}}(\theta) = \int \frac{1}{1-\theta x} P_{(\theta_1,\nu)}(dx),$$

for $\theta \in \Theta = \{\theta > 0;\ M_{P_{(\theta_1,\nu)}(dx)}(\theta) < \infty\}$. The CSK family generated by $Q_{(m_1,\nu)} = P_{(\theta_1,\nu)}$ is

$$\mathcal{K}_+(P_{(\theta_1,\nu)}) = \{\overline{P}_{(\theta, P_{(\theta_1,\nu)})}(dx)\} = \left\{ \frac{1}{M_{P_{(\theta_1,\nu)}}(\theta)(1-\theta x)} P_{(\theta_1,\nu)}(dx),\ \theta \in \Theta \right\}.$$

Proposition 2.2. *[7, Proposition 2.2]*

(i) $\Theta = (0, \theta_+(\nu))$
(ii) *For* $\theta \in \Theta$, *we have*

$$M_{P_{(\theta_1,\nu)}}(\theta) = \begin{cases} \dfrac{\theta M_\nu(\theta) - \theta_1 M_\nu(\theta_1)}{M_\nu(\theta_1)(\theta - \theta_1)} & \text{if } \theta \neq \theta_1; \\[2ex] \dfrac{M_\nu(\theta_1) + \theta_1 M'_\nu(\theta_1)}{M_\nu(\theta_1)} & \text{if } \theta = \theta_1. \end{cases} \tag{2.19}$$

(iii) *For* $\theta \in \Theta$, *we set* $k_\nu(\theta) = \int x P_{(\theta,\nu)}(dx)$ *the mean of* $P_{(\theta,\nu)}$, *and* $k_{P_{(\theta_1,\nu)}}(\theta) =$

$\int x\overline{P}_{(\theta,P_{(\theta_1,\nu)})}(dx)$, *the mean of* $\overline{P}_{(\theta,P_{(\theta_1,\nu)})}(dx)$. *Then*

$$k_{P_{(\theta_1,\nu)}}(\theta)=\begin{cases}\dfrac{\theta k_\nu(\theta)-\theta_1 k_\nu(\theta_1)}{(\theta-\theta_1)+\theta\theta_1(k_\nu(\theta)-k_\nu(\theta_1))} & \text{if } \theta\neq\theta_1;\\[2ex] \dfrac{k_\nu(\theta_1)+\theta_1 k'_\nu(\theta_1)}{1+\theta_1^2 k'_\nu(\theta_1)} & \text{if } \theta=\theta_1.\end{cases} \tag{2.20}$$

Next we denote by $\mathcal{D}_+(\nu)$ and $\mathbb{V}_\nu$ the domain of means and the pseudo-variance function of the family $\mathcal{K}_+(\nu)$, and by $\mathcal{D}_+(Q_{(m_1,\nu)})$ and $\mathbb{V}_{Q_{(m_1,\nu)}}$ the domain of means and the pseudo-variance function of $\mathcal{K}_+(Q_{(m_1,\nu)})$. Recall that $\mathcal{D}_+(\nu)=k_\nu((0,\theta_+))$ and $\mathcal{D}_+(Q_{(m_1,\nu)})=k_{P_{(\theta_1,\nu)}}((0,\theta_+))$. We set $m=k_\nu(\theta)$ and $\overline{m}=k_{P_{(\theta_1,\nu)}}(\theta)$. We will also use the inverse ψ_ν of the function $\theta\longmapsto k_\nu(\theta)$ from $(0,\theta_+)$ into (m_0,m_+), and the inverse $\psi_{P_{(\theta_1,\nu)}}$ of the function $\theta\longmapsto k_{P_{(\theta_1,\nu)}}(\theta)$ from $(0,\theta_+)$ into its image $(\overline{m}_0,\overline{m}_+)$.

Theorem 2.3. *[7, Theorem 2.3] Let ν be a probability measure with support bounded from above, and let $\mathcal{K}_+(\nu)$ be the CSK family generated by ν. Fix $m_1\in(m_0,m_+)$ and let $B=B(\nu)$ be given by (2.9). With the notations introduced above, we have*

(i)

$$\overline{m}=k_{P_{(\theta_1,\nu)}}(\psi_\nu(m))=\begin{cases}\dfrac{m^2\mathbb{V}_\nu(m_1)-m_1^2\mathbb{V}_\nu(m)}{m\mathbb{V}_\nu(m_1)-m_1\mathbb{V}_\nu(m)} & \text{if } m\neq m_1;\\ \dfrac{2m_1\mathbb{V}_\nu(m_1)-m_1^2\mathbb{V}'_\nu(m_1)}{\mathbb{V}_\nu(m_1)-m_1\mathbb{V}'_\nu(m_1)} & \text{if } m=m_1.\end{cases} \tag{2.21}$$

(ii) The (one sided) domain of means is

$$\mathcal{D}_+(Q_{(m_1,\nu)})=(\overline{m}_0,\overline{m}_+)=\left(m_1,\frac{m_+G_\nu(B)-m_1^2/\mathbb{V}_\nu(m_1)}{G_\nu(B)-m_1/\mathbb{V}_\nu(m_1)}\right).$$

(interpreted as the limit $b\to B^+$.)

(iii)

$$\frac{\mathbb{V}_{Q_{(m_1,\nu)}}(\overline{m})}{\overline{m}}+\overline{m}=\frac{\mathbb{V}_\nu(m)}{m}+m. \tag{2.22}$$

Note that the function $m\longmapsto\overline{m}$ is a bijection from $\mathcal{D}_+(\nu)$ into $\mathcal{D}_+(Q_{(m_1,\nu)})$, so that to get explicitly the pseudo-variance function of the CSK family $\mathcal{K}_+(Q_{(m_1,\nu)})$, we need to express m in terms of $\overline{m}$ from (2.21) and insert it in (2.22).

Note that as the probability measure $Q_{(m_1,\nu)}$ has a finite first moment $\overline{m}_0=m_1$, the variance function $V_{Q_{(m_1,\nu)}}(.)$ of the CSK family $\mathcal{K}_+(Q_{(m_1,\nu)})$ exists and from (2.14) we have

$$\mathbb{V}_{Q_{(m_1,\nu)}}(\overline{m})=\frac{\overline{m}}{\overline{m}-m_1}V_{Q_{(m_1,\nu)}}(\overline{m}).$$

The following two special cases are of interest as they exhibit the iterated CSK families generated by two laws of importance in free probability. We consider the Wigner semicircle distribution. It is named after the Hungarian theoretical physicist Eugene Wigner who contributed to mathematical physics. The Wigner semicircle distribution arises as the limiting distribution of eigenvalues of a random symmetric matrices as the size of the matrix approaches infinity. In free probability theory, the role of Wigner's semicircle distribution is analogous to that of the Gaussian distribution in classical probability theory.

Example 2.4. The Wigner's semicircle (free Gaussian) law

$$\nu(dx) = \frac{\sqrt{4-x^2}}{2\pi} 1_{(-2,2)}(x)dx,$$

generates the CSK family with a constant variance function $V_\nu(m) = 1 = \mathbb{V}_\nu(m)$ and the (one-sided) domain of means is $\mathcal{D}_+(\nu) = (0,1)$. (The full two-sided domain of means is of course $(-1,1)$.) For $m_1 \in \mathcal{D}_+(\nu)$, the probability measure

$$Q_{(m_1,\nu)}(dx) = \frac{\sqrt{4-x^2}}{2\pi(1+m_1(m_1-x))} 1_{(-2,2)}(x)dx,$$

generates the CSK family with pseudo-variance function

$$\mathbb{V}_{Q_{(m_1,\nu)}}(\overline{m}) = \frac{\overline{m}}{\overline{m}-m_1}(-m_1\overline{m} + m_1^2 + 1),$$

and with the domain of means $\mathcal{D}_+(Q_{(m_1,\nu)}) = (m_1, 1+m_1)$. The corresponding variance function is

$$V_{Q_{(m_1,\nu)}}(\overline{m}) = -m_1\overline{m} + m_1^2 + 1. \tag{2.23}$$

Up to an affine transformation, this is the Marchenko-Pastur law, see next example. In fact, in the mathematical theory of random matrices the Marchenko-Pastur distribution or Marchenko-Pastur law is introduced by the Ukrainian mathematicians Vladimir Alexandrovich Marchenko and Leonid Andreevich Pastur. It describes the asymptotic behavior of singular values of large rectangular random matrices, (see [26] for more details).

Example 2.5. For $0 < a^2 < 1$, the (absolutely continuous) centered Marchenko-Pastur (free Poisson) law

$$\nu(dx) = \frac{\sqrt{4-(x-a)^2}}{2\pi(1+ax)} 1_{(a-2,a+2)}(x)dx$$

generates the CSK family with quadratic variance function $V(m) = 1 + am = \mathbb{V}(m)$, and the domain of means is $\mathcal{D}_+(\nu) = (0,1)$.

For $m_1 \in \mathcal{D}_+(\nu)$, the probability measure

$$Q_{(m_1,\nu)}(dx) = \frac{(1+am_1)\sqrt{4-(x-a)^2}}{2\pi(1+m_1(a+m_1-x))(1+ax)} 1_{(a-2,a+2)}(x)dx$$

generates CSK family with pseudo-variance function

$$\mathbb{V}_{Q_{(m_1,\nu)}}(\overline{m}) = \frac{\overline{m}}{(1+am_1)(\overline{m}-m_1)}(1+a\overline{m})\left(1+m_1\left(a+m_1-\overline{m}\right)\right).$$

The domain of means is

$$\mathcal{D}_+(Q_{(m_1,\nu)}) = (m_1, 1+(a+1)m_1).$$

The variance function is

$$V_{Q_{(m_1,\nu)}}(\overline{m}) = \frac{(1+a\overline{m})\left(1+m_1\left(a+m_1-\overline{m}\right)\right)}{1+am_1}.$$

3. Free additive convolution

Free convolution is the free probability analog of the classical notion of convolution of probability measures. Due to the non-commutative nature of free probability theory, one has to talk separately about additive and multiplicative free convolution, which arise from addition and multiplication of free random variables. Free convolution can be used to compute the laws and spectra of sums or products of random variables which are free independents. Such examples include: random walk operators on free groups (Kesten measures) and asymptotic distribution of eigenvalues of sums or products of independent random matrices. In this section, we are interested in the study of free additive convolution from the perspective of CSK families.

Denote by $\mathcal{M}$ (respectively by $\mathcal{M}_+$) the set of Borel probability measures on $\mathbb{R}$ (respectively on $\mathbb{R}_+$). For $\nu \in \mathcal{M}$, its Cauchy transform G_ν is defined by (2.11). Note in particular that $\Im(G_\nu(z)) < 0$ for any $z \in \mathbb{C}^+$, and hence we may consider the reciprocal Cauchy transform $F_\nu : \mathbb{C}^+ \longrightarrow \mathbb{C}^+$ given by $F_\nu(z) = 1/G_\nu(z)$ for $z \in \mathbb{C}^+$. According to [5], for any probability measure $\nu \in \mathcal{M}$ and any $\lambda \in (0,+\infty)$, there exists positive numbers α, β and M such that F_ν is univalent on the set $\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}^+ \ : \ \Im(z) > \beta,\ |\Re(z)| < \alpha\Im(z)\}$ and such that $F_\nu(\Gamma_{\alpha,\beta}) \supset \Gamma_{\lambda,M}$. Therefore the right inverse F_ν^{-1} of F_ν exists on $\Gamma_{\lambda,M}$, and the free cumulant transform R_ν can be defined by

$$R_\nu(z) = zF_\nu^{-1}(1/z) - 1, \quad \text{for all } z \text{ such that } 1/z \in \Gamma_{\lambda,M}. \tag{3.1}$$

The name refers to the fact that R_ν linearizes free additive convolution (see [5]). Variants of R_ν (with the same linearizing property) are the $\mathcal{R}$-transform $\mathcal{R}_\nu$ and the Voiculescu transform v_ν related by the following equalities:

$$R_\nu(z) = z\mathcal{R}_\nu(z) = zv_\nu(1/z). \tag{3.2}$$

The free additive convolution $\mu \boxplus \nu$ of the probability measures μ, ν on Borel sets of the real line is a uniquely defined probability measure $\mu \boxplus \nu$ such that

$$\mathcal{R}_{\mu\boxplus\nu}(z) = \mathcal{R}_\mu(z) + \mathcal{R}_\nu(z). \tag{3.3}$$

A probability measure $\nu \in \mathcal{M}$ is $\boxplus$-infinitely divisible, if for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathcal{M}$ such that

$$\nu = \underbrace{\nu_n \boxplus ..... \boxplus \nu_n}_{n \text{ times}}.$$

Our interest in the $\mathcal{R}$-transform stems from its linear property to free additive convolution. Let $\nu^{\boxplus\alpha}$ denote the α-fold free additive convolution of ν with itself. In contrast to classical convolution, this operation is well defined for all real $\alpha \geq 1$, (see [29]) and we have

$$\mathcal{R}_{\nu^{\boxplus\alpha}}(z) = \alpha \mathcal{R}_\nu(z). \tag{3.4}$$

Probability measure ν is $\boxplus$-infinitely divisible if its free additive convolution power $\nu^{\boxplus\alpha}$ is well-defined for all real $\alpha > 0$.

3.1. Free additive convolution and variance function

In this paragraph, we present further similarities of CSK families with exponential families and reproductive exponential models. Next, we give the formula of pseudo-variance function (and variance function in case of existence) by the effect of free additive convolution power. It is the same as the formula for variance function of a NEF under the effect of classical additive convolution power. More precisely:

Proposition 3.1. *[10, Proposition 3.10] Let $\mathbb{V}_\nu$ be the pseudo-variance function of the one sided CSK family generated by a probability measure ν with support bounded from above and with the mean $-\infty \leq m_0 < \infty$. Then for $\alpha > 0$ such that $\nu^{\boxplus\alpha}$ is defined, the support of $\nu^{\boxplus\alpha}$ is bounded from above and for $m > \alpha m_0$ close enough to αm_0,*

$$\mathbb{V}_{\nu^{\boxplus\alpha}}(m) = \alpha \mathbb{V}_\nu(m/\alpha). \tag{3.5}$$

Furthermore, if $m_0 < +\infty$, then the variance functions of the CSK families generated by ν and $\nu^{\boxplus\alpha}$ exists and

$$V_{\nu^{\boxplus\alpha}}(m) = \alpha V_\nu(m/\alpha). \tag{3.6}$$

We remark that the restriction of (3.5) to m close enough to αm_0 cannot be easily avoided, as we do not have a general formula for the upper end of the domain of means for $\nu^{\boxplus\alpha}$. The study of the action of free additive convolution power on the domain of means is given in what follows, (see [7] for more details).

A property which in [3, (3.16)] is indeed called the reproductive property of an exponential family states that if $\mu \in F$ with variance function V_F, then for all $n \in \mathbb{N}$ the law of the sample mean, $D_{1/n}(\mu^{*n})$ is in the NEF with variance function V_F/n. The analogue of this result for the CSK families is given in [6], [10]. More precisely we have,

Proposition 3.2. *Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_+(\nu)$ generated by a non degenerate probability measure ν with support bounded from above and mean $m_0(\nu)$, then for $\alpha \geq 1$ measure*

$$\nu_\alpha := D_{1/\alpha}(\nu^{\boxplus\alpha}) \tag{3.7}$$

has also support bounded from above and there is $\varepsilon > 0$ such that the pseudo-variance function of the one sided CSK family generated by ν_α is

$$\mathbb{V}_{\nu_\alpha}(m) = \mathbb{V}_\nu(m)/\alpha,$$

for all $m \in (m_0, m_0 + \varepsilon)$.

If ν is $\boxplus$-infinitely divisible, then the above holds for every $\alpha > 0$. Conversely, if for every $\alpha > 0$, there is $\delta = \delta(\alpha) > 0$ such that $\mathbb{V}_\nu(m)/\alpha$ is a pseudo-variance function of some CSK family on $(m_0, m_0 + \delta)$, then ν is $\boxplus$-infinitely divisible.

Recall that if ν is a compactly supported measure, the $\mathcal{R}$-transform is analytic at $z = 0$

$$\mathcal{R}_\nu(z) = \sum_{n=1}^{\infty} c_n(\nu) z^{n-1}. \tag{3.8}$$

The coefficients $c_n = c_n(\nu)$ are called free cumulants of measure ν. The following result give the link between free cumulants and variance function of a CSK family, (see [6] for more details).

Theorem 3.3. *Suppose V is analytic in a neighborhood of m_0, $V(m_0) > 0$, and ν is a probability measure with finite all moments, such that $\int x\nu(dx) = m_0$. Then the following conditions are equivalent.*

- *(i) ν is non-degenerate, compactly supported, and there exists an interval $(A, B) \ni m_0$ such that (2.8) defines a family of probability measures parameterized by the mean with the variance function V.*
- *(ii) The free cumulants (3.8) of ν are $c_1 = m_0$, and for $n \geq 1$*

$$c_{n+1} = \frac{1}{n!} \frac{d^{n-1}}{dx^{n-1}} (V(x))^n \bigg|_{x=m_0}. \tag{3.9}$$

We now use (3.9) to relate certain free cumulants to Catalan numbers, see [6, Corollary 3.4].

Corollary 3.4. *If ν is the standardized free gamma Meixner law, i.e. it generates the free exponential family with $m_0 = 0$ and variance function $V(m) = (1 + am)^2$, then its free cumulants are*

$$c_{k+1}(\nu) = \frac{1}{k+1} \begin{pmatrix} 2k \\ k \end{pmatrix} a^{k-1}, \quad k \geq 1.$$

3.2. Marchenko-Pastur approximation

Let

$$\omega_{a,\sigma}(dx) = \frac{\sqrt{4\sigma^2 - (x-a)^2}}{2\pi\sigma^2}\mathbf{1}_{|x-a|<2\sigma}(dx).$$

denote the semicircle law of mean a and variance σ^2. Up to affine transformations, this is the free Meixner law which generates the CSK family with the variance function $V_{\omega_{a,\sigma}}(m) = \sigma^2$.

Following the analogy with NEFs, the CSK family $\mathcal{K}(\omega_{a,\sigma})$ can be thought as a free analog of the NEF generated by the normal distribution. Somewhat surprisingly, this family does not contain all semicircle laws, but instead it contains affine transformations of the (absolutely continuous) Marchenko-Pastur laws.

Example 3.5 (Semi-circle CSK family)**.** For $\lambda > 0$, let

$$\pi_{m,\lambda}(dx) = \frac{\sqrt{\lambda}\sqrt{4-\lambda x^2}}{2\pi(1+\lambda m(m-x))}\mathbf{1}_{x^2<4/\lambda}(dx).$$

Function $V(m) = 1/\lambda$ is the variance function of the CSK family

$$\mathcal{K}(\omega_{0,1/\sqrt{\lambda}}) = \left\{\pi_{m,\lambda}(dx) \ : \ |m| < 1/\sqrt{\lambda}\right\} \tag{3.10}$$

To verify that the expression integrates to 1 for $m \neq 0$, we use the explicit form of the density [24, (3.3.2)] to note that $\pi_{m,\lambda} = \mathcal{L}(m + 1/(\lambda m) - mX)$ is the law of the affine transformation of a free Poisson (Marchenko-Pastur) random variable X with parameter $1/(\lambda m^2)$. From the properties of Marchenko-Pastur law we see that $\int \pi_{m,\lambda}(dx) = 1$ if and only if f $m^2 \leq 1/\lambda$.

We have the following analogue of [3, Theorem 3.4].

Theorem 3.6. *[6, Theorem 4.1](Marchenko-Pastur approximation) Consider $\mathcal{K}_+(\nu)$ the CSK family generated by a probability measure ν with mean m_0. Suppose the variance function V of $\mathcal{K}_+(\nu)$ is analytic and strictly positive in a neighborhood of m_0. Then there is $\delta > 0$ such that if $\mathcal{L}(Y_\lambda)$ is in the CSK family with variance function V/λ has mean $\mathbb{E}(Y_\lambda) = m_0 + m/\sqrt{\lambda}$ with $|m| < \delta$, then*

$$\sqrt{\lambda}(Y_\lambda - m_0) \xrightarrow{\lambda\to+\infty} \pi_{m,1/V(m_0)} \qquad \text{in distribution.}$$

By Example 3.5, if $0 < |m| \leq \sigma$, then up to affine transformation $\pi_{m,1/\sigma^2}$ is a Marchenko-Pastur law. Thus in this case Theorem 3.6 gives a Marchenko-Pastur approximation to $\mathcal{L}(Y_\lambda)$.

Of course, every compactly supported mean-zero measure ν is an element of the CSK family that it generates. Since $\pi_{0,1/\sigma^2} = \omega_{0,\sigma}$ is the semicircle law, combining Proposition 3.2 with Theorem 3.6 we get the following Free Central Limit Theorem; see [11], [35].

Corollary 3.7. *If a probability measure ν is compactly supported and centered, then with $\sigma^2 = \int x^2\nu(dx)$, we have*

$$D_{1/\sqrt{n}}(\nu^{\boxplus n}) \xrightarrow{n\to+\infty} \omega_{0,\sigma} \qquad \text{in distribution.}$$

3.3. Extending the domain for parametrization by the mean

We investigate when the domain of means can be extended beyond the natural domain. This is a property in CSK families that have no counterpart in NEFs. Given a probability measure ν with support bounded from above, equation (2.10) tells us how to determine the one-sided domain of means (m_0, m_+) and formula (2.13) tell us how to compute the pseudo-variance function $\mathbb{V}_\nu(m)$ for $m \in (m_0, m_+)$. But the pseudo-variance function is often well defined for other values of m, too. So it is natural to ask whether the corresponding "family of measures" can also be enlarged preserving the pseudo-variance function. The following examples illustrates the idea.

Example 3.8. Consider the (two-sided) CSK family generated by the semicircle law

$$\nu = \frac{1}{2\pi}\sqrt{4-x^2}1_{|x|<2}dx$$

with the variance function $V_\nu(m) = \mathbb{V}_\nu(m) = 1$, the domain of means $(-1, 1)$ and

$$\mathcal{K}(\nu) = \left\{ \frac{\sqrt{4-x^2}}{2\pi(1+m(m-x))}1_{|x|<2}dx : m \in (-1,1) \right\}.$$

This is a family of atomless Marchenko-Pastur laws, which can be naturally enlarged to include all Marchenko-Pastur laws:

$$\overline{\mathcal{K}}(\nu) = \Big\{ \pi_m(dx) = \frac{\sqrt{4-x^2}}{2\pi(1+m(m-x))}1_{|x|<2}dx \\ + (1-\frac{1}{m^2})^+\delta_{m+\frac{1}{m}};\ m \in (-\infty,\infty) \Big\}.$$

Noting that

$$\int \pi_m(dx) = 1,\ \int x\pi_m(dx) = m,\ \int (x-m)^2\pi_m(dx) = 1,$$

we see that $V_\nu(m) = 1$ is the variance function of this enlarged family.

Of course, it may also happen that the extension beyond the natural domain of means is not possible. Here is a simple example when this happens.

Example 3.9. Let $\nu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ be the symmetric Bernoulli distribution. Then $M_\nu(\theta) = \frac{1}{1-\theta^2}$ and $k_\nu(\theta) = \theta$. The (two-sided) range of parameter is $\Theta = (-1, 1)$. So the domain of means here is $(-1, 1)$, and with $m_0(\nu) = 0$ the pseudo-variance function is equal to the variance function,

$$V_\nu(m) = \mathbb{V}_\nu(m) = 1 - m^2.$$

In this case, the variance function is negative outside the domain of means, so we cannot extend the family $\{Q_{(m,\nu)} : m \in (-1, 1)\}$ beyond the original domain of means.

Our next example shows that the extension sometimes may proceed in two separate steps.

Example 3.10. Consider the inverse semicircle law

$$\nu(dx) = \frac{\sqrt{-1-4x}}{2\pi x^2}\mathbf{1}_{(-\infty,-\frac{1}{4})}(x)dx.$$

Since $m^2 + m \geq -1/4$, it is clear that measure $Q_{(m,\nu)}$ is non-negative and well defined for all m. Since the integral $\int Q_{(m,\nu)}(dx)$ is an analytic function of $m < -1/2$, it must be 1, so $Q_{(m,\nu)}$ is a probability measure for all $m < -1/2$. This is the "first part" of the extension, from $(-\infty, -1)$ to a larger interval $(-\infty, -1/2)$.

Inspecting the original definition of $P_{(\theta,\nu)}$ we see that the kernel $1/(1-\theta x)$ is positive on the support of ν for θ from $\Theta = (0,\infty)$, which was the set used in the definition, but it is also well defined for $\theta < -1/4$. So this extension "includes" this second set, with $m = -1$ corresponding to infinite values of θ.

At $m = -1/2$ the integrand has singularity at $x = -1/4$ but the integral is still 1, see the calculation below. For $m > -1/2$, the mass becomes less then one, as $\int Q_m(dx) = m^2/(1+m)^2$. So for $m > -1/2$ we can define a new probability measure

$$\begin{aligned}\overline{Q}_{(m,\nu)}(dx) &= Q_{(m,\nu)}(dx) + \left(1 - \frac{m^2}{(1+m)^2}\right)\delta_{m+m^2}(dx) \\ &= Q_{(m,\nu)}(dx) + \frac{(1+2m)}{(1+m)^2}\delta_{m+m^2}(dx) \qquad (3.11)\end{aligned}$$

with the extra mass in the atomic part, which located at $m + m^2$ so that the mean is preserved.

We now prove the above two claims in Example 3.10.

Proof. By the change of variable $t = \sqrt{-1-4x}$ in

$$\int Q_{(m,\nu)}(dx) = \int_{-\infty}^{-1/4} \frac{m^2\sqrt{-1-4x}}{2\pi x^2(m^2+m-x)}dx,$$

we obtain

$$\int Q_{(m,\nu)}(dx) = \frac{16m^2}{\pi}\int_0^{+\infty} \frac{t^2}{(t^2+1)^2((2m+1)^2+t^2)}dt.$$

The integrand can be decomposed as follows

$$\begin{aligned}\frac{t^2}{(t^2+1)^2((2m+1)^2+t^2)} &= \frac{(2m+1)^2}{((2m+1)^2-1)^2(t^2+1)} \\ &- \frac{1}{((2m+1)^2-1)(t^2+1)^2}\end{aligned}$$

$$-\frac{(2m+1)^2}{((2m+1)^2-1)^2(t^2+(2m+1)^2)}.$$

For real numbers $a,\ b,\ r\neq 0$, we denote $J_n=\int_a^b \frac{dx}{(x^2+r^2)^n}$. Then we have

$$J_{n+1}=\frac{1}{2nr^2}\left((2n-1)J_n+\left[\frac{x}{(x^2+r^2)^n}\right]_a^b\right).$$

Using this, we get:

For $m=-1/2$,

$$\int Q_{(-1/2,\nu)}(dx)=\frac{4}{\pi}\int_0^{+\infty}\frac{1}{(t^2+1)^2}dt=\frac{4}{\pi}\left(\frac{1}{2}\left(\frac{\pi}{2}+\left[\tfrac{x}{1+x^2}\right]_0^{+\infty}\right)\right)=1.$$

For $m\neq -1/2$

$$\begin{aligned}\int Q_{(m,\nu)}(dx) &= \frac{16m^2}{\pi}\int_0^{+\infty}\frac{t^2}{(t^2+1)^2((2m+1)^2+t^2)}dt\\ &= \frac{16m^2}{\pi}\Big[\int\frac{(2m+1)^2}{((2m+1)^2-1)^2(t^2+1)}dt\\ &- \int\frac{1}{((2m+1)^2-1)(t^2+1)^2}dt\\ &- \int\frac{(2m+1)^2}{((2m+1)^2-1)^2(t^2+(2m+1)^2)}dt\Big]\\ &= \frac{16m^2}{\pi}\Big[\frac{(2m+1)^2}{((2m+1)^2-1)^2}[\arctan(t)]_0^{+\infty}\\ &- \frac{1/2}{(2m+1)^2-1}\left(\frac{\pi}{2}+\left[\tfrac{t}{(t^2+1)^2}\right]_0^{+\infty}\right)\\ &- \frac{(2m+1)}{((2m+1)^2-1)^2}\left[\arctan(\tfrac{t}{2m+1})\right]_0^{+\infty}\Big].\end{aligned}$$

If $m<-1/2$

$$\begin{aligned}\int Q_{(m,\nu)}(dx) &= \frac{16m^2}{\pi}(\frac{(2m+1)^2}{((2m+1)^2-1)^2}\frac{\pi}{2}-\frac{1}{((2m+1)^2-1)}\frac{\pi}{4}\\ &- \frac{(2m+1)}{((2m+1)-1)^2}(-\frac{\pi}{2}))=1.\end{aligned}$$

If $m>-1/2$

$$\begin{aligned}\int Q_{(m,\nu)}(dx) &= \frac{16m^2}{\pi}\Big(\frac{(2m+1)^2}{((2m+1)^2-1)^2}\frac{\pi}{2}-\frac{1}{((2m+1)^2-1)}\frac{\pi}{4}\\ &- \frac{(2m+1)}{((2m+1)-1)^2}\frac{\pi}{2}\Big)=\frac{m^2}{(1+m)^2}.\end{aligned}$$

We now verify that the atomic part works as needed. By the change of variable $t=\sqrt{-1-4x}$ from (3.11) we get

$$\begin{aligned}
\int xQ_{(m,\nu)}(dx) &= \int_{-\infty}^{-1/4} \frac{m^2\sqrt{-1-4x}}{2\pi x(m^2+m-x)}dx \\
&= -\frac{4m^2}{\pi}\int_0^{+\infty} \frac{t^2}{(t^2+1)((2m+1)^2+t^2)}dt \\
&= -\frac{4m^2}{\pi}\Big(-\int_0^{+\infty} \frac{1}{4m(1+m)(t^2+1)}dt \\
&+ \int_0^{+\infty} \frac{(2m+1)^2}{4(m^2+m)((2m+1)^2+t^2)}dt\Big) \\
&= -\frac{4m^2}{\pi}\left(\left[\frac{-1}{4m(1+m)}\arctan(t)\right]_0^{+\infty} \right. \\
&+ \left. \left[\frac{(2m+1)}{4m(1+m)}\arctan\left(\frac{t}{2m+1}\right)\right]_0^{+\infty} \right) \\
&= -\frac{4m^2}{\pi}\left(\frac{-1}{4m(1+m)}\frac{\pi}{2}+\frac{(2m+1)}{4m(1+m)}\frac{\pi}{2}\right) = -\frac{m^2}{1+m}.
\end{aligned}$$

So

$$\int x\overline{Q}_{(m,\nu)}(dx) = -\frac{m^2}{1+m}+\frac{1-2m}{(1+m)^2}m(1+m)=m$$

as expected. □

We now give a general theory that shows how the two-step extension works.

3.3.1. The first extension

Suppose that the pseudo-variance function $\mathbb{V}_\nu(.)$ extends as a real analytic function to $(m_0,+\infty)$. Denote by $A=A(\nu)=\sup\operatorname{supp}(\nu)$, recall notation (2.9) and define

$$\mathbf{m}_+(\nu)=\inf\left\{m>m_0 : m+\frac{\mathbb{V}(m)}{m}=A(\nu)\right\}. \tag{3.12}$$

We know that $\mathbf{m}_+(\nu)\geq m_+$ is well defined. We will verify that one can use $Q_{(m,\nu)}(dx)=f_\nu(x,m)\nu(dx)$ given by (2.17) to extend the domain of means to $(m_0,\mathbf{m}_+(\nu))$, preserving the pseudo-variance function. (The definition (2.13) of pseudo-variance is not directly applicable beyond $m>m_+$, so we use an equivalent definition).

Theorem 3.11. *[7, Theorem 3.4] Formula (2.17) defines the family of probability measures*

$$\left\{Q_{(m,\nu)}(dx)=f_\nu(x,m)\nu(dx) : m\in(m_0,\mathbf{m}_+)\right\},$$

parameterized by the mean $m = \int xQ_{(m,\nu)}(dx)$. *The Cauchy-Stieltjes transform of the generating measure* ν *satisfies* (2.16) *with* z *given by* (2.15) *for all* $m \in (m_0, \mathbf{m}_+)$. *In particular, if* ν *has finite first moment* m_0 *then for* $m \in (m_0, \mathbf{m}_+)$ *the variance of* $Q_{(m,\nu)}(dx)$ *is given by* (2.13).

The rest of this paragraph contains proof of Theorem 3.11. We consider the set Θ for which the transform (2.6) exists. In fact, if $A(\nu) \geq 0$, then $\Theta = (0, \theta_+)$ with $\theta_+ = \frac{1}{B}$, and if $A(\nu) < 0$, then

$$\Theta = \left(-\infty, \tfrac{1}{A(\nu)}\right) \cup (0, \infty). \tag{3.13}$$

One can always write

$$\Theta = \left(\frac{sign(A(\nu))}{B}, \frac{1}{A(\nu)}\right) \cup \left(0\ , \frac{1}{B}\right)$$

with

$$sign(A(\nu)) = \begin{cases} 1, \text{ if } \ A(\nu) \geq 0 & ; \\ -1, \text{ if } \ A(\nu) < 0 & . \end{cases}$$

One can then define the first extension of $\mathcal{K}_+(\nu)$ as

$$\overline{\mathcal{K}}_+(\nu) = \Bigg\{ P_{(\theta,\nu)}(dx) = \frac{1}{M_\nu(\theta)(1-\theta x)}\nu(dx)\ ;$$
$$\theta \in \left(\frac{sign(A(\nu))}{B}, \frac{1}{A(\nu)}\right) \cup \left(0, \frac{1}{B}\right) \Bigg\}.$$

Note that $\overline{\mathcal{K}}_+(\nu) = \mathcal{K}_+(\nu)$ when $A(\nu) \geq 0$, because in this case

$$\left(\frac{sign(A(\nu))}{B}, \frac{1}{A(\nu)}\right) = \emptyset.$$

Therefore, the first extension is non-trivial only when $A(\nu) < 0$.

Proposition 3.12. *[7, Proposition 3.5] Suppose* $A(\nu) < 0$. *For* $\theta \in \Theta = \left(-\infty, \frac{1}{A(\nu)}\right) \cup (0, \infty)$ *the mean*

$$k(\theta) = \int xP_{(\theta,\nu)}(dx) = \frac{M_\nu(\theta) - 1}{\theta M_\nu(\theta)}, \tag{3.14}$$

is strictly increasing on $(0, \infty)$ *and on* $\left(-\infty, \frac{1}{A(\nu)}\right)$.

Proof. It is known ([10]) that the function $k_\nu(.)$ is strictly increasing on $(0, \infty)$, we will use the same reasoning to show that it is also increasing on $\left(-\infty, \frac{1}{A(\nu)}\right)$. We first observe that for $\theta \in \left(-\infty, \frac{1}{A(\nu)}\right)$, the expression $(1 - \theta x)$ is negative for all x in the support of ν. In fact, $x < A(\nu)$ implies that $\theta x > \theta A(\nu) > 1$, that is $1 - \theta x < 1 - \theta A(\nu) < 0$. Hence

$$\int \frac{|x|}{(1-\theta x)^2}\nu(dx) \quad = \quad \frac{1}{|\theta|}\int \frac{|\theta x - 1 + 1|}{(1-\theta x)^2}\nu(dx)$$

$$\begin{aligned}
&\leq -\frac{1}{\theta}\int \frac{|\theta x-1|}{(1-\theta x)^2}\nu(dx) + -\frac{1}{\theta}\int \frac{1}{(1-\theta x)^2}\nu(dx)\\
&\leq \frac{M_\nu(\theta)}{\theta} + (-\frac{1}{\theta})\frac{M_\nu(\theta)}{1-\theta A(\nu)} < \infty.
\end{aligned}$$

Now fix $-\infty < \alpha < \beta < 1/A(\nu)$. For $x \in supp(\nu) \subset (-\infty, 0)$, the function

$$\theta \longmapsto \frac{\partial}{\partial\theta}\left(\frac{1}{1-\theta x}\right) = \frac{x}{(1-\theta x)^2}$$

is decreasing on $\left(-\infty, \frac{1}{A(\nu)}\right)$, so for all $\theta \in [\alpha,\ \beta]$,

$$\frac{x}{(1-\beta x)^2} \leq \frac{x}{(1-\theta x)^2} \leq \frac{x}{(1-\alpha x)^2}.$$

We define for $x \in supp(\nu)$

$$g(x) = \frac{|x|}{(1-\alpha x)^2} + \frac{|x|}{(1-\beta x)^2}.$$

Then $g \geq 0$, and g is ν-integrable, because α and β are in $\left(-\infty, \frac{1}{A(\nu)}\right)$, and $\frac{\partial}{\partial\theta}\left(\frac{1}{1-\theta x}\right) = \frac{x}{(1-\theta x)^2} \leq g(x)$, for all $\theta \in [\alpha,\ \beta]$. Thus, one can differentiate $M_\nu(\theta)$ under the integral sign and formula (2.7) gives

$$k'_\nu(\theta) = \frac{M_\nu(\theta) + \theta M'_\nu(\theta) - M_\nu(\theta)^2}{(\theta M_\nu(\theta))^2}.$$

The fact that

$$M_\nu(\theta) + \theta M'_\nu(\theta) - M_\nu(\theta)^2 = \int \frac{1}{(1-\theta x)^2}\nu(dx) - \left(\int \frac{1}{1-\theta x}\nu(dx)\right)^2 \geq 0$$

implies that the function $\theta \longmapsto k_\nu(\theta)$ is increasing on $\left(-\infty, \frac{1}{A(\nu)}\right)$.

We have that

$$\begin{aligned}
\lim_{\theta\to-\infty} k_\nu(\theta) &= \lim_{\theta\to-\infty} \frac{M_\nu(\theta)-1}{\theta M_\nu(\theta)} = \lim_{\theta\to-\infty} \frac{\frac{1}{\theta}G_\nu(\frac{1}{\theta})-1}{G_\nu(\frac{1}{\theta})}\\
&= 0 - \frac{1}{G_\nu(0)} = B - \frac{1}{G_\nu(B)} = m_+.
\end{aligned}$$

For the proof of Theorem 3.11 instead of using (3.12), we define

$$\mathbf{m}_+(\nu) = \lim_{\theta\to\frac{1}{A(\nu)}} k_\nu(\theta). \tag{3.15}$$

(We will later verify that this coincides with (3.12) when $A(\nu) < 0$.) Then, the function $k_\nu(.)$ realizes a bijection from $\left(-\infty, \frac{1}{A(\nu)}\right)$ onto its image $(m_+, \mathbf{m}_+(\nu))$.

We then define the function ψ_ν on (m_0, m_+) as the inverse of the restriction of $k_\nu(.)$ to $(0,\infty)$, and on $(m_+, \mathbf{m}_+(\nu))$ as the inverse of the restriction of $k_\nu(.)$ to $\left(-\infty, \frac{1}{A(\nu)}\right)$. This leads to the parametrization by the mean $m \in (m_0, m_+) \cup (m_+, \mathbf{m}_+(\nu))$ of the family $\overline{\mathcal{K}}_+(\nu)$. The definition of the pseudo-variance function can also be extended using the function ψ_ν. Following (2.13), we define $\mathbb{V}_\nu(.)$ for $m \in (m_0, m_+) \cup (m_+, \mathbf{m}_+(\nu))$ as

$$\mathbb{V}_\nu(m) = m\left(\frac{1}{\psi_\nu(m)} - m\right).$$

We have that

$$\lim_{m \longrightarrow (m_+)^-} \frac{1}{\psi_\nu(m)} = 0 = \lim_{m \longrightarrow (m_+)^+} \frac{1}{\psi_\nu(m)},$$

so that we define $\mathbb{V}_\nu(.)$ at m_+ by $\mathbb{V}_\nu(m_+) = -m_+^2$. Note that $Q_{(m_+,\nu)}(dx) = \frac{m_+}{x}\nu(dx)$ is well defined for $A(\nu) < 0$.

The explicit parametrization by the means of the enlarged family can then be given by

$$\overline{\mathcal{K}}_+(\nu) = \left\{Q_{(m,\nu)}(dx) = f_\nu(x,m)\nu(dx)\ ;\ m \in (m_0, \mathbf{m}_+(\nu))\right\}.$$

The function $m \longmapsto \psi_\nu(m) = \frac{1}{\mathbb{V}_\nu(m)/m+m}$ is increasing on $(m_+, \mathbf{m}_+(\nu))$, so the function $m \longmapsto \mathbb{V}_\nu(m)/m + m$ is decreasing on $(m_+, \mathbf{m}_+(\nu))$ and

$$\lim_{m \longrightarrow \mathbf{m}_+(\nu)} \mathbb{V}_\nu(m)/m + m = A(\nu).$$

This implies that (3.12) holds when $A(\nu) < 0$.

If $A(\nu) \geq 0$, then (3.12) gives $\mathbf{m}_+(\nu) = m_+$ because $m_+ + \frac{\mathbb{V}_\nu(m_+)}{m_+} = \frac{1}{\theta_+} = B = A(\nu)$, and then $\overline{\mathcal{K}}_+(\nu) = \mathcal{K}_+(\nu)$. This ends the proof of Theorem 3.11. □

3.3.2. Domain of means under affine transformation

Let φ an affine transformation. It is well known that the lower end of the one sided domain of means of the family $\mathcal{K}_+(\nu)$ behave nicely under the action of affine transformation φ, that is $m_0(\varphi(\nu)) = \varphi(m_0(\nu))$. But we do not have a general formula of the upper end for the natural domain of means for $\mathcal{K}_+(\nu)$. The following examples show that there is no simple formula for $m_+(\nu)$ under affine transformation.

Example 3.13. Consider the inverse semicircle distribution

$$\nu(dx) = \frac{\sqrt{-1-4x}}{2\pi x^2}\mathbf{1}_{(-\infty,-\frac{1}{4})}(x)dx. \tag{3.16}$$

It generates the CSK family with pseudo-variance function $\mathbb{V}_\nu(m) = m^3$, and the domain of means is $\mathcal{D}_+(\nu) = (m_0(\nu), m_+(\nu)) = (-\infty, -1)$. The image $\varphi(\nu)$

of ν by the map $x \longmapsto \varphi(x) = x + 1/2$ is

$$\varphi(\nu)(dx) = \frac{\sqrt{1-4x}}{2\pi(x-1/2)^2}\mathbf{1}_{(-\infty,\frac{1}{4})}(x)dx. \tag{3.17}$$

and it generates the CSK family with pseudo-variance function $\mathbb{V}_{\varphi(\nu)}(m) = m(m-1/2)^2$. We have that

$$\psi_{\varphi(\nu)}(m) = \frac{1}{m^2+1/4} \qquad \text{and} \qquad k_{\varphi(\nu)}(\theta) = -\frac{\sqrt{\theta(4-\theta)}}{2\theta}$$

for all θ in $\Theta(\varphi(\nu)) = (0,4)$. The domain of means is $(m_0(\varphi(\nu)), m_+(\varphi(\nu))) = (-\infty, 0)$. In this case we have $m_+(\varphi(\nu)) = 0 \neq -1/2 = \varphi(m_+(\nu))$.

Example 3.14. Consider the (two-sided) CSK family generated by the semi-circle law

$$\nu(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{|x|<2}(dx) \tag{3.18}$$

with the variance function $V_\nu(m) = \mathbb{V}_\nu(m) = 1$ and domain of means $(-1,1)$, that is

$$\mathcal{K}(\nu) = \left\{\frac{\sqrt{4-x^2}}{2\pi(1+m(m-x))}\mathbf{1}_{|x|<2}dx : m \in (-1,1)\right\}.$$

The image $\varphi(\nu)$ of ν by the map $x \longmapsto \varphi(x) = x - 3$ is

$$\varphi(\nu)(dx) = \frac{\sqrt{-(x+1)(x+5)}}{2\pi}\mathbf{1}_{(-5,-1)}(x)dx. \tag{3.19}$$

With $B(\varphi(\nu)) = \max\{0, \sup supp(\varphi(\nu))\} = 0$, the (two-sided) range of parameter is $(\theta_-, \theta_+) = (-1/5, +\infty)$. The probability measure $\varphi(\nu)$ generates the (two-sided) CSK family

$$\begin{aligned}\mathcal{K}(\varphi(\nu)) &= \{P_{(\theta,\varphi(\nu))}(dx); \theta \in (-1/5, +\infty)\} \\ &= \{Q_{(m,\nu)}(dx), m \in (m_-(\varphi(\nu)), m_+(\varphi(\nu)))\}\end{aligned}$$

with pseudo-variance function $\mathbb{V}_{\varphi(\nu)}(m) = \frac{m}{m+3}$. We have that

$$\psi_{\varphi(\nu)}(m) = \frac{m+3}{m^2+3m+1} \qquad \text{and} \qquad k_{\varphi(\nu)}(\theta) = \frac{(1-3\theta) - \sqrt{(\theta+1)(5\theta+1)}}{2\theta}.$$

We have

$$\begin{aligned}m_+(\varphi(\nu)) &= \lim_{\theta \longrightarrow +\infty} k_{\varphi(\nu)}(\theta) = \lim_{\theta \longrightarrow +\infty} \frac{(1-3\theta) - \sqrt{(\theta+1)(5\theta+1)}}{2\theta} \\ &= -\frac{3+\sqrt{5}}{2} \neq -2 = \varphi(m_+(\nu)).\end{aligned}$$

The purpose is to give a more natural definition for the domain of means of a CSK family that behave nicely under affine transformation. In several references, we consider the range of the parameter θ such that $1/\theta \in (\sup supp\ \nu, \infty) \cap [0, \infty)$. In fact authors in [10] have pushed forward the theory of CSK families by extending the results in [6] to allow measures ν with unbounded support. In such situation, the family is parameterized by a 'one-sided' range of θ of a fixed sign, so that generating measures have support bounded from above and the CSK families are parameterized by $\theta > 0$, which gives the domain of means (m_0, m_+). We can include additional range of θ which is possible only when the support of ν is in $(-\infty, 0)$. In this case we can include additional range of $1/\theta \in (\sup supp\ \nu, 0)$, so the extended range of θ would have a simpler description

$$\Theta(\nu) = \{\theta;\ 1/\theta \in (\sup supp\ \nu, \infty)\},$$

that is, $\Theta(\nu)$ is the set for which the transform M_ν exists and, with $A = A(\nu) = \sup \mathrm{supp}(\nu)$, it can be written as

$$\Theta(\nu) = \begin{cases} (-\infty, 1/A) \cup (0, \infty), & \text{if } A < 0; \\ (0, 1/A), & \text{if } A \geq 0. \end{cases}$$

This extension for the range of the parameter θ was considered in the first extension.

It is worth mentioning here that if ν is the inverse semicircle distribution, from example 3.13, given by (3.16). The image $\varphi(\nu)$ of ν by the map $x \longmapsto \varphi(x) = x + 1/2$ is given by (3.17). We have that $\Theta(\nu) = (-\infty, -4) \cup (0, +\infty)$. We have that

$$\overline{\mathcal{K}}_+(\nu) = \{P_{(\theta,\nu)}(dx),\ \theta \in \Theta(\nu)\} = \{Q_{(m,\nu)}(dx), m \in (m_0, \mathbf{m}_+(\nu))\},$$

with $\mathbf{m}_+(\nu) = -1/2$. We have that

$$\mathbf{m}_+(\varphi(\nu)) = m_+(\varphi(\nu)) = 0 = \varphi(\mathbf{m}_+(\nu)).$$

Also, if ν is the semicircle distribution, from example 3.14, given by (3.18). The image $\varphi(\nu)$ of ν by the map $x \longmapsto \varphi(x) = x - 3$ is given by (3.19). We have that $\mathbf{m}_+(\nu) = m_+(\nu) = 1$ and $\Theta(\varphi(\nu)) = (-\infty, -1) \cup (-1/5, +\infty)$. The CSK family generated by $\varphi(\nu)$ is

$$\begin{aligned} \overline{\mathcal{K}}_+(\varphi(\nu)) &= \{P_{(\theta,\varphi(\nu))}(dx); \theta \in (-\infty, -1) \cup (-1/5, +\infty)\} \\ &= \{Q_{(m,\nu)}(dx), m \in (m_-(\varphi(\nu)), \mathbf{m}_+(\varphi(\nu)))\}, \end{aligned}$$

with

$$\begin{aligned} \mathbf{m}_+(\varphi(\nu)) &= \lim_{\theta \longrightarrow -1} k_{\varphi(\nu)}(\theta) = \lim_{\theta \longrightarrow -1} \frac{(1-3\theta) - \sqrt{(\theta+1)(5\theta+1)}}{2\theta} \\ &= -2 = \varphi(\mathbf{m}_+(\nu)). \end{aligned}$$

Given a probability measure ν with support bounded from above and an affine transformation φ, the following result gives the link between the mean function of the CSK family generated by $\varphi(\nu)$ and the the mean function of the CSK family generated by ν.

Proposition 3.15. *[18, Proposition 3.3] Let ν be a non degenerate probability measure with support bounded from above and let $\varphi(\nu)$ be the image of ν by the map $\varphi : x \longmapsto \alpha x + \beta$. If $\Theta(\nu)$ and $\Theta(\varphi(\nu))$ are respectively the sets for which the transforms M_ν and $M_{\varphi(\nu)}$ exists and $h : x \longmapsto 1/x$, then*

$$h(\Theta(\varphi(\nu))) = \varphi(h(\Theta(\nu))), \tag{3.20}$$

and for $\theta \in \Theta(\varphi(\nu))$,

$$k_{\varphi(\nu)}(\theta) = \begin{cases} \varphi(-1/G_\nu(0)), & if \quad \theta = 1/\beta \quad ; \\ \varphi\left(k_\nu\left(\dfrac{\theta\alpha}{1-\theta\beta}\right)\right), & if \quad \theta \neq 1/\beta \quad . \end{cases} \tag{3.21}$$

The following result prove that the domain of means of the extended CSK family behave nicely under affine transformation in a manner analogous to the domain of means for NEFs. We also consider how $m_+(\nu)$ gets transformed under affine transformation, (see [18] for more details).

Theorem 3.16. *[18, Theorem 3.4] Consider a probability measure ν with support bounded from above and let $\varphi(\nu)$ be the image of ν by the map $\varphi : x \longmapsto \alpha x + \beta$, where $\alpha \in \mathbb{R}\backslash\{0\}$ and $\beta \in \mathbb{R}$.*

(A) Suppose that $\alpha > 0$. The domain of means of $\overline{\mathcal{K}}_+(\varphi(\nu))$ is $(m_0(\varphi(\nu)), \mathbf{m}_+(\varphi(\nu)))$ with $\mathbf{m}_+(\varphi(\nu)) = \varphi(\mathbf{m}_+(\nu))$. Furthermore:

(a) If $\beta = 0$, then $m_+(\varphi(\nu)) = \varphi(m_+(\nu))$.

(b) If $\beta \neq 0$, we have that

(i) If $A \geq 0$ and $\alpha A + \beta \geq 0$, then $m_+(\varphi(\nu)) = \varphi(m_+(\nu))$.
(ii) If $A \geq 0$ and $\alpha A + \beta < 0$, then $m_+(\varphi(\nu)) = \alpha k_\nu(-\alpha/\beta) + \beta$.
(iii) If $A < 0$ and $\alpha A + \beta \geq 0$, then $m_+(\varphi(\nu)) = \varphi(\mathbf{m}_+(\nu))$.
(iv) If $A < 0$ and $\alpha A + \beta < 0$, then $m_+(\varphi(\nu)) = \alpha k_\nu(-\alpha/\beta) + \beta$.

(B) Suppose that $\alpha < 0$. The probability measure $\varphi(\nu)$ has support bounded from below and we are dealing with left sided CSK family. The domain of means of $\overline{\mathcal{K}}_-(\varphi(\nu))$ is $(\mathbf{m}_-(\varphi(\nu)), m_0(\varphi(\nu)))$ with $\mathbf{m}_-(\varphi(\nu)) = \varphi(\mathbf{m}_+(\nu))$. Furthermore,

(a) If $\beta = 0$, then $m_-(\varphi(\nu)) = \varphi(m_+(\nu))$.

(b) If $\beta \neq 0$, we have that

(i) If $A \geq 0$ and $\alpha A + \beta \geq 0$, then $m_-(\varphi(\nu)) = \alpha k_\nu(-\alpha/\beta) + \beta$.
(ii) If $A \geq 0$ and $\alpha A + \beta < 0$, then $m_-(\varphi(\nu)) = \varphi(m_+(\nu))$.
(iii) If $A < 0$ and $\alpha A + \beta \geq 0$, then $m_-(\varphi(\nu)) = \alpha k_\nu(-\alpha/\beta) + \beta$.
(iv) If $A < 0$ and $\alpha A + \beta < 0$, then $m_-(\varphi(\nu)) = \varphi(\mathbf{m}_+(\nu))$.

3.3.3. The second extension

As indicated by Example 3.14 and Example 3.10, family $\overline{\mathcal{K}}_+(\nu)$ may have a further extension. This extension is possible if the density (2.17) is non-negative, that is, if $m + \mathbb{V}_\nu(m)/m \geq A$ and $\mathbb{V}_\nu(m)/m \geq 0$ for all $m > m_0$. Define

$$\mathbf{M}_+ = \inf\{m > m_0 : \mathbb{V}_\nu(m)/m < 0\}. \tag{3.22}$$

It is clear that $\mathbf{M}_+ \geq m_+$. In fact, $\mathbf{M}_+ \geq \mathbf{m}_+$. This can be seen from (3.12): since the mean must be smaller than $A(\nu)$ we have $\mathbf{m}_+ \leq A(\nu)$, so $\mathbb{V}_\nu(m)/m \geq 0$ for all $m < \mathbf{m}_+$. It is easy to see that $\mathbf{M}_+ = \infty > \mathbf{m}_+$ in Example 3.14 and in Example 3.10 while $\mathbf{M}_+ = \mathbf{m}_+ = m_+$ in Example 3.9.

We now introduce the second extension of $\mathcal{K}_+(\nu)$ as the family of measures

$$\overline{\overline{\mathcal{K}}}_+(\nu) = \{\overline{Q}_{(m,\nu)}(dx) : \ m \in (m_0, \mathbf{m}_+) \cup (\mathbf{m}_+, \mathbf{M}_+(\nu))\},$$

with $\overline{Q}_{(m,\nu)}$ given by

$$\overline{Q}_{(m,\nu)}(dx) = f_\nu(x,m)\nu(dx) + p(m)\delta_{m+\mathbb{V}_\nu(m)/m}, \tag{3.23}$$

where the weight of the atom is

$$p(m) = \begin{cases} 0 & \text{if } \ m < m_+ := B - \frac{1}{G_\nu(B)}, \\ 1 - \frac{\mathbb{V}_\nu(m)}{m} G_\nu\left(m + \frac{\mathbb{V}_\nu(m)}{m}\right) & \text{if } \ m > m_+ \text{ and } \ \mathbb{V}_\nu(m)/m \geq 0. \end{cases}$$

It is clear that the expression on the right hand side of (3.23) is well defined at all m such that $m + \mathbb{V}_\nu(m)/m > A$. We need to show that the expression is well defined also at the points where $m + \mathbb{V}_\nu(m)/m = A$; one such point is of course $\mathbf{m}_+$. The argument here relies on the fact that G_ν is analytic in the slit plane $\mathbb{C}\backslash(-\infty, A)$. Furthermore, $G_\nu(a)$ is decreasing to 0 and convex on (A, ∞). In particular $\lim_{a\longrightarrow A^+} G_\nu(a)$ exists, and is either ∞ or $\mathbf{m}_+/\mathbb{V}_\nu(\mathbf{m}_+)$. Furthermore, if the limit is ∞, then $\mathbb{V}(\mathbf{m}_+)/\mathbf{m}_+ = 0$, which implies that $\mathbf{M}_+ = \mathbf{m}_+$. So without loss of generality we may assume that $\lim_{a\longrightarrow A^+} G_\nu(a) = \mathbf{m}_+/\mathbb{V}_\nu(\mathbf{m}_+) < \infty$. and that the integral defining $G_\nu(A)$ converges.

Suppose $m_1 < \mathbf{M}_+$ such that $m_1 + \mathbb{V}_\nu(m_1)/m_1 = A$. Then $\mathbb{V}_\nu(m_1)/m_1 = A - m_1 > 0$ and, taking the limit,

$$p(m_1) = 1 - \frac{A - m_1}{A - \mathbf{m}_+} = \frac{m_1 - \mathbf{m}_+}{A - \mathbf{m}_+} \in [0, 1).$$

On the other hand,

$$\frac{\mathbb{V}_\nu(m_1)}{\mathbb{V}_\nu(m_1) + m_1(m_1 - x)} = \frac{A - m_1}{A - x}.$$

Therefore, for $m \in [\mathbf{m}_+, \mathbf{M}_+)$ such that $m + \mathbb{V}_\nu(m)/m = A$, the right hand side of (3.23) is well defined and simplifies to

$$\overline{Q}_{(m,\nu)}(dx) = \frac{A - m}{A - x}\nu(dx) + \frac{m - \mathbf{m}_+}{A - \mathbf{m}_+}\delta_A.$$

Then, the second extension of the family is given by

$$\overline{\overline{\mathcal{K}}}_+(\nu) = \{\overline{Q}_{(m,\nu)}(dx) : \ m \in (m_0, \mathbf{M}_+(\nu))\},$$

Since formula (2.16) holds for all $m \in (m_0, \mathbf{m}_+)$, it is clear that $\overline{\mathcal{K}}_+(\nu) \subset \overline{\overline{\mathcal{K}}}_+(\nu)$. We now verify that the extension satisfies desired conditions.

Theorem 3.17. *[7, Theorem 3.6]Suppose $\mathbb{V}_\nu(\cdot)$ is such that $m+\mathbb{V}_\nu(m)/m \geq A$ for all $m > m_0$. Let $\mathbf{m}_+ < m < \mathbf{M}_+$. Then (3.23) defines a probability measure $\overline{Q}_{(m,\nu)}(dx)$ with mean m, and if ν has finite first moment m_0 then the variance of $\overline{Q}_{(m,\nu)}$ is*

$$\int (x-m)^2 \overline{Q}_{(m,\nu)}(dx) = \frac{(m-m_0)\mathbb{V}_\nu(m)}{m}. \tag{3.24}$$

Here the use of $\mathbb{V}_\nu(m)$ is based on the assumption the pseudo-variance function $\mathbb{V}_\nu$ extends as a real analytic function to $(m_0, +\infty)$.

Example 3.18. Consider the (absolutely continuous, centered) Marchenko-Pastur law

$$\nu(dx) = \frac{\sqrt{4-(x-a)^2}}{2\pi(1+ax)} 1_{(a-2,a+2)}(x)dx$$

with $a^2 < 1$. The variance function is $V_\nu(m) = 1+am = \mathbb{V}_\nu(m)$, and the domain of means is $D_+(\nu) = (0,1)$, with $\Theta = (0, \theta_+) = (0, 1/(2+a))$ and $A = 2+a$. Here $\mathbf{m}_+(\nu) = m_+ = 1$ and the function $m \mapsto m + \mathbb{V}_\nu(m)/m$ is convex on $(0,\infty)$ with minimum at $m = \mathbf{m}_+$. Since $\mathbb{V}_\nu(m)/m > 0$ for all $m > m_0 = 0$, we have $\mathbf{M}_+ = \infty$.

3.3.4. Domain of means under free additive convolution power

One notes that the lower end of the one sided domain of means of the family $\mathcal{K}_+(\nu)$ satisfies the relation, for $\alpha > 0$

$$m_0(\nu^{\boxplus\alpha}) = \alpha m_0(\nu),$$

but, we do not have a general formula of the upper end of the natural domain of the means for $\nu^{\boxplus\alpha}$. The following examples show that there is no "one simple formula" for m_+ under free additive convolution power.

Example 3.19. Let ν be the symmetric Bernoulli distribution. Consider $\mu = \nu \boxplus \nu$, (Then μ is the arcsine law.)

Here, $B(\mu) = 2$ and $\mathbb{V}_\mu(m) = 2\mathbb{V}_\nu(m/2) = 2 - m^2/2$. We compute G_μ the Cauchy Stieltjes transform of μ from $\mathbb{V}_\mu$ and (2.16). Solving the equation

$$z = z(m) = m + \mathbb{V}_\mu(m)/m = \frac{4+m^2}{2m},$$

we obtain $m = z - \sqrt{z^2-4}$. This gives that

$$G_\mu(z) = \frac{1}{\sqrt{z^2-4}}$$

so from (2.10) we get $m_+(\mu) = \lim_{B\to 2}(B - 1/G_\mu(B)) = 2$. So in this case

$$m_+(\nu^{\boxplus 2}) = 2m_+(\nu).$$

Example 3.20. Consider the semicircle law

$$\nu(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\ \mathbf{1}_{|x|<2}dx.$$

We have that $\mathbb{V}_\nu(m) = 1$ and $m_+(\nu) = 1$. Then $\mu = \nu \boxplus \nu$ is the semicircle law with density

$$\mu(dx) = \frac{1}{4\pi}\sqrt{8-x^2}\ \mathbf{1}_{|x|<2\sqrt{2}}dx.$$

The CSK family generated by μ has a pseudo-variance function $\mathbb{V}_\mu(m) = 2$. Since $\mathbb{V}_\mu(m)$ is quadratic and $m_0 = 0$, formula (2.10) gives $m_+(\mu) = \sqrt{2}$. In this case, we have

$$m_+(\nu^{\boxplus 2}) \neq 2m_+(\nu).$$

It is well known that the domain of means for exponential families scales nicely under classical additive convolution power, and it is satisfying to note that the domain of means of the extended CSK family $\overline{\overline{\mathcal{K}}}_+(\nu)$ lead to the analogous formula:

$$\mathbf{M}_+(\nu^{\boxplus\alpha}) = \alpha\mathbf{M}_+(\nu).$$

Indeed, since $\mathbb{V}_{\nu^{\boxplus\alpha}}(m) = \alpha\mathbb{V}_\nu(m/\alpha)$, the result follows from (3.22).

4. Boolean additive convolution

Let $\nu \in \mathcal{M}$. The boolean additive convolution is determined by the K-transform K_ν of ν which is given by

$$K_\nu(z) = z - \frac{1}{G_\nu(z)}, \qquad \text{for } z \in \mathbb{C}^+. \tag{4.1}$$

The function K_ν is usually called self energy and it represent the analytic backbone of boolean additive convolution. For two probability measures μ and ν in $\mathcal{M}$, the boolean additive convolution $\mu \uplus \nu$ is determined by

$$K_{\mu\uplus\nu}(z) = K_\mu(z) + K_\nu(z), \qquad \text{for } z \in \mathbb{C}^+, \tag{4.2}$$

and $\mu \uplus \nu$ is again a probability measure.

According to [32], we call a probability measure $\nu \in \mathcal{M}$ is infinitely divisible in the boolean sense, if for each $n \in \mathbb{R}$, there exists $\nu_n \in \mathcal{M}$ such that

$$\nu = \underbrace{\nu_n \uplus \ldots\ldots \uplus \nu_n}_{n \text{ times}}.$$

Note that all probability measure $\nu \in \mathcal{M}$ are $\uplus$-infinitely divisible, see [32, Theorem 3.6].

4.1. Boolean additive convolution and variance function

In this paragraph, we deal with boolean additive convolution from the perspective of CSK families. Next, we give the formula for pseudo-variance function (and variance function V_ν in case of existence) under boolean additive convolution power.

Theorem 4.1. *[16, Theorem 2.3] Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_+(\nu)$ generated by a non degenerate probability measure ν with support bounded from above and mean $m_0(\nu)$. For $\alpha > 0$, we have that:*

(i) The support of $\nu^{\uplus\alpha}$ is bounded from above.
(ii) For m close enough to $\alpha m_0(\nu)$,

$$\mathbb{V}_{\nu^{\uplus\alpha}}(m) = \alpha\mathbb{V}_\nu(m/\alpha) + m^2(1/\alpha - 1). \tag{4.3}$$

Furthermore, if $m_0 < +\infty$, then the variance functions of the CSK families generated by ν and $\nu^{\uplus\alpha}$ exists and

$$V_{\nu^{\uplus\alpha}}(m) = \alpha V_\nu(m/\alpha) + m(m - \alpha m_0)(1/\alpha - 1). \tag{4.4}$$

Remark 4.2. Let $\nu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ be the symmetric Bernoulli distribution, its Cauchy transform and self energy are respectively

$$G_\nu(z) = \frac{z}{z^2 - 1} \qquad \text{and} \qquad K_\nu(z) = \frac{1}{z}.$$

With $B(\nu) = \max\{0, \ \sup supp(\nu)\} = 1$, we have from (2.10) $m_+(\nu) = 1$. Consider $\mu = \nu^{\uplus 2}$, then we have $K_\mu(z) = K_{\nu^{\uplus 2}}(z) = 2K_\nu(z) = 2/z$ and $G_\mu(z) = \frac{z}{z^2-2}$. So $\mu = \frac{1}{2}\delta_{-\sqrt{2}} + \frac{1}{2}\delta_{\sqrt{2}}$. With $B(\mu) = \max\{0, \ \sup supp(\mu)\} = \sqrt{2}$, we have that $m_+(\mu) = \sqrt{2}$. This implies that $m_+(\nu^{\uplus 2}) \neq 2m_+(\nu)$. So there is no "simple formula" for m_+ under additive boolean convolution power. For this reason, in theorem 4.1 we restrict ourself to m close enough to $\alpha m_0(\nu)$.

The following result gives formulas for pseudo-variance functions (and variance functions in case of existence) under both free additive convolution and boolean additive convolution power.

Proposition 4.3. *[16, Proposition 2.6] Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_+(\nu)$ generated by a non degenerate probability measure ν with support bounded from above. For $\alpha > 0$ such that probability measures $\left(\nu^{\boxplus 1/\alpha}\right)^{\uplus\alpha}$ and $\left(\nu^{\uplus 1/\alpha}\right)^{\boxplus\alpha}$ are well defined, their support are bounded from above and they generates CSK families with pseudo-variance functions*

$$\mathbb{V}_{\left(\nu^{\boxplus 1/\alpha}\right)^{\uplus\alpha}}(m) = \mathbb{V}_\nu(m) + (1/\alpha - 1)m^2, \tag{4.5}$$

and

$$\mathbb{V}_{\left(\nu^{\uplus 1/\alpha}\right)^{\boxplus\alpha}}(m) = \mathbb{V}_\nu(m) + (1 - 1/\alpha)m^2. \tag{4.6}$$

respectively, for m close enough to m_0. Furthermore, if $m_0 < +\infty$, then the variance functions of the CSK families generated respectively by ν, $\left(\nu^{\boxplus 1/\alpha}\right)^{\uplus\alpha}$ and $\left(\nu^{\uplus 1/\alpha}\right)^{\boxplus\alpha}$ exists and for m close enough to m_0 we have

$$V_{\left(\nu^{\boxplus 1/\alpha}\right)^{\uplus\alpha}}(m) = V_\nu(m) + (1/\alpha - 1)m(m - m_0). \tag{4.7}$$

and

$$V_{\left(\nu^{\uplus 1/\alpha}\right)^{\boxplus\alpha}}(m) = V_\nu(m) + (1 - 1/\alpha)m(m - m_0). \tag{4.8}$$

Authors in [2] consider the transformation $\mathbb{B}_t : \mathcal{M} \longmapsto \mathcal{M}$ defined by, for every $t \geq 0$

$$\mathbb{B}_t(\mu) = \left(\mu^{\boxplus(1+t)}\right)^{\uplus\frac{1}{1+t}}, \qquad \mu \in \mathcal{M}. \tag{4.9}$$

They prove that for $t = 1$ the transformation $\mathbb{B}_1$ coincides with the canonical bijection $\mathbb{B} : \mathcal{M} \longmapsto \mathcal{M}_{Inf-div}$ discovered by Bercovici and Pata in their study of the relations between infinite divisibility in free and in Boolean probability. Here $\mathcal{M}_{Inf-div}$ stands for the set of probability distributions in $\mathcal{M}$ which are infinitely divisible with respect to the operation $\boxplus$. As a consequence, we have that $\mathbb{B}_t(\mu)$ is $\boxplus$-infinitely divisible for every $\mu \in \mathcal{M}$ and every $t \geq 1$. The following result gives the pseudo-variance function (and variance function in case of existence) of the CSK family generated by $\mathbb{B}_t(\mu)$. In fact this easily follows from (4.5) and (4.7) by choosing $\alpha = \frac{1}{1+t}$.

Proposition 4.4. *[16, Proposition 2.7] Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_+(\nu)$ generated by a non degenerate probability measure ν with support bounded from above. For $t \geq 0$, the probability measure*

$$\mathbb{B}_t(\nu) = \left(\nu^{\boxplus(1+t)}\right)^{\uplus\frac{1}{1+t}} \tag{4.10}$$

has support bounded from above and it generates the CSK family with pseudo-variance function

$$\mathbb{V}_{\mathbb{B}_t(\nu)}(m) = \mathbb{V}_\nu(m) + tm^2. \tag{4.11}$$

Furthermore, if $m_0 < +\infty$, then the variance functions of the CSK families generated by ν and $\mathbb{B}_t(\nu)$ exists and

$$V_{\mathbb{B}_t(\nu)}(m) = V_\nu(m) + tm(m - m_0). \tag{4.12}$$

Denote by $\mathcal{V}$ the class of variance functions corresponding to probability measures ν such that ν is compactly supported, centered: $\int x\nu(dx) = 0$, with variance $\int x^2\nu(dx) = 1$, so that $V_\nu(0) = 1$. Denote $\mathcal{V}_\infty$ the class of those $V \in \mathcal{V}$ that the function $m \mapsto V(cm)$ is in $\mathcal{V}$ for every real c. It was proved in [8, Corollary 1.1], that the map $V(m) \mapsto V(m) - m^2$ is a bijection of $\mathcal{V}_\infty$ onto $\mathcal{V}$ (also the map $V_\nu(m) \longmapsto V_\nu(m) + m^2$ is the inverse bijection). We will see that this bijection between variance functions correspond to the boolean Bercovici-Pata bijection between probability measures.

Proposition 4.5. *[16, Proposition 2.8] Suppose $V_\nu(.)$ is the variance function of the CSK family generated by a non degenerate probability measure ν with mean 0 and variance 1. For $\alpha > 0$ such that probability measures $\left(\nu^{\boxplus 1/\alpha}\right)^{\uplus\alpha}$ and $\left(\nu^{\uplus 1/\alpha}\right)^{\boxplus\alpha}$ are well defined, we have*

(i) The bijection $V_\nu(m) \longmapsto V_\nu(m) + m^2$ from $\mathcal{V}$ onto $\mathcal{V}_\infty$ correspond to boolean Bercovici-Pata bijection between probability measures $\nu \longmapsto \mathbb{B}_1(\nu)$, in addition

$$\left(\nu^{\uplus 1/\alpha}\right)^{\boxplus\alpha} \xrightarrow{\alpha\to+\infty} \mathbb{B}_1(\nu), \qquad \text{in distribution.} \tag{4.13}$$

(ii) The bijection $V_\nu(m) \longmapsto V_\nu(m) - m^2$ from $\mathcal{V}_\infty$ onto $\mathcal{V}$ correspond to the inverse boolean Bercovici-Pata bijection between probability measures $\nu \longmapsto \mathbb{B}_1^{-1}(\nu)$, in addition

$$\left(\nu^{\boxplus 1/\alpha}\right)^{\uplus\alpha} \xrightarrow{\alpha\to+\infty} \mathbb{B}_1^{-1}(\nu), \qquad \text{in distribution.} \tag{4.14}$$

If ν is a compactly supported probability measure on the real line, the K-transform K_ν of ν admit a Laurent expansion. From [32], one sees that

$$K_\nu(z) = \sum_{n=1}^{\infty} r_n(\nu)\frac{1}{z^{n-1}}. \tag{4.15}$$

The coefficients $r_n = r_n(\nu)$ are called the boolean cumulants of the measure ν. In particular $r_0 = 0$, $r_1 = \int x\nu(dx) = m_0$. The following result gives the connection between boolean cumulants and variance functions of CSK families.

Theorem 4.6. *[16, Theorem 3.1] Suppose V_ν is analytic in a neighborhood of m_0, $V_\nu(m_0) > 0$, and ν is a probability measure with finite all moments, such that $\int x\nu(dx) = m_0$. Then the following conditions are equivalent.*

(i) ν is non degenerate, compactly supported and there exists an interval $(A, B) \ni m_0$ such that $\{Q_{(m,\nu)}(dx) = f_\nu(x, m)\nu(dx) \ : \ m \in (A, B)\}$, with $f_\nu(x, m)$ given by (2.17), define a family of probability measures parameterized by the mean with variance function $V_\nu(.)$.

(ii) The boolean cumulants of the measure ν are $r_0 = 0$, $r_1 = m_0$ and for all $n \geq 1$

$$r_{n+1} = \frac{1}{n!}\,\frac{d^{n-1}}{dm^{n-1}}\left(V_\nu(m) + m(m - m_0)\right)^n\bigg|_{m=m_0}. \tag{4.16}$$

In the following we relate boolean cumulants of the Marchenko Pastur distribution to Catalan numbers. The centered Marchenko-Pastur distribution is given by

$$\nu(dx) = \frac{\sqrt{4-(x-a)^2}}{2\pi(1+ax)}\mathbf{1}_{(a-2,a+2)}(x)dx + p_1\delta_{x_1}.$$

The discrete part is absent except for $a^2 > 1$, in this case $p_1 = 1 - 1/a^2$ and $x_1 = -1/a$. It generates the *CSK family* with variance function $V_\nu(m) = 1 + am = \mathbb{V}_\nu(m)$.

Corollary 4.7. *[16, Corollary 3.2] If ν is the centered standardized Marchenko Pastur distribution with parameter $a = 2$, i.e. it generates the CSK family with $m_0 = 0$ and variance function $V_\nu(m) = 1 + 2m$, then its boolean cumulants are $r_0 = 0$, $r_1 = m_0 = 0$ and for $n \geq 1$,*

$$r_{n+1}(\nu) = \frac{1}{1+n}\begin{pmatrix} 2n \\ n \end{pmatrix}. \tag{4.17}$$

Next, we relate boolean cumulants of certain probability distribution to Fuss-Catalan numbers. In combinatorial mathematics and statistics, the Fuss-Catalan numbers are defined in [21] by the Swiss mathematician Fuss, Nicolaus. They are numbers of the form

$$A_n(p, r) = \frac{r}{np + r}\begin{pmatrix} np + r \\ n \end{pmatrix}. \tag{4.18}$$

The Fuss-Catalan represents the number of legal permutations or allowed ways of arranging a number of articles, that is restricted in some way. This means that they are related to the Binomial coefficient.

On the other hand, some examples of variance functions that are polynomial in the mean of arbitrary degree are introduced in [8]. In particular a complete resolution of compactly supported CSK with cubic variance function is given (see [8, Theorem 1.2]). Next, we relate boolean cumulants of certain probability distribution generating a cubic CSK family to Fuss-Catalan numbers of the form (4.18) for $p = 3$ and $r = 1$.

Corollary 4.8. *[16, Corollary 3.3] The function $V(m) = 1 + 3m + 2m^2 + m^3$ is the variance function the CSK family generated by a compactly supported probability measure ν, with mean 0, variance 1 and with boolean cumulants given by: $r_0 = 0$, $r_1 = m_0 = 0$ and for $n \geq 1$,*

$$r_{n+1}(\nu) = \frac{1}{3n+1}\begin{pmatrix} 3n + 1 \\ n \end{pmatrix}. \tag{4.19}$$

4.2. Some approximations in CSK family

In this paragraph, we give an approximation of elements of the CSK family generated by the boolean Gaussian distribution and an approximation of elements of the CSK family generated by the boolean Poisson distribution, (see [20] for more details).

4.2.1. Approximation of boolean Gaussian CSK family

According to [32], the centered boolean Gaussian distribution μ_{0,σ^2} with variance σ^2 (or symmetric Bernoulli distribution)

$$\mu_{0,\sigma^2} = \frac{1}{2}(\delta_{-\sigma} + \delta_\sigma),$$

has a self energy or a Cauchy transform

$$K_{\mu_{0,\sigma^2}}(z) = \frac{\sigma^2}{z} \qquad \text{or} \qquad G_{\mu_{0,\sigma^2}}(z) = \frac{1}{z - \sigma^2/z},$$

respectively. We have, for all $\theta \in (-1/\sigma, 1/\sigma)$

$$M_{\mu_{0,\sigma^2}}(\theta) = \frac{1}{1-\theta^2\sigma^2} \qquad \text{and} \qquad k_{\mu_{0,\sigma^2}}(\theta) = \theta\sigma^2.$$

The inverse of the function $k_{\mu_{0,\sigma^2}}(.)$ is $\psi_{\mu_{0,\sigma^2}}(m) = m/\sigma^2$ for all $m \in (-\sigma, \sigma) = k_{\mu_{0,\sigma^2}}((-1/\sigma, 1/\sigma))$. With $m_0 = 0$, the variance function of the CSK family generated by μ_{0,σ^2} is

$$V_{\mu_{0,\sigma^2}}(m) = \mathbb{V}_{\mu_{0,\sigma^2}}(m) = \sigma^2 - m^2.$$

The two sided CSK family generated by μ_{0,σ^2} is given by

$$\begin{aligned}\mathcal{K}(\mu_{0,\sigma^2}) = \Big\{ & Q_{(m,\mu_{0,\sigma^2})}(dx) = \mu_{m,\sigma^2}(dx) \\ & = \frac{1}{2\sigma}\left[(\sigma - m)\delta_{-\sigma} + (\sigma + m)\delta_{\sigma}\right] : \ m \in (-\sigma, \sigma)\Big\}.\end{aligned}$$

The family $\mathcal{K}(\mu_{0,\sigma^2})$ consists of boolean Gaussian distributions with mean $m \in (-\sigma, \sigma)$. The following result gives an approximation of elements of the CSK family $\mathcal{K}(\mu_{0,\sigma^2})$.

Theorem 4.9. *[20] Suppose the variance function V_ν of a CSK family $\mathcal{K}(\nu)$ is analytic and strictly positive in a neighborhood of $m_0 = 0$. Then there is $\delta > 0$ such that if, for $\alpha > 0$, $\mathcal{L}(Y_\alpha) \in \mathcal{K}(\nu_\alpha)$, with $\nu_\alpha = D_{1/\alpha}(\nu^{\uplus\alpha})$, has mean $\mathbb{E}(Y_\alpha) = m/\sqrt{\alpha}$ with $|m| < \delta$, then*

$$\sqrt{\alpha}Y_\alpha \xrightarrow{\alpha \to +\infty} \mu_{m,\sigma^2} \qquad \textit{in distribution,}$$

where $\sigma^2 = V_\nu(0)$.

From Theorem 4.9, we get the boolean central limit theorem (see [32, Theorem 3.4].)

Corollary 4.10 (boolean central limit theorem)**.** *If ν is a probability measure with mean $m_0 = 0$ and variance $\sigma^2 = V_\nu(0)$, then*

$$D_{1/\sqrt{n}}(\nu^{\uplus n}) \xrightarrow{n \to +\infty} \mu_{0,\sigma^2} \qquad \textit{in distribution.}$$

4.2.2. Approximation of boolean Poisson CSK family

For $N \in \mathbb{N}$, $s > 0$ and $0 < \lambda < N$, consider

$$\mu_N = (1 - \frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\delta_s.$$

We have that for all $\theta \in (-\infty, \frac{1}{s})$,

$$M_{\mu_N}(\theta) = 1 - \frac{\lambda}{N} + \frac{\lambda/N}{1-\theta s} \qquad \text{and} \qquad k_{\mu_N}(\theta) = \frac{\lambda s}{N - N\theta s + \lambda\theta s}.$$

As the inverse of the function $k_{\mu_N}(.)$, we have that for all $m \in (0,s) = k_{\mu_N}((-\infty, \frac{1}{s}))$,

$$\psi_{\mu_N}(m) = \frac{\lambda s - Nm}{sm(\lambda - N)}.$$

Formula (2.13) implies that the pseudo-variance function of the two sided CSK family $\mathcal{K}(\mu_N)$ is

$$\mathbb{V}_{\mu_N}(m) = \frac{Nm^2(m-s)}{\lambda s - Nm}.$$

With $m_0(\mu_N) = \lambda s/N$, we see from (2.14) that the variance function of the two sided CSK family $\mathcal{K}(\mu_N)$ is

$$V_{\mu_N}(m) = m(s-m).$$

The CSK family generated by μ_N is given by

$$\mathcal{K}(\mu_N) = \left\{ Q_{(m,\mu_N)}(dx) = \frac{s-m}{s}\delta_0 + \frac{m}{s}\delta_s : \; m \in (0,s) \right\}.$$

The boolean Poisson distribution $\pi_\lambda^{(s)}$ with jump size s and parameter λ ($s, \lambda \geq 0$) is given by

$$\pi_\lambda^{(s)} = \frac{1}{\lambda+1}[\delta_0 + \lambda \delta_{s(\lambda+1)}].$$

We have for all $\theta \in (-\infty, \frac{1}{s(\lambda+1)})$

$$M_{\pi_\lambda^{(s)}}(\theta) = \frac{1-\theta s}{1-\theta s(1+\lambda)} \qquad \text{and} \qquad k_{\pi_\lambda^{(s)}}(\theta) = \frac{\lambda s}{1-\theta s}.$$

As the inverse of the function $k_{\pi_\lambda^{(s)}}(.)$, we have that for all $m \in (0, s(1+\lambda)) = k_{\pi_\lambda^{(s)}}((-\infty, \frac{1}{s(\lambda+1)}))$,

$$\psi_{\pi_\lambda^{(s)}}(m) = \frac{m - \lambda s}{sm}.$$

Formula (2.13) implies that the pseudo-variance function of the two sided CSK family $\mathcal{K}(\pi_\lambda^{(s)})$ is

$$\mathbb{V}_{\pi_\lambda^{(s)}}(m) = \frac{m^2(s(\lambda+1) - m)}{m - \lambda s}.$$

With $m_0(\pi_\lambda^{(s)}) = \lambda s$, we see from (2.14) that the variance function of the two sided CSK family $\mathcal{K}(\pi_\lambda^{(s)})$ is

$$V_{\pi_\lambda^{(s)}}(m) = m(s(\lambda+1) - m).$$

The CSK family generated by $\pi_\lambda^{(s)}$ is given by

$$\mathcal{K}(\pi_\lambda^{(s)}) = \Big\{ Q_{(m,\pi_\lambda^{(s)})}(dx) = \frac{s(\lambda+1)-m}{(\lambda+1)s}\delta_0 + \frac{m(s(\lambda+1)-m)}{(\lambda+1)s^2}\delta_{s(\lambda+1)} : \ m \in (0, s(\lambda+1)) \Big\}.$$

The following result gives an approximation of elements of the boolean Poisson CSK family. In particular we get the boolean Poisson limit theorem, (see [32, Theorem 3.5]).

Theorem 4.11. *[20] For $N \in \mathbb{N}$, $s > 0$ and $0 < \lambda < N$, let*

$$\mu_N = (1 - \frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\delta_s,$$

and consider the CSK family generated by $\mu_N^{\uplus N}$, with mean $m_0(\mu_N^{\uplus N}) = \lambda s$ and variance function $V_{\mu_N^{\uplus N}}(.)$. We have that

$$Q_{(m,\mu_N^{\uplus N})} \xrightarrow{N\to+\infty} Q_{(m,\pi_\lambda^{(s)})}, \quad \textit{in distribution.}$$

for all m in a neighborhood of $m_0 = \lambda s$. In particular, for $m = m_0 = \lambda s$, we get the boolean Poisson limit theorem

$$\mu_N^{\uplus N} \xrightarrow{N\to+\infty} \pi_\lambda^{(s)}, \quad \textit{in distribution.}$$

5. Free multiplicative convolution

Let $\nu \in \mathcal{M}_+$ such that $\delta = \nu(\{0\}) < 1$, and consider the function

$$\Psi_\nu(z) = \int_0^{+\infty} \frac{zx}{1-zx}\nu(dx), \qquad z \in \mathbb{C} \setminus \mathbb{R}_+. \tag{5.1}$$

The function Ψ_ν is univalent in the left half-plane $i\mathbb{C}^+$ and its image $\Psi_\nu(i\mathbb{C}^+)$ is contained in the circle with diameter $(\nu(\{0\}) - 1, 0)$. Moreover $\Psi_\nu(i\mathbb{C}^+) \cap \mathbb{R} = (\nu(\{0\}) - 1, 0)$. Let $\chi_\nu : \Psi_\nu(i\mathbb{C}^+) \longrightarrow i\mathbb{C}^+$ be the inverse function of Ψ_ν. Then the S-transform of ν is the function

$$S_\nu(z) = \chi_\nu(z)\frac{1+z}{z}. \tag{5.2}$$

The product of S-transforms is an S-transform, so that the multiplicative free convolution $\nu_1 \boxtimes \nu_2$ of the measures ν_1 and ν_2 in $\mathcal{M}_+$ is defined by

$$S_{\nu_1\boxtimes\nu_2}(z) = S_{\nu_1}(z)S_{\nu_2}(z).$$

We say that a probability measure $\nu \in \mathcal{M}_+$ is infinitely divisible with respect to $\boxtimes$, if for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathcal{M}_+$ such that

$$\nu = \underbrace{\nu_n \boxtimes \ldots\ldots \boxtimes \nu_n}_{n \text{ times}}.$$

The multiplicative free convolution power $\nu^{\boxtimes\alpha}$ is defined at least for all $\alpha \geq 1$ (see [1, Theorem 2.17]) by $S_{\nu^{\boxtimes\alpha}}(z) = S_\nu(z)^\alpha$.

5.1. Free multiplicative convolution and variance function

In this paragraph, we deal with free multiplicative convolution from a point of view related to CSK families. We first state the result concerning the effect of the free multiplicative convolution power on a CSK family.

Theorem 5.1. *[22] Let $\mathbb{V}_\nu$ be the pseudo-variance function of the CSK family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability distribution ν concentrated on the positive real line with mean $m_0(\nu)$. Consider $\alpha > 0$ such that $\nu^{\boxtimes\alpha}$ is defined. Then*

(i) $m_-(\nu^{\boxtimes\alpha}) = (m_-(\nu))^\alpha$ and $m_0(\nu^{\boxtimes\alpha}) = (m_0(\nu))^\alpha$, and for $m \in (m_-(\nu^{\boxtimes\alpha}), m_0(\nu^{\boxtimes\alpha}))$,

$$\mathbb{V}_{\nu^{\boxtimes\alpha}}(m) = m^{2-2/\alpha}\mathbb{V}_\nu\left(m^{1/\alpha}\right). \tag{5.3}$$

(ii) If $m_0 < +\infty$, then the variance functions of the CSK families generated by ν and $\nu^{\boxtimes\alpha}$ exist and

$$V_{\nu^{\boxtimes\alpha}}(m) = \frac{m - m_0^\alpha}{m^{1/\alpha} - m_0} m^{1-1/\alpha} V_\nu\left(m^{1/\alpha}\right). \tag{5.4}$$

Several limit theorems involving the free additive convolution, the boolean additive convolution and the free multiplicative convolution have been established in [27] and in [30]. The authors in [22] used variance functions to re-derive these results, this leads to some new variance functions with non usual form.

Theorem 5.2. *[22] Let ν be a non degenerate probability distribution concentrated on the positive real line with mean $m_0(\nu) > 0$. Suppose that ν has a finite second moment. Then denoting $\gamma = \frac{Var(\nu)}{(m_0(\nu))^2} = \frac{V_\nu(m_0)}{m_0^2}$, we have*

(i)

$$D_{1/(nm_0^n)}\left(\nu^{\boxtimes n}\right)^{\boxplus n} \xrightarrow{n\to+\infty} \eta_\gamma \qquad \text{in distribution},$$

where η_γ is such that $m_0(\eta_\gamma) = 1$, $(m_-(\eta_\gamma), m_0(\eta_\gamma)) \subset (0,1)$ and the variance function of the CSK family generated by η_γ is given for $m \in (m_-(\eta_\gamma), m_0(\eta_\gamma))$, by

$$V_{\eta_\gamma}(m) = \frac{m(m-1)}{m_0^2 \ln(m)} V_\nu(m_0) = \frac{\gamma m(m-1)}{\ln(m)}. \tag{5.5}$$

(ii)

$$D_{1/(nm_0^n)}\left(\nu^{\boxtimes n}\right)^{\uplus n} \xrightarrow{n\to+\infty} \sigma_\gamma \qquad \text{in distribution},$$

where σ_γ is such that $m_0(\sigma_\gamma) = 1$, $(m_-(\sigma_\gamma), m_0(\sigma_\gamma)) \subset (0,1)$, and for all $m \in (m_-(\sigma_\gamma), m_0(\sigma_\gamma))$, the variance function of the CSK family generated by σ_γ is given by

$$V_{\sigma_\gamma}(m) = \frac{m(m-1)}{m_0^2 \ln(m)} V_\nu(m_0) + m(1-m) = \frac{\gamma m(m-1)}{\ln(m)} + m(1-m) \tag{5.6}$$

Example 5.3. The Wigner's semicircle (free Gaussian) distribution

$$\mu(dx) = \frac{\sqrt{4-x^2}}{2\pi}\mathbf{1}_{(-2,2)}(x)dx,$$

generates the CSK family with variance function $V_\mu(m) = 1 = \mathbb{V}_\mu(m)$. The one sided domain of means of the family $\mathcal{K}_-(\mu)$ is $(m_-(\mu), m_0(\mu)) = (-1,\ 0)$. By the translation $f : x \longmapsto x+2$, the probability measure

$$\nu(dx) = f(\mu)(dx) = \frac{\sqrt{x(4-x)}}{2\pi}\mathbf{1}_{(0,\ 4)}(x)dx,$$

generates the CSK family with pseudo-variance function $\mathbb{V}_\nu(m) = \frac{m}{m-2}$. We have that $V_\nu(m) = 1$ and $m_0(\nu) = 2$. One see that

$$D_{1/(n2^n)}\left(\nu^{\boxtimes n}\right)^{\boxplus n} \xrightarrow{n\to+\infty} \eta_{1/4} \qquad \text{in distribution,}$$

where $\eta_{1/4}$ is such that $m_0(\eta_{1/4}) = 1$, $(m_-(\eta_{1/4}), m_0(\eta_{1/4})) \subset (0,1)$ and the variance function of the CSK family generated by $\eta_{1/4}$ is given for $m \in (m_-(\eta_{1/4}), m_0(\eta_{1/4}))$, by

$$V_{\eta_{1/4}}(m) = \frac{m(m-1)}{4\ln(m)}. \tag{5.7}$$

We also have,

$$D_{1/(n2^n)}\left(\nu^{\boxtimes n}\right)^{\uplus n} \xrightarrow{n\to+\infty} \sigma_{1/4} \qquad \text{in distribution,}$$

where $\sigma_{1/4}$ is such that $m_0(\sigma_{1/4}) = 1$, $(m_-(\sigma_{1/4}), m_0(\sigma 1/4)) \subset (0,1)$ and the variance function of the CSK family generated by $\sigma_{1/4}$ is given for $m \in (m_-(\sigma_{1/4}), m_0(\sigma_{1/4}))$, by

$$V_{\sigma_{1/4}}(m) = \frac{m(m-1)}{4\ln(m)} + m(1-m). \tag{5.8}$$

In what follows, we give the link between the two limit probability measures η_γ and σ_γ by mean of the boolean Bercovici-Pata transformation.

Proposition 5.4. *[22] Let ν be a non degenerate probability distribution concentrated on the positive real line with mean $m_0(\nu) > 0$. Suppose that ν has a finite second moment. Then denoting $\gamma = \frac{Var(\nu)}{(m_0(\nu))^2} = \frac{V_\nu(m_0)}{m_0^2}$, we have*

$$\eta_\gamma = \mathbb{B}_1(\sigma_\gamma).$$

It is worth mentioning that the $\mathcal{R}$ and R- transforms of the limiting distribution η_γ, is given in [30], in terms of the Lambert's W-function which satisfies the functional equation

$$z = W(z)\exp(W(z)).$$

For more details of the Lambert W-function, see [13]. Let $W_0(z)$ be the principal branch of the Lambert W-function.

Theorem 5.5. *[30, Theorem 4.1]*

(1) The $\mathcal{R}$ and R–transforms of probability measure η_γ are given as follows:

$$\mathcal{R}_{\eta_\gamma}(z) = \frac{-W_0(-\gamma z)}{\gamma z},$$

$$R_{\eta_\gamma}(z) = \frac{-1}{\gamma} W_0(-\gamma z).$$

(2) η_γ is both $\boxplus$-infinitely divisible and $\boxtimes$-infinitely divisible.

(3) The free cumulant sequence of η_γ is $\left\{\frac{(\gamma n)^{n-1}}{n!}\right\}_{n\in\mathbb{N}}$.

(4) The Lévy measure ν_{η_γ} of η_γ is given by

$$\nu_{\eta_\gamma}(ds) = \frac{1}{\gamma\pi} s g^{-1}(\gamma/s) \mathbf{1}_{[0,\gamma e]}(s) ds,$$

where $g(u) = \csc(u)\exp(-u\cot(u))$.

(5) It holds the following formulas:

$$\eta_\gamma^{\boxplus t} = D_t(\eta_\gamma^{\boxtimes 1/t}),$$

$$\eta_\gamma^{\boxtimes t} = D_t(\eta_\gamma^{\boxplus 1/t}).$$

Let $\sigma := \sigma_1$. We have the following result, see [30, Proposition 4.5].

Proposition 5.6. *The probability density ω_σ of the measure σ can be given in the implicit form as:*

$$\omega_\sigma\left(\frac{\sin(v)}{v}\exp(v\cot(v))\right) = \frac{1}{\pi}\frac{v^2\exp(-v\cot(v))}{\sin(v)\left((1-v\cot(v))^2+v^2\right)}, \qquad 0 < v < \pi.$$

5.2. Explicit free multiplicative law of large numbers

The limit probability measure for the free multiplicative law of large numbers was proved by Tucci [33] for probability measures with bounded support. Haagerup and Möller [34] extend Tucci's result to probability measures with unbounded support and at the same time they give a more elementary proof for the case of probability measures with bounded support. In contrast to the classical multiplicative convolution case, the limit measure for the free multiplicative law of large numbers is not a Dirac measure, unless the original measure is a Dirac measure. More precisely we have (see [34, Theorem 2]):

Theorem 5.7. *Let $\nu \in \mathcal{M}_+$ and let $\phi_n : [0,\infty) \longrightarrow [0,\infty)$ be the map $\phi_n(x) = x^{1/n}$. Set $\alpha = \nu(\{0\})$. If we denote*

$$\mu_n = \phi_n(\underbrace{\nu \boxtimes \ldots\ldots \boxtimes \nu}_{n\ \text{times}})$$

then μ_n converge weakly to a probability measure $\mu \in \mathcal{M}_+$. If ν is a Dirac measure on $[0,\infty)$ then $\mu = \nu$. Otherwise μ is the unique probability measure on $[0,\infty)$ characterized by

$$\mu\left(\left[0, \frac{1}{S_\nu(t-1)}\right]\right) = t \tag{5.9}$$

for all $t \in (\alpha, 1)$ and $\mu(\{0\}) = \alpha$. The support of the measure μ is the closure of the interval

$$(a,b) = \left(\left(\int_0^\infty x^{-1}\nu(dx)\right)^{-1}, \int_0^\infty x\nu(dx)\right)$$

where $0 \leq a < b \leq \infty$.

Consider $\mathbb{V}_\nu$ the pseudo-variance function of the CSK family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability measure $\nu \in \mathcal{M}_+$. We give explicitly the law of large numbers μ for free multiplicative convolution in terms of the pseudo-variance function $\mathbb{V}_\nu$.

Theorem 5.8. *[19, Theorem 3.1] Let ν a non degenerate probability measure on $[0,\infty)$ and let $\phi_n : [0,\infty) \longrightarrow [0,\infty)$ be the map $\phi_n(x) = x^{1/n}$. Set $\alpha = \nu(\{0\})$. If we denote*

$$\mu_n = \phi_n(\underbrace{\nu \boxtimes \ldots\ldots \boxtimes \nu}_{n\ times})$$

then μ_n converge weakly to a probability measure μ on $[0,\infty)$ which is given by

$$\mu(dm) = \alpha\delta_0 + \left(\frac{m^2}{\mathbb{V}_\nu(m)}\right)' \mathbf{1}_{(m_-(\nu), m_0(\nu))}(m)dm. \tag{5.10}$$

The following examples illustrate the usefulness of Theorem 5.8 and provide examples of the free multiplicative law of large numbers μ for probability measures ν of importance in free probability. However probability measures ν presented in the following examples generates CSK families having quadratic and cubic pseudo-variance functions.

Example 5.9. Let $\gamma = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ be the symmetric Bernoulli distribution. It generates the CSK family with variance function $V_\gamma(m) = 1 - m^2 = \mathbb{V}_\gamma(m)$ and $m_0(\gamma) = 0$. By the translation $f : x \longmapsto x + 1$, the probability measure $\nu = f(\gamma) = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$ generates the CSK family with $m_0(\nu) = 1$ and pseudo-variance function

$$\mathbb{V}_\nu(m) = \frac{m^2(2-m)}{m-1}.$$

The one sided domain of means of the family $\mathcal{K}_-(\nu)$ is $(m_-(\nu), m_0(\nu)) = (0,\ 1)$. In this case

$$\mu(dm) = \frac{1}{2}\delta_0 + \frac{1}{(2-m)^2}\mathbf{1}_{(0,\ 1)}(m)dm.$$

Example 5.10. The Wigner's semicircle (free Gaussian) distribution

$$\gamma(dx) = \frac{\sqrt{4-x^2}}{2\pi}\mathbf{1}_{(-2,2)}(x)dx,$$

generates the CSK family with variance function $V_\gamma(m) = 1 = \mathbb{V}_\gamma(m)$. The one sided domain of means of the family $\mathcal{K}_-(\gamma)$ is $(m_-(\gamma), m_0(\gamma)) = (-1,\ 0)$. By the translation $f : x \longmapsto x+2$, the probability measure

$$\nu(dx) = f(\gamma)(dx) = \frac{\sqrt{x(4-x)}}{2\pi}\mathbf{1}_{(0,\ 4)}(x)dx,$$

generates the CSK family with $m_0(\nu) = 2$ and pseudo-variance function

$$\mathbb{V}_\nu(m) = \frac{m}{m-2}.$$

Solving $z(m) = m + \mathbb{V}_\nu(m)/m = 0$ for $m > 0$, we obtain that $m = 1$. Using relation (2.16), we get $G_\nu(0) = -1$. The domain of means of the family $\mathcal{K}_-(\nu)$ is $(m_-(\nu), m_0(\nu)) = (1,\ 2)$. In this case

$$\mu(dm) = 2(m-1)\mathbf{1}_{(1,\ 2)}(m)dm.$$

Example 5.11. For $0 < a^2 < 1$, the (absolutely continuous) centered Marchenko-Pastur distribution

$$\gamma(dx) = \frac{\sqrt{4-(x-a)^2}}{2\pi(1+ax)}\mathbf{1}_{(a-2,a+2)}(x)dx$$

generates the CSK family with variance function $V(m) = 1 + am = \mathbb{V}(m)$. This with the affine transformation $f : x \longmapsto ax + 1$ leads to the distribution given by,

$$\nu(dx) = f(\gamma)(dx) = \frac{\sqrt{((a+1)^2 - x)\,(x-(a-1)^2)}}{2\pi a^2 x}\mathbf{1}_{((a-1)^2,(a+1)^2)}(x)dx.$$

It generates the CSK family with $m_0(\nu) = 1$, and pseudo-variance function of the form

$$\mathbb{V}_\nu(m) = \frac{a^2m^2}{m-1}.$$

Solving $z(m) = m + \mathbb{V}_\nu(m)/m = 0$ for $m > 0$, we obtain that $m = 1 - a^2$. Using relation (2.16), we get $G_\nu(0) = -\frac{1}{1-a^2}$. The domain of means of the family $\mathcal{K}_-(\nu)$ is $(m_-(\nu), m_0(\nu)) = (1-a^2,\ 1)$. In this case

$$\mu(dm) = \frac{1}{a^2}\mathbf{1}_{(1-a^2,\ 1)}(m)dm.$$

Example 5.12. For $a^2 > 1$, the Marchenko-Pastur distribution is

$$\gamma(dx) = \frac{\sqrt{4-(x-a)^2}}{2\pi(1+ax)}1_{(a-2,a+2)}(x)dx + (1-1/a^2)\delta_{-1/a}(dx)$$

It generates the CSK family with quadratic variance function $V_\gamma(m) = 1 + am = \mathbb{V}_\gamma(m)$. By the affine transformation $f : x \longmapsto ax+1$, the probability distribution given by

$$\begin{aligned}\nu(dx) &= f(\gamma)(dx)\\ &= \frac{\sqrt{((a+1)^2-x)\,(x-(a-1)^2)}}{2\pi a^2 x}\mathbf{1}_{((a-1)^2,(a+1)^2)}(x)dx + (1-1/a^2)\delta_0\end{aligned}$$

generates the CSK family with $m_0(\nu) = 1$, and pseudo-variance function of the form

$$\mathbb{V}_\nu(m) = \frac{a^2m^2}{m-1}.$$

In this case $m_-(\nu) = 0$ and

$$\mu(dm) = (1-1/a^2)\delta_0 + \frac{1}{a^2}\mathbf{1}_{(0,\ 1)}(m)dm.$$

Example 5.13. If ν is the standard free gamma distribution,

$$\gamma(dx) = \frac{\sqrt{4(1+a^2)-(x-2a)^2}}{2\pi(a^2x^2+2ax+1)}\mathbf{1}_{(2a-2\sqrt{1+a^2},2a+2\sqrt{1+a^2})}(x),$$

for $a \neq 0$, it generate the CSK family with $m_0 = 0$, and pseudo-variance function equal to the variance function $\mathbb{V}_\gamma(m) = V_\gamma(m) = (1+am)^2$. Suppose that $a > 0$. By the affine transformation $f : x \longmapsto ax+1$, the probability distribution

$$\begin{aligned}\nu(dx) &= f(\gamma)(dx)\\ &= \frac{\sqrt{((\sqrt{a^2+1}+a)^2-x)(x-(\sqrt{a^2+1}-a)^2)}}{2\pi a^2x^2}\mathbf{1}_{((\sqrt{a^2+1}-a)^2,(\sqrt{a^2+1}+a)^2)}(x)dx,\end{aligned}$$

generates the CSK family with $m_0(\nu) = 1$, and pseudo-variance function of the form

$$\mathbb{V}_\nu(m) = \frac{a^2m^3}{m-1}.$$

Solving $z(m) = m + \mathbb{V}_\nu(m)/m = 0$ for $m > 0$, we obtain that $m = \frac{1}{1+a^2}$. Using relation (2.16) wee get $G_\nu(0) = -(1+a^2)$. The one sided domain of means of the family $\mathcal{K}_-(\nu)$ is $(m_-(\nu), m_0(\nu)) = \left(\frac{1}{1+a^2},\ 1\right)$. We have that

$$\mu(dm) = \frac{1}{a^2m^2}\mathbf{1}_{\left(\frac{1}{1+a^2},\ 1\right)}(m)dm.$$

Example 5.14. The inverse semicircle distribution

$$\gamma(dx) = \frac{\sqrt{-1-4x}}{2\pi x^2}\mathbf{1}_{\left(-\infty,-\frac{1}{4}\right)}(x)dx,$$

generates the CSK family with pseudo-variance function $\mathbb{V}_\gamma(m) = m^3$, and with domain of means $(m_0(\gamma), m_+(\gamma)) = (-\infty, -1)$. By the transformation $f : x \longmapsto -x$, the probability distribution

$$\nu(dx) = f(\gamma)(dx) = \frac{\sqrt{-1+4x}}{2\pi x^2}\mathbf{1}_{(\frac{1}{4},+\infty)}(x)dx,$$

generates the CSK family $\mathcal{K}_-(\nu)$ with pseudo-variance function $\mathbb{V}_\nu(m) = -m^3$ and the domain of means is $(m_-(\nu), m_0(\nu)) = (1, +\infty)$. In this case

$$\mu(dm) = \frac{1}{m^2}\mathbf{1}_{(1,\ +\infty)}(m)dm.$$

Example 5.15. The free Ressel (or free Kendall) distribution

$$\gamma(dx) = \frac{-1}{\pi x\sqrt{-1-x}}\mathbf{1}_{(-\infty,-1)}(x)dx$$

generates the CSK family with domain of means $(m_0(\gamma), m_+(\gamma)) = (-\infty, -2)$ and pseudo-variance function $\mathbb{V}_\gamma(m) = m^2(m+1)$. With the transformation $f : x \longmapsto -x$ the probability distribution

$$\nu(dx) = f(\gamma)(dx) = \frac{1}{\pi x\sqrt{x-1}}\mathbf{1}_{(1,+\infty)}(x)dx.$$

generates the CSK family $\mathcal{K}_-(\nu)$ with pseudo-variance function $\mathbb{V}_\nu(m) = m^2(1-m)$, and domain of means $(m_-(\nu), m_0(\nu)) = (2, +\infty)$. We have that

$$\mu(dm) = \frac{1}{(1-m)^2}\mathbf{1}_{(2,\ +\infty)}(m)dm.$$

Example 5.16. The Free Abel (or Free Borel-Tanner) distribution

$$\gamma(dx) = \frac{1}{\pi(1-x)\sqrt{-x}}\mathbf{1}_{(-\infty,0)}(x)dx$$

generates the CSK family with domain of means $(m_0(\gamma), m_+(\gamma)) = (-\infty, 0)$ and pseudo-variance function $\mathbb{V}_\gamma(m) = m^2(m-1)$. By the transformation $f : x \longmapsto -x$ the probability distribution

$$\nu(dx) = f(\gamma)(dx) = \frac{1}{\pi(1+x)\sqrt{x}}\mathbf{1}_{(0,+\infty)}(x)dx,$$

generates the CSK family $\mathcal{K}_-(\nu)$ with pseudo-variance function $\mathbb{V}_\nu(m) = -m^2(1+m)$ and domain of means $(m_-(\nu), m_0(\nu)) = (0, +\infty)$. We have that

$$\mu(dm) = \frac{1}{(1+m)^2}\mathbf{1}_{(0,\ +\infty)}(m)dm.$$

Example 5.17. The free strict arcsine distribution

$$\gamma(dx) = \frac{\sqrt{3-4x}}{2\pi(1+x^2)}\mathbf{1}_{(-\infty,3/4)}(x)dx$$

generates the CSK family with domain of means $(m_0(\gamma), m_+(\gamma)) = (-\infty, -1/2)$ and pseudo-variance function $\mathbb{V}_\gamma(m) = m(1+m^2)$. By the affine transformation $f : x \longmapsto -x + 3/4$ the probability distribution

$$\nu(dx) = f(\gamma)(dx) = \frac{\sqrt{x}}{\pi(1+(3/4-x)^2)}\mathbf{1}_{(0,+\infty)}(x)dx,$$

generates the CSK family $\mathcal{K}_-(\nu)$ with pseudo-variance function $\mathbb{V}_\nu(m) = -m$ $(m^2 - \frac{3}{2}m + \frac{25}{16})$. Solving $z(m) = m + \mathbb{V}_\nu(m)/m = 0$ for $m > 0$, we obtain that $m = 5/4$. Using relation (2.16) we get $G_\nu(0) = -\frac{4}{5}$. The domain of means of $\mathcal{K}_-(\nu)$ is $(m_-(\nu), m_0(\nu)) = (5/4, +\infty)$. We have that

$$\mu(dm) = \frac{m^2 - \frac{25}{16}}{(m^2 - \frac{3}{2}m + \frac{25}{16})^2}\mathbf{1}_{(5/4,\ +\infty)}(m)dm.$$

6. Boolean multiplicative convolution

For $\nu \in \mathcal{M}_+$, the η-transform of ν is defined by:

$$\eta_\nu : \mathbb{C}\backslash\mathbb{R}_+ \longrightarrow \mathbb{C}\backslash\mathbb{R}_+; \quad z \mapsto \eta_\nu(z) = \frac{\Psi_\nu(z)}{1+\Psi_\nu(z)}. \tag{6.1}$$

where the function $\Psi_\nu(.)$ is given by (5.1). It is clear that ν is determined uniquely from the function η_ν. For $\nu \in \mathcal{M}_+$, it is known that $\eta_\nu((-\infty,0)) \subset (-\infty, 0)$, $\lim_{x\to 0,\ x<0} \eta_\nu(x) = \eta_\nu(0^-) = 0$ and $\eta_\nu(\overline{z}) = \overline{\eta_\nu(z)}$, for $z \in \mathbb{C}\backslash\mathbb{R}_+$. Also $\arg(z) \leq \arg(\eta_\nu(z)) < \pi$, for $z \in \mathbb{C}^+$.

The analytic function

$$B_\nu(z) = \frac{z}{\eta_\nu(z)} \tag{6.2}$$

is well defined in the region $z \in \mathbb{C}\backslash\mathbb{R}_+$. Now for $\mu,\ \nu \in \mathcal{M}_+$, their multiplicative boolean convolution $\mu \uplus\!\!\!\times \nu$ is defined as the unique probability measure in $\mathcal{M}_+$ that satisfies

$$B_{\mu \uplus\!\!\!\times \nu}(z) = B_\mu(z)B_\nu(z), \qquad \text{for} \ \ z \in \mathbb{C}\backslash\mathbb{R}_+ \tag{6.3}$$

Note that for $\mu,\ \nu \in \mathcal{M}_+$ which satisfies

(i) $\arg(\eta_\mu(z)) + \arg(\eta_\nu(z)) - \arg(z) < \pi \quad \text{for} \ \ z \in \mathbb{C}^+ \cup (-\infty, 0)$
(ii) at least one of the first moments of one of the measures μ or ν exists finitely,

then $\mu \uplus\!\!\!\times \nu \in \mathcal{M}_+$ is well defined.

We now state the result concerning the effect of the boolean multiplicative convolution on a CSK family. According to [4], the boolean multiplicative boolean convolution power $\nu^{\uplus\!\!\!\times\alpha}$ is defined for $0 \leq \alpha \leq 1$ by $B_{\nu^{\uplus\!\!\!\times\alpha}}(z) = B_\nu(z)^\alpha$.

Theorem 6.1. *[17, Theorem 3.2] Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability measure $\nu \in \mathcal{M}_+$ with mean $m_0(\nu) < +\infty$. For $\alpha \in [0,1]$, we have*

(i) $m_-(\nu^{\uplus\alpha}) = (m_-(\nu))^\alpha$ *and* $m_0(\nu^{\uplus\alpha}) = (m_0(\nu))^\alpha$, *and for* $m \in (m_-(\nu^{\uplus\alpha}), m_0(\nu^{\uplus\alpha}))$

$$\mathbb{V}_{\nu^{\uplus_\alpha}}(m) = m^{1+1/\alpha} - m^2 + m^{1-1/\alpha}\mathbb{V}_\nu\left(m^{1/\alpha}\right). \tag{6.4}$$

(ii) The variance functions of the CSK families generated by ν and $\nu^{\uplus\alpha}$ exists and

$$V_{\nu^{\uplus_\alpha}}(m) = \frac{m - m_0^\alpha}{m^{1/\alpha} - m_0}V_\nu\left(m^{1/\alpha}\right) + (m - m_0^\alpha)(m^{1/\alpha} - m). \tag{6.5}$$

The following result show how the permutation of power between free and boolean multiplicative convolutions affect the variance functions.

Theorem 6.2. *[17, Theorem 3.3] Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability measure $\nu \in \mathcal{M}_+$ with mean $m_0(\nu) < +\infty$. For $\alpha > 0$ such that $\left(\nu^{\boxtimes\alpha}\right)^{\uplus 1/\alpha}$ and $\left(\nu^{\uplus\alpha}\right)^{\boxtimes 1/\alpha}$ are defined, they generates respectively CSK families with pseudo-variance functions given by*

$$\mathbb{V}_{\left(\nu^{\boxtimes\alpha}\right)^{\uplus_{1/\alpha}}}(m) = m^{1+\alpha} - m^2 + m^{\alpha-1}\mathbb{V}_\nu(m) \tag{6.6}$$

and

$$\mathbb{V}_{\left(\nu^{\uplus\alpha}\right)^{\boxtimes_{1/\alpha}}}(m) = m^{-\alpha+3} - m^2 + m^{1-\alpha}\mathbb{V}_\nu(m), \tag{6.7}$$

for $m \in (m_-(\nu), m_0(\nu))$. Furthermore, the variance functions of the CSK families generated by ν, $\left(\nu^{\boxtimes\alpha}\right)^{\uplus 1/\alpha}$ and $\left(\nu^{\uplus\alpha}\right)^{\boxtimes 1/\alpha}$ exists and

$$V_{\left(\nu^{\boxtimes\alpha}\right)^{\uplus_{1/\alpha}}}(m) = (m - m_0)(m^\alpha - m) + m^{\alpha-1}V_\nu(m) \tag{6.8}$$

and

$$V_{\left(\nu^{\uplus\alpha}\right)^{\boxtimes_{1/\alpha}}}(m) = (m - m_0)(m^{-\alpha+2} - m) + m^{1-\alpha}V_\nu(m). \tag{6.9}$$

Authors in [31] introduce the analogue of Bercovici-Pata map for the multiplicative convolutions: that is for $t \geq 0$,

$$\begin{aligned} \mathbb{M}_t : \mathcal{M}_+ &\to \mathcal{M}_+ \\ \mu &\mapsto \left(\mu^{\boxtimes(t+1)}\right)^{\uplus\frac{1}{t+1}}. \end{aligned}$$

$\mu^{\uplus t} \in \mathcal{M}_+$ is defined for any probability measure $\mu \in \mathcal{M}_+$ and $0 \leq t \leq 1$. The following result gives the pseudo-variance function and variance function of the CSK family generated by $\mathbb{M}_t(\mu)$. In fact this easily follows from (6.6) and (6.8) by choosing $\alpha = 1 + t$.

Proposition 6.3. *[17, Proposition 3.4] Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability measure $\nu \in \mathcal{M}_+$ with mean $m_0(\nu) < +\infty$. For $t \geq 0$, the probability measure*

$$\mathbb{M}_t(\nu) = \left(\mu^{\boxtimes(t+1)}\right)^{\uplus\frac{1}{t+1}} \tag{6.10}$$

generates the CSK family with pseudo-variance function given by

$$\mathbb{V}_{\mathbb{M}_t(\nu)}(m) = m^t\ \mathbb{V}_\nu(m) + m^2(m^t - 1), \tag{6.11}$$

for $m \in (m_-(\nu), m_0(\nu))$. Furthermore, the variance functions of the CSK families generated by ν and $\mathbb{M}_t(\nu)$ exists and

$$V_{\mathbb{M}_t(\nu)}(m) = m^t\ V_\nu(m) + m^{t+2} - m_0 m^{t+1} - m^2 + m m_0. \tag{6.12}$$

Authors in [8] construct a class of examples which exhausts all cubic variance functions, and provide examples of polynomial variance functions of arbitrary degree. In fact, they use some algebraic operations that allow to build new variance functions from known ones. Formula (6.12) gives a relation between variance function via the multiplicative analog of Belinschi-Nica type semigroup. One see that some of polynomial variance functions can be obtained from known variance functions by applying the multiplicative analog of Belinschi-Nica type semigroup to the generating probability measure.

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