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Sandpile models*

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Abstract: This survey is an extended version of lectures given at the Cornell Probability Summer School 2013. The fundamental facts about the Abelian sandpile model on a finite graph and its connections to related models are presented. We discuss exactly computable results via Majumdar and Dhar's method. The main ideas of Priezzhev's computation of the height probabilities in 2D are also presented, including explicit error estimates involved in passing to the limit of the infinite lattice. We also discuss various questions arising on infinite graphs, such as convergence to a sandpile measure, and stabilizability of infinite configurations.

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1. Introduction

The Abelian sandpile model and close variants were introduced several times in different contexts independently. There is motivation coming from statistical physics, probability and combinatorics. However, we are going to delay a detailed discussion of where the model comes from to Section 3, and start with its definition and some of its basic properties in Section 2. There are a number of reasons why this seems to be a good choice:

1. The basic model is very simple to define, and some of its fundamental properties can be established without any serious pre-requisites. It is hoped that the model will have sufficient appeal on its own without motivation in advance.
2. We do not want to assume prior familiarity with statistical physics models such as percolation, the Ising model, etc. However, since the connection with critical phenomena is very important, it has to be explained, and it

will be easier to do so when the basic model can be used as illustration. We have attempted to organize Section 3 in such a way that a reader unfamiliar with statistical physics has a quick access to some important concepts.

3. As part of the motivation, we will also be ready to state some of the main open questions.

There are a number of excellent surveys already available on sandpile models [12, 33, 37, 56, 57, 61, 67, 79]. Our focus is similar to that of Redig's notes [79], in that we cover rigorous results roughly at the level of beginning PhD students. On the other hand, we have incorporated topics complementary to those in [79] and some results that are more recent. For example, we discuss connections to the Tutte polynomial, the rotor-router walk and a large part of Priezzhev's computation of height probabilities in 2D. Dhar's extensive survey [12], written from the point of view of theoretical physics, will be an invaluable guide to anyone wanting to learn about the model. An aspect of the theory that does not seem to receive much attention in the physics literature, though, is the precise arguments involving the limit of infinite graphs. Here we explain how this can be done based on one-endedness of components of the wired uniform spanning forest [43]. The connection to the rotor-router model is due to [33], that extends many of the basic results to directed graphs. For simplicity, in these notes we restricted attention to undirected graphs.

The outline of the paper is as follows. Section 2 introduces the sandpile Markov chain, recurrent configurations, the sandpile group and Dhar's formula for the average number of topplings. In Section 3 we give a brief introduction to critical phenomena using the percolation model as example. Self-organized criticality is first illustrated with the forest-fire model built on the Erdős-Rényi random graph, that is perhaps the most intuitive example of the concept. Then we discuss self-organized criticality in the Abelian sandpile model in terms of critical exponents. Section 4 starts with the burning bijection of Majumdar and Dhar and the connection to uniform spanning trees. Following this we present the connections to the rotor-router model and the Tutte polynomial. Section 5 is devoted to exactly computable results, and starts with Majumdar and Dhar's method. The scaling limit of the height 0 field is discussed. Section 5.5 is devoted to an exposition of the computation of height probabilities in 2D due to Priezzhev, and is followed by further 2D results in Section 5.6. Section 6 is devoted to questions on infinite graphs and highlights the role that properties of the wired uniform spanning forest play in infinite volume limits. Finally, Section 7 discusses certain questions of stabilizability of infinite configurations.

2. The Abelian sandpile model / chip-firing game on a finite graph

2.1. Definition of the model

Let $G = (V \cup \{s\}, E)$ be a finite, connected multigraph (i.e. we allow multiple edges between vertices). The distinguished vertex s is called the **sink**. We exclude loop-edges for simplicity (their presence would involve only trivial modifications). We write $\deg_G(x)$ for the degree of the vertex x in the graph G, and we write $x \sim y$ to denote that vertices x and y are connected by at least one edge.

Example 2.1. Let $V \subset \mathbb{Z}^d$ be finite. Identify all vertices in $V^c = \mathbb{Z}^d \setminus V$ into a single vertex that becomes the sink s. Then remove all loop-edges at s. This is called the **wired graph** induced by V. Instead of $\mathbb{Z}^d$, we can start from any locally finite, infinite, connected graph.

A **sandpile** is a collection of indistinguishable particles (chips, sand grains, etc.) on the vertices in V. A sandpile is hence specified by a map $\eta : V \to \{0, 1, 2, \dots\}$. We say that η is **stable at** $x \in V$, if $\eta(x) < \deg_G(x)$, and we say that it is **stable**, if it is stable at all $x \in V$.

We now introduce a dynamics that stabilizes any unstable sandpile. If η is unstable at some $x \in V$ (i.e. $\eta(x) \geq \deg_G(x)$), x is **allowed to topple** which means that x sends one particle along each edge incident to it. (In the combinatorics literature it is common to say **the vertex x fires** by sending chips to its neighbours.) On toppling the vertex x, the particles are re-distributed as follows:

$$\begin{aligned} \eta(x) &\longrightarrow \eta(x) - \deg_G(x) \\ \eta(y) &\longrightarrow \eta(y) + a_{xy}, \quad y \in V,\, y \neq x, \end{aligned}$$

where a_{xy} = number of edges between x and y. Regarding η as a row vector, this can be concisely written as

$$\eta \longrightarrow \eta - \Delta'_{x,\cdot}, \tag{1}$$

where

$$\begin{aligned} \Delta'_{xy} &= \begin{cases} \deg_G(x) & \text{if } x = y \in V; \\ -a_{xy} & \text{if } x \neq y,\, x, y \in V; \end{cases} \\ \Delta'_{x,\cdot} &= \text{row } x \text{ of } \Delta'. \end{aligned}$$

That is, if Δ = graph Laplacian of G then Δ' = restriction of Δ to $V \times V$. Particles arriving at the sink are lost, that is, we do not keep track of them. Observe that requiring $\eta(x) \geq \deg_G(x)$ before toppling ensures that we still have a sandpile after toppling (i.e. the number of particles at x is still non-negative after toppling). We also say in this case that toppling x is **legal**.

Toppling a vertex may create further unstable vertices.

Definition 2.2. Given a sandpile ξ, we define its **stabilization**

$$\xi^\circ \in \Omega_G := \{\text{stable sandpiles}\} = \prod_{x \in V} \{0, 1, \dots, \deg_G(x) - 1\},$$

by carrying out all possible legal topplings, in any order, until a stable sandpile is reached.

Theorem 2.3. *[11] The map $\xi \mapsto \xi^\circ$ is well-defined.*

Proof. We need to show:

(a) Only finitely many topplings can occur, regardless of how we choose to topple vertices.

(b) The final stable configuration is independent of the sequence of topplings chosen.

In order to see (a), observe that if $x \sim s$ then x can topple only finitely many times (the system loses particles to s on each toppling of x). It follows by induction that for all $k \geq 1$, if $x \sim x_{k-1} \sim \cdots \sim x_1 \sim s$, then x can topple only finitely many times. Since G is connected, we are done.

We now prove (b) in two steps.

(i) *"Topplings commute"*. If $x, y \in V$, $x \neq y$ and η is unstable at both x and y, then writing T_x to denote the effect of toppling x we claim that

$$T_y T_x \eta = T_x T_y \eta. \tag{2}$$

Observe that in either order, both topplings are legal. Then the claim is immediate from observing that both sides equal $\eta - \Delta'_{x,\cdot} - \Delta'_{y,\cdot}$.

(ii) Suppose now that

$$x_1, x_2, \ldots, x_k \tag{3}$$

and

$$y_1, y_2, \ldots, y_\ell \tag{4}$$

are two sequences of vertices that are both possible stabilizing sequences of η. That is, when carried out in order from left to right, in both sequences each toppling is legal, and the final results are stable configurations. If η is already stable, then $k = \ell = 0$ and there is nothing to prove.

Otherwise, we have $k, \ell \geq 1$ and $\eta(x_1) \geq \deg_G(x_1)$. Therefore, x_1 must occur somewhere in the second sequence, otherwise the second sequence would never reduce the number of particles at x_1. Let $x_1 = y_i$, $1 \leq i \leq \ell$, and suppose that i is the smallest such index. By part (i), the toppling of $y_i = x_1$ can be moved to the front of the second stabilizing sequence. Precisely, we have

$$\begin{aligned} T_{x_1} T_{y_{i-1}} \ldots T_{y_1} \eta &= T_{y_{i-1}} T_{x_1} T_{y_{i-2}} \ldots T_{y_1} \eta \\ &= T_{y_{i-1}} T_{y_{i-2}} T_{x_1} \ldots T_{y_1} \eta \\ &\vdots \\ &= T_{y_{i-1}} T_{y_{i-2}} \ldots T_{y_1} T_{x_1} \eta. \end{aligned}$$

It follows that the sequence

$$x_1, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_\ell \tag{5}$$

also stablizes η. We now remove x_1 from the beginning of the sequences (3) and (5) and repeat the argument for $T_{x_1}\eta$. Iterating gives that $k = \ell$ and the multisets $[x_1, \dots, x_k]$ and $[y_1, \dots, y_\ell]$ are permutations of each other. That is, each vertex topples the same number of times in the two stabilizing sequences, and hence they reach the same final configuration. This completes the proof that the stabilization $\xi \mapsto \xi^\circ$ is well-defined. □

Remark 2.4. Sometimes, especially in the physics literature, a stable sandpile is defined as having possible values $1, \dots, \deg_G(x)$ at x, and a toppling of x is allowed when $\eta(x) > \deg_G(x)$. It is easy to see that this merely amounts to a shift of coordinates, and defines the same model.

Motivating remarks. The sandpile dynamics can be viewed as a toy model of avalanche-type phenomena. Adding a single particle to the pile and stabilizing can induce a complex sequence of topplings, called an "avalanche". However, the model is *not intended* as a realistic model of sand. In order to model sand grains moving down a slope, a more suitable condition for toppling could be that the *discrete gradient* exceeds some fixed critical value $d_c > 0$. It is easy to see that in such models topplings do not commute. In fact, if $y_1 \sim x \sim y_2$ and $\eta(x)-\eta(y_1) = d_c = \eta(x) - \eta(y_2)$, one needs to make a choice in the model if a particle from x will move to y_1 or y_2. We will see later that commutativity in the Abelian sandpile has many nice consequences, which make it more amenable to study. The point is that the Abelian model already possesses important qualitative features of avalanche-like phenomena and as we will see, has very nontrivial behaviour. We will return to this in Section 3.

Exercise 2.5. *(Asymmetric sandpile model)* Let $G = (V \cup \{s\}, E)$ be a *directed* graph with a distinguished vertex s. Find appropriate definitions of "stable" and "toppling". Find a condition on G that ensures that stabilization is well-defined. See [33].

Exercise 2.6. *(Least action principle)* Check that the argument of Theorem 2.3(b) gives the following stronger statement. Suppose that $x_1, x_2, \dots, x_k$ is a stabilizing sequence for η consisting of legal topplings. Suppose that $y_1, y_2, \dots, y_\ell$ is any other sequence of possibly illegal topplings, such that carrying them out results in a stable configuration. Then each vertex is toppled at least as many times in the y-sequence as in the x-sequence. In other words, with legal topplings, each vertex does the minimum amount of required "work" to stabilize the configuration. See [16] for more on this "least action principle".

Definition 2.7. The **addition operators** are the maps on sandpiles defined by adding one particle at x and stabilizing. More formally, $E_x\eta = (\eta + \mathbf{1}_x)^\circ$, where $\mathbf{1}_x$ is the row vector with 1 in position x and 0 elsewhere. The sequence of topplings carried out in stabilizing $\eta + \mathbf{1}_x$ is called the **avalanche** induced by the addition.

Lemma 2.8. *[11] We have $E_xE_y = E_yE_x$ for all $x, y \in V$.*

Proof. We have

$$E_xE_y\eta = ((\eta + \mathbf{1}_y)^\circ + \mathbf{1}_x)^\circ \qquad (6)$$

and

$$E_y E_x \eta = ((\eta + \mathbf{1}_x)^\circ + \mathbf{1}_y)^\circ. \tag{7}$$

We show that both expressions equal

$$(\eta + \mathbf{1}_x + \mathbf{1}_y)^\circ. \tag{8}$$

To see this, start with the configuration $\eta + \mathbf{1}_x + \mathbf{1}_y$, and carry out topplings as in the stabilization of $\eta + \mathbf{1}_y$. The extra particle present at x does not affect the legality of any of the topplings. Hence with the extra particle at x present, we arrive at the configuration $(\eta + \mathbf{1}_y)^\circ + \mathbf{1}_x$. Now carry out any further topplings that are possible, arriving at the right hand side of (6). Due to Theorem 2.3, the final configuration also equals (8). Equality of (8) and (7) is seen similarly. □

So far the dynamics has been determinisitic. We now add randomness and define the **sandpile Markov chain** as follows. We take as state space the set Ω_G of stable sandpiles. Fix a positive probability distribution p on V, i.e. $\sum_{x \in V} p(x) = 1$ and $p(x) > 0$ for all $x \in V$. Given the current state $\eta \in \Omega_G$, pick a random vertex $X \in V$ according to p, add a particle there, and stabilize to obtain the next state of the Markov chain. That is, the Markov chain makes the transition

$$\eta \longrightarrow E_X \eta = (\eta + \mathbf{1}_X)^\circ.$$

If the Markov chain has initial state η_0, we can write the time evolution, using Theorem 2.3 and Lemma 2.8, as

$$\eta_n = \left(\eta_0 + \sum_{i=1}^n \mathbf{1}_{X_i} \right)^\circ = E_{X_n} \dots E_{X_1} \eta_0,$$

where $X_1, X_2, \dots$ are i.i.d. random variables distributed according to p.

We denote by $\eta^{\max}$ the sandpile defined by $\eta^{\max}(x) = \deg_G(x) - 1$, $x \in V$. For sandpiles η, ξ, we write $\eta \geq \xi$, if $\eta(x) \geq \xi(x)$ for all $x \in V$.

Recall the following standard terminology for general Markov chains. For two states η, ξ of a Markov chain we say that ξ *can be reached from* η, if there exists $n \geq 0$ such that $\mathbf{p}^n(\eta, \xi) > 0$, where $\mathbf{p}^n$ is the n-step transition probability. We say that η *and* ξ *communicate*, if they can be reached from each other. This is an equivalence relation, and the equivalence classes are the *communicating classes*. A state η is *recurrent*, if starting from η the chain returns to η with probability 1, and it is *transient* otherwise. Recurrence and transience are *class properties*, that is, if one state in a class is recurrent then all states are.

Theorem 2.9. *[11, 33] Consider the sandpile Markov chain on any finite connected multigraph $G = (V \cup \{s\}, E)$ (satisfying $p(x) > 0$ for all $x \in V$).*
(i) There is a single recurrent class.
(ii) The following are equivalent for $\eta \in \Omega_G$:

- *(a) η is recurrent;*
- *(b) there exists a sandpile $\xi \geq \eta^{\max}$ such that $\xi^\circ = \eta$;*

(c) *for any sandpile σ, it is possible to reach η from σ by adding particles and toppling vertices, i.e. there exists a sandpile ζ such that $\eta = (\sigma + \zeta)^\circ$.*

Proof. (i) The configuration $\eta^{\max}$ is reachable for the Markov chain from any $\zeta \in \Omega_G$ (by addition of particles). Hence $\eta^{\max}$ is recurrent, and the recurrent class containing it is the only recurrent class.

(ii) (a) $\Longrightarrow$ (b). If η is recurrent, it is reachable from $\eta^{\max}$, i.e. there exist $k \geq 0$ and $x_1, \dots, x_k \in V$ such that

$$\eta = E_{x_k} \dots E_{x_1} \eta^{\max} = \left(\eta^{\max} + \sum_{i=1}^{k} \mathbf{1}_{x_i} \right)^\circ .$$

Hence we can take ξ to be the configuration inside the parentheses on the right hand side.

(b) $\Longrightarrow$ (a). If $\xi^\circ = \eta$, $\xi \geq \eta^{\max}$, we can write $\xi = \eta^{\max} + \sum_{i=1}^{k} \mathbf{1}_{x_i}$ with some $x_1, \dots, x_k \in V$, so that $\eta = E_{x_k} \dots E_{x_1} \eta^{\max}$. This shows that η is reachable from $\eta^{\max}$, and hence recurrent.

(c) $\Longrightarrow$ (b). This is obvious by taking $\sigma = \eta^{\max}$.

(b) $\Longrightarrow$ (c). Let $\xi \geq \eta^{\max}$ be such that $\xi^\circ = \eta$. Take $\zeta := \xi - \sigma^\circ \geq \eta^{\max} - \sigma^\circ \geq 0$. Then since $\xi - \sigma^\circ \geq 0$, starting from $\sigma + \zeta = \sigma + \xi - \sigma^\circ$ we can legally topple a sequence of vertices that stabilizes σ, and arrive at the configuration $\sigma^\circ + \xi - \sigma^\circ = \xi$. Now we can legally topple a sequence of vertices that stabilizes ξ, and arrive at the configuration η. This shows that η has property (c). $\square$

Definition 2.10. We denote by $\mathcal{R}_G$ the set of recurrent sandpiles.

2.2. The sandpile group / critical group

Let $G = (V \cup \{s\}, E)$ be a finite connected multigraph. We now define the sandpile group of G. Consider $\mathbb{Z}^V$ as an Abelian group. The integer row span $\mathbb{Z}^V \Delta'_G$ of the matrix Δ'_G forms a subgroup of $\mathbb{Z}^V$. For $\xi, \zeta \in \mathbb{Z}^V$, let us write $\xi \sim \zeta$ if $\xi - \zeta \in \mathbb{Z}^V \Delta'_G$. This is an equivalence relation, and we write $[\xi]$ to denote the equivalence class containing ξ. The equivalence classes form an Abelian group, the factor group

$$K_G := \mathbb{Z}^V / \mathbb{Z}^V \Delta'_G.$$

The group K_G is called the **sandpile group** of G (sometimes called the **critical group** in the combinatorics literature). Any toppling corresponds to subtracting a row of Δ'_G from the configuration (recall (1)). Therefore, during stabilization a configuration is replaced by an equivalent one. Hence we can expect that the group K_G plays a role in understanding the sandpile Markov chain. This is made precise by the following theorem.

Theorem 2.11. *[11] (i) Every equivalence class in K_G contains precisely one recurrent sandpile in $\mathcal{R}_G$. In particular,*

$$|\mathcal{R}_G| = |K_G| = \det(\Delta'_G).$$

(ii) Consequently, the following operation $\oplus : \mathcal{R}_G \times \mathcal{R}_G \to \mathcal{R}_G$ turns $\mathcal{R}_G$ into an Abelian group isomorphic to K_G:

$$\eta \oplus \xi := (\eta + \xi)^\circ.$$

The proof of (i) we give is due to [33]. We will need the following lemma of [33] that provides a configuration with a special property (later it will become clear that this is a representative of the identity of K_G).

Lemma 2.12. *[33] Let $\varepsilon := \delta - \delta^\circ$, where δ is defined by $\delta(x) = \deg_G(x)$, $x \in V$. If η is recurrent, then $(\eta + \varepsilon)^\circ = \eta$.*

Proof. By Theorem 2.9, if η is recurrent, it is possible to add particles to δ and stabilize to get η. That is, there exists $\zeta \geq 0$ such that $\eta = (\delta + \zeta)^\circ$. Consider the configuration

$$\gamma = (\zeta + \delta) + \varepsilon = \delta + \zeta + \delta - \delta^\circ.$$

Since $\varepsilon \geq 0$, we can start from γ and legally topple a sequence of vertices that stabilizes $\zeta + \delta$, arriving at the configuration $\eta + \varepsilon$. Stabilizing further gives $(\eta + \varepsilon)^\circ$. On the other hand, since $\delta - \delta^\circ \geq 0$, we can start from γ, and legally topple a sequence of vertices that stabilizes δ, arriving at the configuration $\delta^\circ + \zeta + \delta - \delta^\circ = \zeta + \delta$. Stabilizing further we obtain η. Comparing the two stabilizing sequences, Theorem 2.3(ii) yields $\eta = (\eta + \varepsilon)^\circ$. □

Proof of Theorem 2.11. (i) We first observe that every equivalence class contains some representative ξ with $\xi \geq \eta^{\max}$. Then $\xi^\circ =: \eta \in \mathcal{R}_G$ by Theorem 2.9(ii), and $\eta \in [\xi]$. It follows that $\mathcal{R}_G$ intersects each equivalence class.

It remains to show that the intersection of $\mathcal{R}_G$ with any equivalence class contains at most one element. To see this, suppose that $\eta_1 \sim \eta_2$, $\eta_1, \eta_2 \in \mathcal{R}_G$, and we show $\eta_1 = \eta_2$. Since $\eta_1 \sim \eta_2$, there exist $c_x \in \mathbb{Z}$, $x \in V$, such that

$$\eta_1 = \eta_2 + \sum_{x \in V} c_x \Delta'_{x,\cdot}.$$

Let $V_- := \{x \in V : c_x < 0\}$ and $V_+ := \{x \in V : c_x > 0\}$, and define

$$\eta := \eta_1 + \sum_{x \in V_-} (-c_x)\Delta'_{x,\cdot} = \eta_2 + \sum_{x \in V_+} c_x \Delta'_{x,\cdot}.$$

Take k large enough such that the configuration η' defined by $\eta' = \eta + k\varepsilon$ satisfies $\eta'(x) \geq |c_x| \deg_G(x)$ for all $x \in V$. This is possible, since each entry of ε is at least 1. Starting from η', we may legally topple $(-c_x)$-times each vertex $x \in V_-$, arriving at the configuration $\eta_1 + k\varepsilon$. This further stabilizes to η_1, by Lemma 2.12. Similarly, we can legally topple c_x-times each vertex $x \in V_+$, arriving at the configuration $\eta_2 + k\varepsilon$. This further stabilizes to η_2, again by Lemma 2.12. Comparing the two stabilzations, Theorem 2.3(ii) yields $\eta_1 = \eta_2$.

The number of elements of K_G is the index of the subgroup $\mathbb{Z}^V \Delta'_G$. It is easy to see that this equals the determinant of Δ'_G.

(ii) It is not difficult to show (see Exercise 2.13) that if $\eta, \xi \in \mathcal{R}_G$, we have $(\eta + \xi)^\circ \in \mathcal{R}_G$. Hence $\oplus$ indeed maps $\mathcal{R}_G \times \mathcal{R}_G$ into $\mathcal{R}_G$. We also have

$$[\eta \oplus \xi] = [(\eta + \xi)^\circ] = [\eta + \xi] = [\eta] + [\xi].$$

This shows that $\oplus$ indeed corresponds to the group operation in K_G. $\square$

Exercise 2.13. Show that if $\eta, \xi \in \mathcal{R}_G$, then $(\eta + \xi)^\circ \in \mathcal{R}_G$. (*Hint:* One way to see this is the criterion of Theorem 2.9(ii)(b).) See [11] and [33]*Corollary 2.16.

The identity element of K_G, that is, the unique recurrent configuration $\eta_0 \in \mathcal{R}_G$ such that $[\eta_0] = [0]$, displays highly non-trivial features. On large rectangular regions in $\mathbb{Z}^2$, the identity element displays both regular and fractal patterns; see the pictures in [6, 57]. Le Borgne and Rossin [6] prove some rigorous results for rectangular regions.

Exercise 2.14. Show that if there is a unique $x \in V$ such that $x \sim s$, then E_x restricted to $\mathcal{R}_G$ is the identity. (*Hint:* Show that $\mathbf{1}_x$ equals the sum of the rows of Δ'_G. This is a special case of Dhar's "multiplication by identity test"; see [11].)

Exercise 2.15. Suppose that $V \subset \mathbb{Z}^d$, and G is the wired graph induced by V. Show that if $p(x) > 0$ for all $x \in V$ such that $x \sim s$, then the sandpile Markov chain is irreducible. (That is, in this case the condition imposed on p in Theorem 2.9 can be substantially relaxed.) See [67].

2.3. The stationary distribution and Dhar's formula

Once the sandpile Markov chain reaches the set $\mathcal{R}_G$ of recurrent sandpiles, it never leaves it. Let us write ν_G for the stationary distribution, which by Theorem 2.9(i) is unique, and is concentrated on $\mathcal{R}_G$. In view of Theorem 2.11, the restriction of the Markov chain to $\mathcal{R}_G$ can be identified with a random walk on the finite group K_G. A transition from the state $\eta \in \mathcal{R}_G$ to $(\eta + \mathbf{1}_x)^\circ$, $x \in V$, is identified with adding to $[\eta] \in K_G$ the group element $[\mathbf{1}_x]$. As the next easy exercise shows, this implies that the stationary distribution of the sandpile Markov chain is *uniform* on $\mathcal{R}_G$.

Exercise 2.16. Let K be a finite group, and let $(X_n)_{n \geq 0}$ be an irreducible random walk on K, that is a Markov chain with transition matrix $P(h, gh) = \mu(g)$, where $\sum_{g \in K} \mu(g) = 1$, and the support of μ generates K. Show that the stationary distribution of $(X_n)_{n \geq 0}$ is uniform on K. See [80].

The following exercise is essentially a triviality. However, we prefer to state it explicitly for two reasons: (i) it will play an important role in Theorem 2.18 below; (ii) its version for infinite graphs is far from trivial.

Exercise 2.17. Check that the additition operators $E_x : \mathcal{R}_G \to \mathcal{R}_G$, $x \in V$, leave the measure ν_G invariant. See [11, 67].

The following theorem due to Dhar [11] gives a formula for the average number of topplings induced by adding a single particle, under stationarity. For a sandpile η and $x, y \in V$, let us write $n(x, y; \eta)$ for the number of topplings occurring at y during the stabilization of $\eta + \mathbf{1}_x$.

Theorem 2.18. *Let $G = (V \cup \{s\}, E)$ be a finite connected multigraph. We have*

$$\mathbf{E}_{\nu_G}[n(x, y, \cdot)] = (\Delta'_G)^{-1}_{xy}, \quad x, y \in V. \tag{9}$$

Proof. From the definition of stabilization we have the relation:

$$(\eta + \mathbf{1}_x)^\circ(y) = \eta(y) + \mathbf{1}_x(y) - \sum_{z \in V} n(x, z; \eta)(\Delta'_G)_{zy}.$$

Now average both sides with respect to the stationary distribution ν_G. We get

$$\mathbf{E}_{\nu_G}[(\eta + \mathbf{1}_x)^\circ(y)] = \mathbf{E}_{\nu_G}[\eta(y)] + \mathbf{1}_x(y) - \sum_{z \in V} \mathbf{E}_{\nu_G}[n(x, z; \eta)](\Delta'_G)_{zy}.$$

Due to Exercise 2.17, the left hand side equals the first term on the right hand side, which gives

$$\sum_{z \in V} \mathbf{E}_{\nu_G}[n(x, z; \eta)](\Delta'_G)_{zy} = \mathbf{1}_x(y).$$

Since this holds for all $x, y \in V$, we get (9). $\square$

The above theorem is extremely useful in estimating topplings in an avalanche. However, it only gives information about the first moment of the toppling numbers.

Open Question 2.19. Find a useful expression for the second moment of the toppling numbers.

3. Motivation from statistical physics

In the physics literature, the sandpile model appears in connection with the notion of "self-organized criticality" (SOC) [11, 12]. In order to explain what SOC is, we first clarify the meaning of "criticality" in the example of percolation in Section 3.1. In Section 3.2 the notion of SOC is illustrated via an example closely related to percolation. In Section 3.3, we discuss SOC in the sandpile model and state some open problems. In Section 4 various connections to other models will be presented as well.

3.1. Percolation — an example of a critical phenomenon

A simple-to-define but deep example of criticality is provided by **bond percolation** on the d-dimensional integer lattice $\mathbb{Z}^d$. Let $0 < p < 1$. Declare each edge of $\mathbb{Z}^d$ (also called a bond) **occupied** with probability p and **vacant** with probability $1-p$, independently. Percolation theory studies the geometry of the connected components (called **clusters**) of the random subgraph of $\mathbb{Z}^d$ induced by the occupied bonds. We are going to write $\mathbf{P}_p$ for the underlying probability measure when the parameter value is p. Let $\mathcal{C}$ denote the connected occupied component containing the origin. A fundamental result in percolation theory is the following theorem due to Broadbent and Hammersley [8, 26, 27].

Theorem 3.1. *Let $d \geq 2$. There exists a critical probability $0 < p_c = p_c(d) < 1$ such that*

$$\begin{aligned} \mathbf{P}_p[|\mathcal{C}| < \infty] &= 1, \qquad \text{if } p < p_c; \\ \mathbf{P}_p[|\mathcal{C}| = \infty] &> 0, \qquad \text{if } p > p_c. \end{aligned}$$

It is easy to see using translation invariance that in the case $p < p_c$ there is no infinite cluster anywhere in the lattice $\mathbf{P}_p$-a.s. It can also be shown that in the case $p > p_c$ there exists a *unique* infinite cluster *somewhere* in the lattice; see [23]. Note the qualitative similarity with extinction/survival for a branching process depending on whether the mean offspring is less than or greater than 1. One says that a **phase transition** occurs at the **critical value** $p = p_c$ as the parameter p is increased. The critical value separates the **subcritical phase** $p < p_c$ where all clusters are finite a.s., and the **supercritical phase** $p > p_c$ where there exists an infinite cluster a.s.

Percolation at the critical point $p = p_c$ has features that set it apart from the sub- and supercritical phases. For example, it is known that for percolation with $p \neq p_c$, the probabilities $\mathbf{P}_p[|\mathcal{C}| = k]$ decay fast with k. The results [23]*Theorem 6.78, [23]*Theorem 8.65 say that when $d \geq 2$, we have

$$\begin{aligned} \mathbf{P}_p\Big[|\mathcal{C}| \geq k\Big] &\leq C_1(p) \exp(-c_1(p) k) \qquad \text{when } p < p_c; \\ \mathbf{P}_p\Big[k \leq |\mathcal{C}| < \infty\Big] &\leq C_2(p) \exp\big(-c_2(p) k^{(d-1)/d}\big) \qquad \text{when } p > p_c; \\ \mathbf{P}_p\Big[x \in \mathcal{C},\ |\mathcal{C}| < \infty\Big] &\leq \exp\big(-c_3(p)|x|\big) \qquad \text{when } p \neq p_c. \end{aligned}$$

While these results are already not easy to establish, the case of $p = p_c$ is even more challenging. For example, it is a major conjecture that $\mathbf{P}_{p_c}[|\mathcal{C}| < \infty] = 1$ in all dimensions $d \geq 2$. So far, this has only been established in the planar case $d = 2$ [32, 48] and when d is sufficiently large ($d \geq 15$) [3, 20, 21, 29]. A more detailed conjecture is that in all dimensions $d \geq 2$, the behaviour at and close to $p = p_c$ is characterized by power laws. For example, it is expected that there exist **critical exponents** $\beta, \delta, \rho, \gamma, \eta \geq 0$, depending on the dimension d, such that

$$\begin{aligned} \mathbf{P}_p[|\mathcal{C}| = \infty] &= (p - p_c)^{\beta + o(1)} \qquad \text{as } p \downarrow p_c; \\ \mathbf{P}_{p_c}[|\mathcal{C}| \geq k] &= k^{-1/\delta + o(1)} \qquad \text{as } k \to \infty; \\ \mathbf{P}_{p_c}[\mathrm{rad}(\mathcal{C}) \geq n] &= n^{-1/\rho + o(1)} \qquad \text{as } n \to \infty; \\ \mathbf{E}_p[|\mathcal{C}|; |\mathcal{C}| < \infty] &= |p - p_c|^{-\gamma + o(1)} \qquad \text{as } p \to p_c; \\ \mathbf{P}_{p_c}[x \in \mathcal{C}] &= \frac{1}{|x|^{d-2+\eta+o(1)}} \qquad \text{as } |x| \to \infty; \end{aligned} \tag{10}$$

where $\mathrm{rad}(\mathcal{C}) = \sup\{|x| : x \in \mathcal{C}\}$. A further conjecture of **universality** states that the values of the exponents are not sensitive to the structure of the lattice. In particular, they would not change if the cubic lattice is replaced by any other d-dimensional periodic lattice.

Most progress on the conjectures (10) has been made in $d = 2$ and in high dimensions. In the planar case, replace bond percolation on $\mathbb{Z}^2$ by so-called **site percolation** on the triangular lattice. Here the *vertices* of the triangular lattice are declared occupied/vacant with probabilities p and $1 - p$, and the nearest neighbour occupied connected components are considered. The combined result of the papers [49, 53, 82, 83] is that for site percolation on the triangular grid we have $\beta = 5/36$, $\delta = 91/5$, $\rho = 48/5$, $\gamma = 43/18$, $\eta = 5/24$. In sufficiently high dimensions ($d \geq 11$) it has been established that $\beta = 1, \delta = 2$, $\rho = 1/2$, $\gamma = 1$, $\eta = 0$; [3, 20, 21, 28–31, 50]. It is conjectured that these are the values of the exponents for all $d > 6$. (The exponents have been established for all $d > 6$ in a modified model where long bonds are allowed; see the above references.)

3.2. Self-organized criticality

At first it seems that the intriguing properties of critical percolation are very sensitive to the fact that we are at the critical point: $p = p_c$ exactly, or (in large finite systems) $p \approx p_c$. However, there are interesting examples of *dynamically evolving* models where criticality (i.e. power law behaviour of various distributions) occurs in a rather robust fashion.

An example can be built on top of the Erdős-Rényi random graph model [7], that can be viewed as percolation on the complete graph with n vertices. It will be useful to consider the random graph in a dynamical fashion. At time $t = 0$ we have n vertices and no edges. Between any pair of vertices, independently, an edge is added at rate $1/n$. This choice of rates ensures that locally around each vertex $O(1)$ edges appear per unit time. Write

$$v_k^n(t) = \text{proportion of vertices in clusters of size } k \text{ at time } t,$$

where the 'size' of a cluster is the number of its vertices. In the limit $n \to \infty$ there is a phase transition. A giant component containing a positive fraction of all vertices emerges at the critical time $t_c = 1$. One way to formalize this statement is that $v_k^n(t)$ converges in probability to a deterministic limit $v_k(t)$, as $n \to \infty$, where the limit satisfies:

$$\theta(t) := 1 - \sum_{k \geq 1} v_k(t) \begin{cases} = 0 & \text{if } t \leq 1; \\ > 0 & \text{if } t > 1. \end{cases}$$

Compare with Theorem 3.1. At the critical time $t_c = 1$, we have the power law $v_k(t_c) \sim ck^{-3/2}$, as $k \to \infty$.

Forest-fire model. Let us modify the dynamics in a way that prevents the giant component from emerging, and keeps the system "at criticality". Return to the finite n model, and let $\lambda(n)$ be a function satisfying $1/n \ll \lambda(n) \ll 1$. Suppose that "lightning" hits each vertex, independently, at rate $\lambda(n)$. When a vertex is hit by lightning, the cluster containing it disintegrates into individual vertices, that is, all edges within the cluster become vacant instantaneously. Heuristically, this mechanism should prevent clusters to reach size of order n, since then it is extremely likely that they are hit by lightning ($n\lambda(n) \gg 1$). On the other hand, our assumption $\lambda(n) \ll 1$ implies that small clusters are not likely to be hit by lightning, so they will be growing more-or-less as in the Erdős-Rényi model. The heuristic suggests that after the critical time the system remains critical forever. Ráth and Tóth [78] proved that this is indeed the case.

Theorem 3.2. *[78] Suppose that $1/n \ll \lambda(n) \ll 1$. Let $\bar{v}_k^n(t)$ be the proportion of vertices in clusters of size k at time t in the forest-fire evolution.*
(i) There is a deterministic limit in probability $\bar{v}_k(t) = \lim_{n\to\infty} \bar{v}_k^n(t)$.
(ii) If $t \le t_c = 1$, $\bar{v}_k(t) = v_k(t)$.
(iii) If $t \ge t_c = 1$ we have $\sum_{\ell \ge k} \bar{v}_\ell(t) \asymp k^{-1/2}$.

Remark 3.3. Analogous statements hold for general initial conditions satisfying a moment condition; see [78].

The phenomenon that a state characterized by power laws is reached and then maintained by the dynamics is called **self-organized criticality**. The term was introduced by Bak, Tang and Wiesenfeld [2], who suggested that this mechanism would be present in many physical systems for which power laws had been observed empirically. Examples include: energy release in earthquakes, avalanche sizes in rice- and sandpiles, areas of forest-fires and many others; see the book [45]. Bak, Tang and Wiesenfeld used the sandpile dynamics to illustrate their idea via numerical simulations [2]. After Dhar [11] discovered the Abelian property and established the fundamental results discussed in Section 2, the Abelian sandpile became the primary theoretical example of SOC [12].

3.3. Self-organized criticality in the Abelian sandpile model

The following heuristic suggests that on a large graph, avalanches in the stationary sandpile Markov chain will occur on all scales up to the size of the system. Start from an empty pile. Initially, when not many particles have been added yet, avalanches will be small. As more particles get added, the typical size of avalanches grows. The only limit to this growth is that particles are lost to the sink. Hence when stationarity is reached, we can expect to see avalanches on all scales up to the size of the system.

Numerical simulations [12, 65] of the model on subsets of $\mathbb{Z}^d$ suggest that the above heuristic is correct, and various avalanche characteristics have power law distributions, up to a cut-off that grows with the system size. In this section we state some conjectures that quantify this in terms of critical exponents. In our discussions, we consider the model on the wired graph G_n constructed from a finite box $V_n = [-n, n]^d \cap \mathbb{Z}^d$, as in Example 2.1.

First we briefly comment on the case $d = 1$. Here one can explicitly compute the set of recurrent configurations and the sandpile group; see Exercise 4.6. The sandpile group of G_n is isomorphic to $\mathbb{Z}_{2n+2}$; in particular, the number of recurrent configurations grows only linearly in n. This is in contrast with $d \geq 2$, where the number of recurrent configurations grows exponentially in n^d. There is significantly "less randomness" in $d = 1$ than in $d \geq 2$. Avalanches can be computed explicitly in $d = 1$, and it is found that with probability approaching 1 all avalanches reach the sink. On the other hand, for $d \geq 2$ most avalanches do not reach the sink. Below we restrict our attention to $d \geq 2$, and refer to [12, 62] for more details on the one-dimensional case.

We write ν_n for the stationary distribution of the sandpile Markov chain on G_n. Given $z \in V_n$, we define the **avalanche cluster at** z as the set of vertices that are toppled when we add a particle at z to the sandpile ξ:

$$\mathrm{Av}_{z,V_n} = \mathrm{Av}_{z,V_n}(\xi) := \{x \in V_n : n(z, x, \xi) > 0\}. \tag{11}$$

We also define the **size** of the avalanche as the number of topplings, with multiplicity:

$$S_{z,V_n} = S_{z,V_n}(\xi) := \sum_{x \in V_n} n(z, x; \xi). \tag{12}$$

The **radius** of the avalanche is:

$$\mathrm{rad}(\mathrm{Av}_{z,V_n}(\xi)) := \max\{|x - z| : x \in \mathrm{Av}_{z,V_n}(\xi)\}. \tag{13}$$

3.3.1. Two easy critical exponents

We start with computing two easy exponents. Recall Dhar's formula from Section 2.3. Observe that

$$\frac{1}{2d}(\Delta'_{V_n})_{zx} = I_{zx} - \mathbf{p}_n^1(z, x),$$

where $\mathbf{p}_n^1(z, x)$ is the transition matrix of simple random walk on V_n stopped on the first exit from V_n, and I is the identity matrix. Denote

$$\begin{aligned} \mathbf{p}_n^k(z, x) &:= k\text{-step transition probability from } z \text{ to } x; \\ G_n(z, x) &:= \left(\Delta'_{V_n}\right)^{-1}_{zx}, \quad z, x \in V_n. \end{aligned}$$

The matrix G_n has a well-known interpretation in terms of the simple random walk.

Exercise 3.4. (i) Show that

$$2d\, G_n(z, x) = \sum_{k=0}^{\infty} \mathbf{p}_n^k(z, x) = \mathbf{E}^z[\text{number of visits to } x \text{ before exiting } V_n].$$

(ii) Show that if $z \in [-n/2, n/2]^d \cap \mathbb{Z}^d$, we have

$$\sum_{x \in V_n} G_n(z,x) \asymp n^2, \quad n \geq 1.$$

See [52]*Lemma 4.6.1 and Proposition 6.2.6

Dhar's formula in the present setting says that

$$\mathbf{E}_{\nu_n}[n(z,x;\cdot)] = G_n(z,x). \tag{14}$$

Exercise 3.4(ii) gives that

$$\mathbf{E}_{\nu_n}[S_{o,V_n}] \asymp n^2, \tag{15}$$

where we write o to denote the origin in $\mathbb{Z}^d$. In particular, in the stationary sandpile, the expected size of an avalanche started at o diverges as $n \to \infty$. Compare this with the divergence of the expected cluster size for critical percolation.

Let $d \geq 3$. Then

$$\lim_{n\to\infty} G_n(z,x) = G(z,x) = (2d)^{-1}\mathbf{E}^z[\text{number of visits to } x] < \infty \tag{16}$$

exists. The function $2d\,G(z,x)$ is called the **Green function** of the random walk. It is known that for all $d \geq 3$ the Green function is asymptotic to $a_d|x - z|^{2-d}$ as $|x-z| \to \infty$; see [52]*Theorem 4.3.1. Hence, by (14) and (16), we have

$$\lim_{n\to\infty} \mathbf{E}_{\nu_n}[n(o,x;\cdot)] = G(o,x) \sim \frac{(2d)^{-1}a_d}{|x|^{d-2}} \quad \text{as } |x| \to \infty. \tag{17}$$

In order to neatly formulate asymptotic results as $n \to \infty$, the following theorem will be useful.

Theorem 3.5. *[1, Theorem 1] Let $d \geq 2$. There is a measure ν on the space $\{0,1,\dots,2d-1\}^{\mathbb{Z}^d}$ such that $\nu_n \Rightarrow \nu$ in the sense of weak convergence.*

Now (17) can be rephrased in the simpler form (see [40]): for all $d \geq 3$ we have

$$\mathbf{E}_{\nu}[n(o,x;\cdot)] \sim \frac{(2d)^{-1}a_d}{|x|^{d-2}} \quad \text{as } |x| \to \infty. \tag{18}$$

Here the precise meaning of the random variable $n(o,x;\cdot)$ is as follows. Draw a sample configuration from the limiting measure ν. Add a particle at o, and attempt to stabilize by toppling all unstable sites simultaneously, whenever there are such. Then $n(o,x;\cdot)$ is the number of induced topplings at x. We write $\mathrm{Av}_z = \{x \in \mathbb{Z}^d : n(z,x;\cdot) > 0\}$ and $S_z = \sum_{x\in\mathbb{Z}^d} n(z,x;\cdot)$.

3.3.2. Further critical exponents

The relation (18) gives the average number of topplings at x induced by adding a particle at o. We now state a theorem and a conjecture concerning the *probability* that x topples, if we add a particle at o.

Theorem 3.6. *[41] For all $d \geq 5$ there are constants $c = c(d), C = C(d) > 0$ such that*

$$\frac{c}{|x|^{d-2}} \leq \nu[x \in \mathrm{Av}_o] \leq \frac{C}{|x|^{d-2}}, \quad x \neq 0. \tag{19}$$

Note that the upper bound follows easily from Dhar's formula (18) and Markov's inequality, so the real content of the theorem is the lower bound.

Conjecture 3.7. For $2 \leq d \leq 4$ there exists $\eta = \eta(d) \geq 0$ such that

$$\nu[x \in \mathrm{Av}_o] = \frac{1}{|x|^{d-2+\eta+o(1)}} \quad \text{as } |x| \to \infty.$$

We have intentionally written the exponent in the form $d-2+\eta$, in order to highlight the comparison with (19) and (10). Upper bounds for the exponent η in dimensions $d = 2, 3, 4$ were recently proved in [5].

The second conjecture we state concerns the number of topplings in an avalanche. This can be measured by the total number of topplings, with or without multiplicity.

Conjecture 3.8. For all $d \geq 2$ there exist exponents $\tau = \tau(d)$, $\tau' = \tau'(d) \geq 0$ depending on d such that

$$\nu[S_o \geq k] = k^{1-\tau+o(1)}, \quad \text{as } k \to \infty;$$
$$\nu[|\mathrm{Av}_o| \geq k] = k^{1-\tau'+o(1)}, \quad \text{as } k \to \infty;$$

Since $\mathbf{E}_\nu S_o = \infty$, we must have $\tau \leq 2$, if it exists. It is also plausible that $\mathbf{E}_\nu |\mathrm{Av}_o| = \infty$, so we should also have $\tau' \leq 2$, if it exists. Manna [65] presents numerical evidence suggesting that $\tau = \tau'$. Theorem 3.6 gives rigorous support to this conjecture when $d \geq 5$. Indeed, (19) shows that

$$\mathbf{E}_\nu[n(o, x; \cdot) \,|\, n(o, x; \cdot) \geq 1] = O(1),$$

that is, the average number of topplings at x, conditional on the event that x topples, is $O(1)$. Heuristic arguments suggesting that $\tau = \tau' = 3/2$ when $d > 4$ were given by Priezzhev [76]. This has recently been proved Hutchcroft [35] in great generality that goes well beyond $\mathbb{Z}^d$.

Finally, we state a conjecture regarding the radius of an avalanche.

Conjecture 3.9. For all $d \geq 2$ there exists $\alpha \geq 0$ such that

$$\nu[\mathrm{rad}(\mathrm{Av}_o) > r] = r^{-\alpha+o(1)}, \quad \text{as } r \to \infty.$$

When $d > 4$, it was conjectured in [76] that $\alpha = 2$. This has recently been proved by Hutchcroft [35]. Upper bounds on the exponent α in $d = 2, 3, 4$, and lower bounds in $d = 3, 4$ were given in [5].

It is a well-known heuristic in statistical physics that the behaviour of a lattice model in sufficiently high dimensions can be approximated by its behaviour on an infinite k-regular tree with $k \geq 3$, called the **Bethe lattice**. In general, there exists a **critical dimension** d_c such that for $d > d_c$ the values of the critical exponents do not depend on d and take on the same values as on the k-regular tree. The conjectured critical dimension for the sandpile model is $d_c = 4$ (for the percolation model of Section 3.1 it is conjectured to be $d_c = 6$). Rigorous support to the idea that $d_c = 4$ for the Abelian sandpile model is provided by Theorem 6.3, showing that there is a clear change in behaviour at dimension 4 for the closely related wired spanning forest measure.

When $\mathbb{Z}^d$ is replaced by a k-regular tree, the distribution of the random set Av_o has been computed explicitly by Dhar and Majumdar [13] using combinatorial methods. In particular, they obtain $\tau = 3/2$. This has been recently extended to supercritical Galton-Watson trees [42]. It is natural to define the Euclidean distance between vertices z, x of the tree as the square root of their graph distance. With respect to this distance Dhar and Majumdar [13] also obtain $\alpha = 2$.

4. Connections to other models

One of the appeals of the Abelian sandpile is its close relationship with other probability models on graphs. In this section we present connections to: spanning trees; the rotor-router walk; the random cluster model and the Tutte polynomial.

4.1. The burning bijection

As in Section 2, let $G = (V \cup \{s\}, E)$ be a finite connected multigraph. The **burning algorithm** introduced by Dhar [11] provides an efficient way to check whether a given stable sandpile is recurrent. This gives a combinatorial characterization of recurrent sandpiles. The algorithm leads to the **burning bijection** between recurrent sandpiles on V and spanning trees of G. This bijection is due to Majumdar and Dhar [64].

Definition 4.1. Let $\eta \in \Omega_G$, and let $\emptyset \neq F \subset V$. We say that η is **ample** for F, if there exists $x \in F$ such that $\eta(x) \geq \deg_F(x)$ (the degree of x in the subgraph induced by the set of vertices F).

Lemma 4.2. *If $\eta \in \mathcal{R}_G$, then η is ample for all $\emptyset \neq F \subset V$.*

Proof. Due to Theorem 2.9(ii)(c), there exists ξ such that $\xi^\circ = \eta$ and $\xi(x) \geq \deg_G(x)$ for all $x \in F$. Fix a stabilizing sequence for ξ. Observe that each vertex in F must topple in this stabilizing sequence. Let x be the first vertex among the vertices in F that finishes toppling. After x finishes toppling, it receives particles from each of its neighbours in F (as each of these neighbours will still topple). The number of particles received is altogether $\sum_{y \in F,\, y \neq x} a_{yx} = \deg_F(x)$. Hence $\eta(x) \geq \deg_F(x)$, as required. ☐

The burning algorithm. The input of the algorithm is a stable sandpile $\eta \in \Omega_G$.

At time $t = 0$, we declare s "burnt". We set $B_0 = \{s\}$, the set of vertices burnt at time 0, and set $U_0 = V$, the set of vertices unburnt at time 0.

At time $t = 1$, we declare burnt all $x \in U_0$ such that $\eta(x) \geq \deg_{U_0}(x)$. That is, we set

$$\begin{aligned} B_1 &= \{x \in U_0 : \eta(x) \geq \deg_{U_0}(x)\} \\ U_1 &= U_0 \setminus B_1. \end{aligned}$$

At a generic time $t \geq 1$, we declare burnt all vertices in $x \in U_{t-1}$ such that $\eta(x) \geq \deg_{U_{t-1}}(x)$. That is, we set

$$\begin{aligned} B_t &= \{x \in U_{t-1} : \eta(x) \geq \deg_{U_{t-1}}(x)\} \\ U_t &= U_{t-1} \setminus B_t. \end{aligned}$$

We must eventually have $U_T = U_{T+1} = \dots$ for some $1 \leq T < \infty$. If $U_T \neq \emptyset$, then the relation $U_T = U_{T+1}$ (equivalently: $B_T = \emptyset$) shows that η is not ample for U_T. Hence Lemma 4.2 shows that η is not recurrent in this case. We will see shortly the converse, that is, when $U_T = \emptyset$ we can conclude that η is recurrent.

The burning bijection. Denote $\mathcal{T}_G = \{\text{spanning trees of } G\}$. We use the burning algorithm to define a map $\varphi : \mathcal{R}_G \to \mathcal{T}_G$. For every $x \in V$ fix an ordering $\prec_x$ of the edges incident with x. It will be useful to think of these edges as being oriented from x towards the corresponding neighbours. Let $\eta \in \mathcal{R}_G$. The spanning tree $\varphi(\eta)$ will be defined by assigning to each $x \in V$ an edge incident with x, and oriented outwards from x. The construction will guarantee that the edges form a spanning tree oriented towards s. We just saw that running the burning algorithm on η burns all vertices. Therefore, given any vertex $x \in V$, there is a unique $t = t(x) \geq 1$ such that $x \in B_t$. Let

$$\begin{aligned} F_x &= \{f : \mathrm{tail}(f) = x,\ \mathrm{head}(f) \in B_{t-1}\} \\ m_x &= \left| \left\{ f : \mathrm{tail}(f) = x,\ \mathrm{head}(f) \in \bigcup_{r \leq t-1} B_r \right\} \right|. \end{aligned}$$

Here $|\cdot|$ denotes the number of elements of a set. Observe the following properties:

(i) We have $F_x \neq \emptyset$. This is because a vertex becomes burnable in the algorithm precisely because a sufficient number of its neighbours have burnt to satisfy the inequality $\eta(x) \geq \deg_{U_{t-1}}(x)$. Hence there always is at least one neighbour of x that burned in the previous step.

(ii) We have $\deg_G(x) - m_x \leq \eta(x) < \deg_G(x) - m_x + |F_x|$. The first inequality holds because the left hand side is $\deg_{U_{t-1}}(x)$. The second inequality holds because if it was violated, then x should have burnt at a time $\leq t-1$.

Supposing $\eta(x) = \deg_G(x) - m_x + i$, with $0 \leq i < |F_x|$, and $F_x = \{f_0 \prec_x f_1 \prec_x \prec_x \cdots \prec_x f_{|F_x|-1}\}$, we set $e_x = f_i$. Now put $\varphi(\eta) := \tau := \{e_x : x \in V\}$. Since $\mathrm{head}(e_x) \in B_{t(x)-1}$ for each x, the collection τ does not contain cycles. Therefore, it is a spanning tree of G (oriented towards s). Now forget the orientation of the edges to obtain an unoriented spanning tree. (Note: there is no loss of information in doing so, since the orientation is uniquely recovered by following paths leading to s).

Exercise 4.3. Show that φ is injective. *Hint:* If $\eta_1 \neq \eta_2$, there is a first time t when "different things happen" in the constructions of $\varphi(\eta_1)$ and $\varphi(\eta_2)$. Check that at this time some e_x is assigned differently for the two configurations. See [64].

A well-known result in combinatorics is the Matrix-Tree Theorem [7], that states that $|\mathcal{T}_G| = \det(\Delta'_G)$. We saw in Theorem 2.11(i) that also $|\mathcal{R}_G| = \det(\Delta'_G)$. Therefore Exercise 4.3 implies the following corollary.

Corollary 4.4. *The mapping $\varphi : \mathcal{R}_G \to \mathcal{T}_G$ is a bijection.*

Exercise 4.5. Deduce the following combinatorial characterization of recurrent states:

a sandpile η is recurrent if and only if it is ample for any $\emptyset \neq F \subset V$.

Hint: The spanning tree $\varphi(\eta)$ is well-defined, and injectivity still holds, whenever η "passes" the burning algorithm, that is, when all vertices burn. See [33, Lemma 4.2].

Exercise 4.6. Show that if $G_n = (V_n \cup \{s\}, E_n)$ is the wired graph constructed from $V_n = \{1, \dots, n\} \subset \mathbb{Z}$, then $\mathcal{R}_{G_n}$ consists of those sandpiles for which there is at most one vertex with no particles (in particular, $|\mathcal{R}_{G_n}| = n+1$). Show that the sandpile group of G_n is isomorphic to $\mathbb{Z}_{n+1}$, and it is generated by E_1. See [14, 62].

Exercise 4.7. Specify explicitly the inverse map $\varphi^{-1} : \tau \mapsto \eta$. *Hint:* $x \in B_t$ if and only if $\mathrm{dist}_\tau(s, x) = t$, where dist_τ is the graph-distance in the tree τ. See [1].

Recall that the stationary measure ν_G is the uniform distribution on recurrent sandpiles. The bijection φ maps this measure to the uniform distribution on the set of spanning trees of G. This is called the **uniform spanning tree** measure, denoted UST_G. See [59] and [4] for the rich theory of uniform spanning trees.

What makes the burning bijection a very useful tool, is that there is a simple (and indeed very efficient) algorithm due to Wilson [85] to generate a uniformly random element of $\mathcal{T}_G$, that is, a sample from UST_G. Mapping this random tree back via the map $\varphi^{-1} : \mathcal{T}_G \to \mathcal{R}_G$, one can analyze the measure ν_G. In order to describe Wilson's algorithm, we need the procedure of loop-erasure. Given a path $\pi = [w_0, w_1, \dots, w_k]$ in G, we define the **loop-erasure** $\mathrm{LE}(\pi) = [v_0, \dots, v_\ell]$ of π by chronologically erasing loops from π, as they are created. That is, we follow the steps of π until the first time t, if any, when $w_t \in \{w_0, \dots, w_{t-1}\}$. Suppose $w_t = w_i$. We remove the loop $[w_i, w_{i+1}, \dots, w_t = w_i]$ from π, and continue tracing π. The process stops when there are no more loops to remove, yielding a self-avoiding path denoted $\mathrm{LE}(\pi)$. If π is obtained from a random walk process on G, its loop-erasure is called the loop-erased random walk (LERW) [52].

Wilson's algorithm. Fix a vertex r of G (for example, in the sandpile context $r = s$ turns out to be a natural choice), and let $v_1, v_2, \dots, v_K$ be an arbitrary enumeration of the remaining vertices of G. Let $\tau_0 = \{r\}$. Start a simple random walk on G at the vertex v_1, and stop it when r is first hit. We attach to τ_0 the loop-erasure of the path from v_1 to r, and call the resulting path τ_1. Now we start a second simple random walk from v_2, stop it when it hits τ_1, and attach the loop-erasure to τ_1. This gives a tree τ_2. When we have visited all the vertices, we have a spanning tree τ_K of G. Wilson's theorem [85] shows that this tree is uniformly distributed over all spanning trees of G.

The LERW and Wilson's algorithm are also very useful when we pass to infinite graphs (see [4]). In $\mathbb{Z}^d$, $d \geq 3$, the loop-erasure of an infinite simple random walk path is still well-defined, because the path visits any vertex only finitely often, due to transience. In $\mathbb{Z}^2$ the definition of the infinite LERW is not as straightforward. One possible definition is take a LERW from the origin to the boundary of a ball of radius n, and take the weak limit of these paths as $n \to \infty$ [52].

Exercise 4.8. Give a direct proof (without appealing to the Matrix-Tree Theorem) that $\varphi : \mathcal{R}_G \to \mathcal{T}_G$ is a bijection. *Hint:* See Exercises 4.3 and 4.7, the hint for Exercise 4.5 and [33, Lemma 4.2].

4.2. The rotor-router model

The rotor-router model, invented by Jim Propp, is a deterministic analogue of random walk [33]. It has also been discovered independently in the physics literature, where it is called the Eulerian walkers model [77]. In the sandpile model, each vertex x has to "wait" until it has collected $\deg(x)$ chips before it can send them to its neighbours. The rotor-router mechanism allows us to send chips one-by-one. The principal reference for this section is [33].

The natural setting for the rotor-router walk is directed graphs. However, since the emphasis in this section is the connection to Abelian sandpiles, we will restrict to connected graphs of the form $G = (V \cup \{s\}, E)$ as in Section 2, and regard each edge of E being present with both orientations. For each $x \in V$, fix a cyclic ordering of the edges incident with x, and orient these edges outward from x. If e is one of these edges ($\mathrm{tail}(e) = x$), then we denote by e^+ the next edge in the cyclic ordering.

Definition 4.9. A **rotor configuration** is a choice of edges

$$\rho = (\rho(x) : x \in V),$$

such that $\rho(x) \in E$ and $\mathrm{tail}(\rho(x)) = x$ for each $x \in V$. We think of $\rho(x)$ as the state of a rotor placed at the vertex x. A **single-chip-and-rotor state** is a pair (ρ, w), where $w \in V$. We think of w as the location of a chip placed on the graph. The **rotor-router operation** advances the rotor at the current position w according to the cyclic ordering, and then moves the chip following the new direction of the rotor. That is, we assign to (ρ, w) the new state (ρ^+, w^+), where

$$\begin{aligned}\rho^+(y) &= \begin{cases}(\rho(w))^+ & \text{if } y = w;\\ \rho(y) & \text{if } y \neq w;\end{cases}\\ w^+ &= \text{head}(\rho^+(x)).\end{aligned}$$

Iterating the rotor-router operation gives the **rotor-router walk**. When the chip arrives at the sink, we stop the walk, and remove the chip.

Lemma 4.10. *Starting from any single-chip-and-rotor state (ρ, w), the rotor-router walk eventually arrives at the sink.*

Proof. If $x \sim s$, then after at most $\deg_G(x)$ visits to s, the walk will arrive at s. Inducting along a path to s, we obtain that any vertex can only be visited finitely many times before the walk arrives at the sink. □

Definition 4.11. A **chip-and-rotor state** is a pair $t = (\rho, \eta)$, where ρ is a rotor configuration and η is a chip configuration (sandpile) on V. If $\eta(x) \geq 1$, we say that x **is active**. In this case, by **firing** x we mean letting a single chip at x take one rotor-router step. t is stable, if there is no active vertex (all have moved to the sink).

The next lemma shows that this model has an Abelian property.

Lemma 4.12. *Starting from any chip-and-rotor state (ρ, η), we reach the same stable state eventually (that is when all chips arrived at the sink), regardless of what rotor-router steps we choose.*

Proof. This can be proved using similar ideas as Theorem 2.3. □

Definition 4.13. We denote by $\eta(\rho)$ the result of adding chips to the rotor configuration ρ according to η and stabilizing. The **chip addition operator** E_x is defined on a rotor configuration ρ as the result of adding a single chip at x and stabilizing, that is: $\mathbf{1}_x(\rho)$.

Definition 4.14. A rotor configuration ρ is **acyclic**, if the edges $(\rho(x) : x \in V)$ form a spanning tree of G (oriented towards s, necessarily).

Lemma 4.15. *For any chip configuration η, the map $\rho \mapsto \eta(\rho)$ permutes the collection of acyclic rotor configurations.*

Perhaps the cleanest way to prove this is to consider unicycles on strongly connected graphs; see [33]. For our specific setting, the next two exercises sketch a proof.

Exercise 4.16. Show that in $\eta(\rho)$, each rotor is either in its original position $\rho(x)$, or it points in the direction of the last chip emitted from x. Conclude: if ρ is acyclic, so is $\eta(\rho)$. See [33, Section 3].

Exercise 4.17. Show that $\rho' \mapsto E_x(\rho')$, as a map from the collection of acyclic rotor configurations to itself, is surjective for any $x \in V$. The following steps can be used: Given ρ, add an oriented edge (s, x) to ρ.

(i) There is an oriented cycle starting at s, let (y_1, s) be its last edge. Place a chip at y_1, and move back the rotor at y_1 by one step.
(ii) Now there is an oriented cycle starting at y_1, let (y_2, y_1) be its last edge. Move the chip back to y_2, and move back the rotor at y_2 by one step.
(iii) Show that eventually the chip arrives at x and if the rotor configuration at that time is ρ', then we have $E_x \rho' = \rho$.

See [33, Section 3].

Theorem 4.18.
(i) The map $(\rho, [\eta]) \mapsto \eta(\rho)$ *defines an action of the sandpile group on acyclic rotor configurations.*
(ii) The action is transitive, that is, for any acyclic ρ, ρ' *there exists* η *such that* $\eta(\rho) = \rho'$.
(iii) The action is free, that is, if $\eta(\rho) = \rho$ *for some acyclic* ρ *then* $[\eta] = [0]$.

Proof. (i) From Lemma 4.12 it is clear that $\eta_2(\eta_1(\rho)) = (\eta_1 + \eta_2)(\rho)$. Suppose $\eta_1 \sim \eta_2$. We show that $\eta_1(\rho) = \eta_2(\rho)$. If $\eta(x) \geq \deg_G(x)$, we can advance $\deg_G(x)$ chips at x, one along each edge incident with x, and leave the rotor at x unchanged. It follows from this that $\eta(\rho) = \eta^\circ(\rho)$ for any chip configuration η. Let $I \in \mathcal{R}_G$ be the sandpile corresponding to the identity (i.e. $[I] = [0]$). Then we have

$$I(I(\rho)) = (I + I)(\rho) = (I + I)^\circ(\rho) = I(\rho)$$

for all acyclic rotor configurations ρ. Due to Lemma 4.15, $\{I(\rho) : \rho \text{ acyclic}\} = \{\rho : \rho \text{ acyclic}\}$, and it follows that $I(\rho) = \rho$ for all acyclic ρ. Now we have $(\eta_1 + I)^\circ = (\eta_2 + I)^\circ$, and

$$\eta_i(\rho) = I(\eta_i(\rho)) = (I + \eta_i)(\rho) = (I + \eta_i)^\circ, \quad i = 1, 2.$$

This implies the claim.

(ii) Given ρ, ρ', let $0 \leq \alpha(x) < \deg_G(x)$ be the number of turns the rotor at x has to make from position $\rho(x)$ to $\rho'(x)$. Adding chips to ρ according to α and letting each chip take a single step we obtain a chip-and-rotor state of the form: (ρ', β). Choose a chip configuration σ such that $[\sigma] = [-\beta]$ (the inverse of β in the sandpile group). Let $\eta = \alpha + \sigma$. Then we have

$$\eta(\rho) = (\sigma + \alpha)(\rho) = (\sigma + \beta)(\rho') = \rho'.$$

This proves transitivity of the action.

(iii) Suppose η is a chip configuration, ρ is acyclic, and $\eta(\rho) = \rho$. This means that adding chips according to η the rotor at x makes a non-negative integer c_x number of full turns during stabilization. Since all chips arrive at the sink, $\eta(x)$ equals the number of chips emitted from x minus the number of chips received at x, for each $x \in V$. Therefore:

$$\eta(x) = \deg_G(x) c_x - \sum_{y \in V} a_{yx} c_y = \sum_{y \in V} c_y \Delta'_{yx}, \quad x \in V.$$

This shows that $[\eta] = [0]$. $\square$

Remark 4.19. The above proof does not rely on the Matrix-Tree Theorem, and in fact provides a new proof of it; see [33, Corollary 3.18].

Remark 4.20. Regarding acyclic rotor configurations as spanning trees of G, the action of K_G allows one to view the sandpile Markov chain as a dynamics on trees. This dynamics on trees seems more transparent and explicit than the one obtained using the burning bijection.

Open Question 4.21. Is there a meaningful link between avalanches in the Abelian sandpile and either the rotor-router dynamics on spanning trees or the dynamics induced by the burning bijection?

4.3. The random cluster model / Tutte polynomial

The uniform spanning tree measure UST_G is a limiting case of the so-called random cluster measure. The random cluster model is a generalization of percolation. The relationship between sandpiles and the random cluster measure leads to a formula for the generating function of recurrent sandpiles enumerated by their total number of particles. In this section again $G = (V \cup \{s\}, E)$ is a finite connected multigraph.

Definition 4.22. If $\eta \in \mathcal{R}_G$, the **mass of** η is defined as $m(\eta) = \sum_{x \in V} \eta(x)$. We put $N_m = |\{\eta \in \mathcal{R}_G : m(\eta) = m\}|$, and let $\mathcal{N}(y) = \sum_m N_m y^m$ be the generating function according to mass.

Exercise 4.23. Show that for all $\eta \in \mathcal{R}_G$ we have:

$$|E| - \deg_G(s) \le m(\eta) \le 2|E| - \deg_G(s) - |V|.$$

Show that the lower bound is achieved for any $\eta \in \mathcal{R}_G$ that is *minimal* in the sense that $\eta - \mathbf{1}_x \notin \mathcal{R}_G$ for all $x \in V$. *Hint for the lower bound:* Use the burning algorithm. See [12, Section 7.2].

The **random cluster model** on G has two parameters: $0 < p < 1$ and $q > 0$. It is specified by a probability measure $\mathbf{P}_{p,q}$ on the space $\{E' : E' \subset E\}$, that is given by:

$$\mathbf{P}_{p,q}[E'] = \frac{1}{Z_{p,q}} p^{|E'|}(1-p)^{|E|-|E'|} q^{k(E')},$$

where $k(E')$ denotes the number of connected clusters in the edge-configuration E', and $Z_{p,q}$ is a normalizing factor to make $\mathbf{P}_{p,q}$ a probability measure. Observe that when $q = 1$, we get the percolation model.

Exercise 4.24 (See [24, Theorem 1.23])**.** As $p \to 0$ and $q/p \to 0$ we have $\mathbf{P}_{p,q} \to \mathsf{UST}_G$.

Abbreviating $v = p/(1-p)$, we have

$$\begin{aligned} Z_{p,q} &= \sum_{E' \subset E} p^{|E'|}(1-p)^{|E|-|E'|} q^{k(E')} = (1-p)^{|E|} \sum_{E' \subset E} v^{|E'|} q^{k(E')} \\ &=: (1-p)^{|E|} Z'_{v,q}. \end{aligned}$$

Letting $q \downarrow 0$, the dominant terms are the ones with $k(E') = 1$, that is the ones where E' is connected. The number of edges in such a graph is at least $|V|$ (with equality for spanning trees). Hence we can write:

$$Z'_{v,q} = qv^{|V|}H(v) + O(q^2), \quad \text{as } q \downarrow 0,$$

where $H(v)$ is a polynomial. Note that $H(0)$ equals the number of spanning trees of G. The following theorem is due to Merino López [68].

Theorem 4.25. *We have*

$$\mathcal{N}(y) = y^{|E|-\deg_G(s)}H(y-1).$$

See [12] for a proof that follows the ideas of the burning algorithm to associate a recurrent configuration to groups of connected graphs E'.

One has $H(y-1) = T(1, y; G)$, where $T(x, y; G)$ is the so-called Tutte polynomial of G, a well-known graph invariant in combinatorics [7]. More generally, $Z_{p,q}$ can be expressed in terms of the Tutte polynomial; see [24, Section 3.6].

Exercise 4.26. According to Theorem 4.25, the number of recurrent sandpiles that have a minimal number of particles is $\mathcal{N}(0) = H(-1) = T_G(1, 0)$. It is known [7] that $T_G(1, 0)$ counts the number of *acyclic orientations* of G with a unique sink at a fixed vertex of G. Taking the unique sink to be at s, use the burning algorithm to construct an explicit bijection between *minimal sandpiles* and *acyclic orientations* of G with unique sink at s.

5. Determinantal formulas and exact computations

In this section we will see that certain sandpile probabilities can be expressed in terms of determinants, and in some cases these can be evaluated explicitly. The fundamental fact behind this is that all finite-dimensional marginals of the uniform spanning tree admit a determinantal formula.

5.1. The Transfer-Current Theorem

Let G be a finite connected (unoriented) graph. Write T_G for a random spanning tree of G chosen uniformly. The following theorem is due to Burton and Pemantle.

Theorem 5.1 (**Transfer Current Theorem** [9]). *There exists a matrix Y_G such that for any $k \geq 1$ and distinct edges $e_1, \dots, e_k$ of G we have*

$$\mathbf{P}[e_1, \dots, e_k \in T_G] = \det(Y_G(e_i, e_j))_{i,j=1}^k. \tag{20}$$

The simplest definition of the **transfer-current matrix** Y_G, is in terms of random walk. (See [59] for a definition of Y_G in terms of electrical networks.) Given *oriented* edges e, f of G, consider the simple random walk on G started at tail(e) and stopped when it first hits head(e). Let $J^e(f)$ be the expected net usage of f by the walk, i.e. the number of times f was used minus the number of times the reversal of f was used. Then $Y_G(e,f) = J^e(f)$. Note that this requires us to chose an orientation for each edge appearing in the right hand side of the Transfer Current Theorem, whereas in the left hand side the edges are unoriented. It is part of the statement of the theorem that the right hand side is independent of what orientations are chosen.[1] Due to the structure present in (20), the random collection of edges T_G is called a **determinantal process** with **kernel** Y_G. There is an extension of (20) to all cylinder events, also due to [9]. A simple case of it that we will use later is:

$$\mathbf{P}[e_1,\dots,e_k \notin T_G] = \det(K_G(e_i,e_j))_{i,j=1}^k,$$

where $K_G = I_G - Y_G$, with I_G the identitiy matrix. See the survey [34] for more information on determinantal processes.

5.2. *The height* **0** *probability*

No simple formula like (20) is known for the finite-dimensional marginals of the sandpile measure ν_G. However, there is a method due to Majumdar and Dhar [63] for the computation of the probabilities of *minimal configurations*. The simplest example is computing the probability that $\eta(o) = 0$, that we now explain.

Consider $V_n = [-n,n]^2 \cap \mathbb{Z}^2$, and let G_n be the wired graph obtained from V_n. We write $\nu_n = \nu_{G_n}$ for short. We will obtain a formula for $\nu_n[\eta(o) = 0]$ that makes it possible to compute its limit as $n \to \infty$.

Let j_1, j_2, j_3 denote the south, west, north neighbours of the origin, respectively. Let G'_n be the graph obtained from G_n by deleting the edges $\{o, j_i\}$, $i = 1,2,3$. Given $\eta \in \mathcal{R}_{G_n}$, let

$$\eta'(y) := \begin{cases} \eta(y) - 1 & \text{if } y = j_1, j_2, j_3; \\ \eta(y) & \text{otherwise.} \end{cases}$$

Exercise 5.2. Show that

$$\eta \in \mathcal{R}_{G_n},\ \eta(o) = 0 \qquad \text{if and only if} \qquad \eta' \in \mathcal{R}_{G'_n}.$$

Hint: Use the burning algorithm. See [63].

We write $\Delta'_{G'_n}$ in the form $\Delta'_{G'_n} = \Delta'_{G_n} + B$. Note that the matrix B has nonzero entries only in the rows and columns corresponding to $\{o, j_1, j_2, j_3\}$, and these are:

$$\begin{array}{cc} \begin{array}{cccc} o & j_1 & j_2 & j_3 \end{array} & \\ \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} & \begin{array}{c} o \\ j_1 \\ j_2 \\ j_3 \end{array} \end{array} \tag{21}$$

[1] This can also be checked directly using the time reversal of the simple random walk on G.

The above allows us to write

$$\nu_n[\eta(o)=0] = \frac{|\{\eta \in \mathcal{R}_{G_n} : \eta(o)=0\}|}{|\mathcal{R}_{G_n}|} \overset{\text{Ex. 5.2}}{=} \frac{|\mathcal{R}_{G'_n}|}{|\mathcal{R}_{G_n}|} = \frac{\det(\Delta'_{G_n}+B)}{\det(\Delta'_{G_n})} \\ = \det(I + B(\Delta'_{G_n})^{-1}). \tag{22}$$

Due to the fact that B is 0 apart from the entries shown in (21), the determinant on the right hand side of (22) reduces to a 4×4 determinant. Recall that $(\Delta'_{G_n})^{-1}_{xy} = G_n(z,w)$. Since the random walk is recurrent in two dimensions, $\lim_{n\to\infty} G_n(z,w) = \infty$. Hence in order to take the limit $n\to\infty$, we need to rely on cancellations.

In two dimensions the **recurrent random walk potential kernel** is defined as

$$a(x) = \lim_{N\to\infty} \sum_{k=0}^{N} \left[\mathbf{p}^k(o,o) - \mathbf{p}^k(o,x)\right],$$

where $\mathbf{p}^k(z,x)$ is the k-step transition probability of simple random walk on $\mathbb{Z}^2$. See [52, Section 4.4.1] for a proof that the limit exists and for further background. Note that $a(o)=0$. It holds that $\frac{1}{4}(\Delta a)(x) = -\mathbf{1}_o(x)$ (see [52, Proposition 4.4.2]); in particular, a is a discrete harmonic function in $\mathbb{Z}^2\setminus\{o\}$.

The potential kernel is related to G_n by the following lemma that is well-known.

Lemma 5.3. *For all $x\in\mathbb{Z}^2$, we have*

$$A(x) := \frac{1}{4}a(x) = \lim_{n\to\infty}[G_n(o,o) - G_n(o,x)].$$

Since we are going to prove a stronger version of this statement in Lemma 5.17, Eqn. (49), we omit the proof.

The values of $A(x)$ can be computed recursively from symmetry considerations and the facts that:
(i) $\frac{1}{4}\Delta A(x) = -\frac{1}{4}\mathbf{1}_o(x)$;
(ii) $A((n,n)) = \frac{1}{\pi}\left[1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right]$; see for example [59] or [84, Section 15].
In particular,

$$\begin{array}{lll} A(o)=0 & A(j_1)=\dfrac{1}{4} & A(j_1+j_2)=\dfrac{1}{\pi} \\ A(j_1-j_3) = 1-\dfrac{2}{\pi} & & A(j_1+j_2-j_3) = \dfrac{-1}{4}+\dfrac{2}{\pi}. \end{array} \tag{23}$$

Let us return to the limit of the determinant in (22). Since the row sums of B are 0, the computation can be recast in terms of A. For example, the o,o entry of $I+B(\Delta'_{G_n})^{-1}$ equals

$$1 - 3G_n(o,o) + G_n(o,j_1) + G_n(o,j_2) + G_n(o,j_3) \overset{n\to\infty}{\longrightarrow} 1 - 3A(j_1) = \frac{1}{4}.$$

Straightforward calculations using the values (23) and symmetry yield:

$$p(0) := \lim_{n\to\infty} \nu_n[\eta(o) = 0] = \nu[\eta(o) = 0] = \frac{2}{\pi^2} - \frac{4}{\pi^3}. \tag{24}$$

When $d \geq 3$, similar arguments apply. As $\lim_{n\to\infty} G_n(z,x) = G(z,x)$, we have that $\nu[\eta(o) = 0]$ is expressed in terms of the Green function $G(z,x)$.

5.3. The height **0-0** *correlation*

The idea of Majumdar and Dhar presented in the previous section also gives a formula for the covariance between the events $\{\eta(o) = 0\}$ and $\{\eta(y) = 0\}$; see [63]. Consider first two dimensions. This time we modify the graph both near o and y, by removing the edges leading from o to j_1, j_2, j_3, and the edges leading from y to neighbours j_1', j_2', j_3'. Then similarly to the previous section, $\nu_n[\eta(o) = 0,\, \eta(y) = 0]$ can be written as an 8×8 determinant, that arises from four blocks of size 4×4. Since the row sums of B are 0, row and column operations can be used to reduce the size of the blocks to 3×3. The result of this can be written in the form (see for example [15]):

$$\nu_n[\eta(o) = 0,\, \eta(y) = 0] = \det\left(I_{v=w} - K_n(v,w)\right)_{v,w\in\{o,y\}}, \tag{25}$$

where $I_{v=w}$ is the 3×3 identity matrix when $v = w$ and the 3×3 null matrix when $v \neq w$. The 3×3 matrix $K_n(v,w)$ is given by:

$$K_n(v,w) = \left(\partial_e^{(1)} \partial_f^{(2)} G_n(v,w)\right)_{e,f}, \tag{26}$$

where for any function $h : \mathbb{Z}^2 \to \mathbb{R}$ and vector $e \in \mathbb{Z}^2$ we define:

$$\begin{aligned} \partial_e^{(1)} h(v,w) &= h(v+e,w) - h(v,w), \\ \partial_f^{(2)} h(v,w) &= h(v,w+f) - h(v,w). \end{aligned}$$

In the formula (26), the vectors e and f range over the unit vectors: $(0,-1)$, $(-1,0)$ and $(0,1)$ (these are the vectors pointing from o to j_1, j_2 and j_3).

Letting $n \to \infty$, we obtain an expression

$$\nu[\eta(o) = 0,\, \eta(y) = 0] = \det\left(I_{v=w} - K(v,w)\right)_{v,w\in\{o,y\}}, \tag{27}$$

with

$$K(v,w) = \left(\partial_e^{(1)} \partial_f^{(2)} A(w-v)\right)_{e,f}. \tag{28}$$

Here $\det(K(o,o)) = \det(K(y,y)) = p(0)$, from the previous section. In order to understand how correlations decay as $|y| \to \infty$, let us examine the order of magnitude of the entries of $K(o,y)$. It is well know (see [52, Theorem 4.4.4]) that there exists a constant c_0 such that

$$A(y) = \frac{1}{2\pi} \log|y| + c_0 + O\left(|y|^{-2}\right), \quad \text{as } |y| \to \infty. \tag{29}$$

This shows that the entries of $K(o,y)$ (and that of $K(y,o)$) are $O(|y|^{-2})$, and hence

$$\nu[\eta(o)=0,\,\eta(y)=0]-\nu[\eta(o)=0]\nu[\eta(y)=0]=O\left(|y|^{-4}\right), \quad \text{as } |y|\to\infty. \tag{30}$$

One can compute the constant in this asymptotics using more precise information on the error term in (29). Indeed, regarding y as a complex number, the error term in (29) is of the form

$$\frac{\mathfrak{Re}\, y^4}{|y|^6}+O\left(|y|^{-4}\right);$$

see [22] or [51]. Therefore, after taking second differences, the error term of (29) does not contribute to the $|y|^{-4}$ term in (30). This yields the result obtained by Majumdar and Dhar [63]:

$$\nu[\eta(o)=0,\,\eta(y)=0]-\nu[\eta(o)=0]\nu[\eta(y)=0]\sim-\frac{p(0)^2}{2|y|^4}, \quad \text{as } |y|\to\infty. \tag{31}$$

In dimensions $d\geq 3$ a similar computation can be carried out showing that the covariance between two 0's decays as $-c|y|^{-2d}$, as $|y|\to\infty$, with $c=c(d)>0$.

5.4. Scaling limit of the height 0 field

The second differences of discrete Green functions considered in the previous section converge, under rescaling, to partial derivatives of continuous Green functions. This allows to get formulas for the scaling limit of the covariance functions between heights 0. The result is especially interesting in two dimensions, as there the continuous Green function is conformally invariant, which implies that the covariance functions transform in a nice way under conformal maps. Although this fact seems to be well-known by physicists (see for example [37, Section 3.3] and [44]), we are not aware of a mathematically precise formulation of it in the physics literature. We state below a theorem of Dürre [15] that provides such a formulation.

Let $U\subset\mathbb{C}$ be a bounded connected domain with smooth boundary. Let $U_\varepsilon=(U/\varepsilon)\cap\mathbb{Z}^2$, and for $v\in U$ let $v_\varepsilon\in U_\varepsilon$ be such that $|v/\varepsilon-v_\varepsilon|\leq 2$. Denote $h_\varepsilon(v)=\mathbf{1}_{\eta(v_\varepsilon)=0}$, which is a random field, indexed by $v\in U$, under the measure ν_{U_ε}.

Theorem 5.4 ([15, Theorem 1]). *Let $V\subset U$ be a finite set of points in the interior of U. Then as $\varepsilon\to 0$, the rescaled joint moments*

$$\varepsilon^{-2|V|}\mathbf{E}_{\nu_{U_\varepsilon}}\left[\prod_{v\in V}\left[h_\varepsilon(v_\varepsilon)-\mathbf{E}_{\nu_{U_\varepsilon}}h_\varepsilon(v)\right]\right]$$

have a finite limit $E_U(v:v\in V)$, which is conformally covariant with scale dimension 2.

Here **conformally covariant** means that if $f : U \to U'$ is a conformal map, then

$$E_U(v : v \in V) = E_{U'}(f(v) : v \in V) \cdot \prod_{v \in V} |f'(v)|^2 ,$$

and the exponent 2 is the **scale dimension**. When $V = \{v, w\}$, the limit is:

$$\begin{aligned} &E_U(v,w) \\ &\quad = -c\left[\left(\partial_x^{(1)}\partial_x^{(2)} g_U\right)^2 + \left(\partial_y^{(1)}\partial_y^{(2)} g_U\right)^2 + \left(\partial_x^{(1)}\partial_y^{(2)} g_U\right)^2 + \left(\partial_y^{(1)}\partial_x^{(2)} g_U\right)^2\right], \end{aligned}$$

where $g_U(v,w) = g_U((x_1,y_1),(x_2,y_2))$ is the continuous Green function in U for the Laplacian with Dirichlet boundary conditions.

Summability of the covariance function $-c/|y|^4$ of (31) suggests that if the random field h_ε is integrated against smooth test functions then we get a Gaussian limit. This is indeed the case.

Theorem 5.5 ([15, Theorem 3])**.** *There is a constant $\mathcal{V} > 0$ such that the following holds. Let $f_i \in C_0^\infty(U)$, $1 \le i \le n$. Then the random variables*

$$f_i \diamond h_\varepsilon := \frac{\varepsilon}{\sqrt{\mathcal{V}}} \sum_{v \in U_\varepsilon} f_i(\varepsilon v) \left(h_\varepsilon(v) - \mathbf{E}_{\nu_{U_\varepsilon}} h_\varepsilon(v)\right)$$

converge in distribution to a multivariate normal random variable with covariance

$$C_{ij} = \int_U f_i(x,y) f_j(x,y)\, dx\, dy, \quad i,j = 1,\dots,n.$$

5.5. The probabilities of heights 1, 2, 3 in two dimensions

The probabilities of heights different from 0 are, in general, more difficult to compute. In the case of an infinite regular tree all height probabilities can be computed using combinatorial methods; see [13]. However, on Euclidean lattices of dimension at least 2, exact results are only known when $d = 2$. The goal of this section is to sketch the main ideas of Priezzhev [75] that yield the probabilities of heights 1, 2, 3 on $\mathbb{Z}^2$. This section is quite long and technical in many parts, so the reader might want to skip some of the proofs on first reading.

5.5.1. Background

Let us denote

$$p(i) := \nu[\eta(o) = i], \quad i = 0, 1, \dots, 2d-1.$$

In the case $d = 2$, Priezzhev [75] gave exact formulas for $p(1), p(2), p(3)$. He was able to express them in terms of explicit rational polynomials in $1/\pi$ and two multiple integrals. Grassberger evaluated the integrals numerically (see [12, Section 9.3.1]), and observed that mysteriously the *average height* $\zeta = \sum_{i=0}^{3} i\, p(i)$ appears to be the simple rational number 17/8.

Jeng, Piroux and Ruelle [44] extended Priezzhev's ideas, and in particular, were able to express the $p(i)$'s in terms of a single integral. They noticed that numerical evaluation of the unknown integral gave $1/2 \pm 10^{-12}$, and conjectured that this integral is exactly equal to 1/2. Assuming this conjecture, and combined with Priezzhev's work, they obtained the remarkable formulas:

$$\begin{aligned}
p(0) &= \frac{2}{\pi^2} - \frac{4}{\pi^3} \\
p(1) &= \frac{1}{4} - \frac{1}{2\pi} - \frac{3}{\pi^2} + \frac{12}{\pi^3} \\
p(2) &= \frac{3}{8} + \frac{1}{\pi} - \frac{12}{\pi^3} \\
p(3) &= 1 - p(0) - p(1) - p(2) = \frac{3}{8} - \frac{1}{2\pi} + \frac{1}{\pi^2} + \frac{4}{\pi^3}.
\end{aligned} \tag{32}$$

These values indeed yield $\zeta = 17/8$ as the average height.

Poghosyan and Priezzhev [73] observed that the average height can be rephrased in terms of the LERW. Let

$$\xi = \mathbf{E}[\text{number of neighbours of } o \text{ visited by the infinite LERW}].$$

Then the statement $\zeta = 17/8$ is equivalent to $\xi = 5/4$ (this equivalence will be explained below). Levine and Peres [55] called ξ the **looping constant of** $\mathbb{Z}^2$, and proved further relations between ξ, the number of spanning uni-cycles and the Tutte-polynomial. The relations they prove hold in all dimensions $d \geq 2$.

Kenyon and Wilson [47], and independently, Poghosyan, Priezzhev and Ruelle [74] gave different proofs that $\xi = 5/4$, which in turn gives a rigorous confirmation that the aforementioned integral is exactly 1/2 and proves the values (32). A direct evaluation of the integral was given by Caracciolo and Sportiello [10]. Subsequently, Kassel and Wilson [46] gave a more direct proof of the sandpile density. Kenyon and Wilson develop a general method for calculating the probability that the infinite LERW in two dimensions passes through any given vertex or any given oriented edge of $\mathbb{Z}^2$. In principle, their method can be used to calculate all finite-dimensional marginals of ν. The proof of Poghosyan, Priezzhev and Ruelle proceeds via a connection to monomer-dimer coverings. They reduce the problem of $\xi = 5/4$ to calculating the probabilities of certain local events in the monomer-dimer model that can be expressed in terms of finite determinants akin to the calculations in Section 5.2.

5.5.2. The looping constant

Let us see the connection between the average height and the looping constant. It will be convenient at this point to introduce a slightly different version of the burning bijection. In what follows, let $G_n = (V_n \cup \{s\}, E_n)$ be the wired graph obtained from $V_n = [-n, n]^d \cap \mathbb{Z}^d$, $d \geq 2$.

Burning bijection anchored at the origin. The burning process will consist of two Phases.

Phase I. We burn all vertices we can *without* burning the origin. That is, we define $B_0^{(I)} = \{s\}$, $U_0^{(I)} = V_n$, and for $t \geq 1$ we set:

$$B_t^{(I)} := \left\{ x \in U_{t-1}^{(I)} \setminus \{o\} : \eta(x) \geq \deg_{U_{t-1}^{(I)}}(x) \right\}$$
$$U_t^{(I)} := U_{t-1}^{(I)} \setminus B_t^{(I)}.$$

At some finite time no more vertices can be burnt, that is, $B_J^{(I)} = \emptyset$ for some $1 \leq J < \infty$.

Phase II. Burn all the remaining vertices in the usual way. That is, we start with $B_0^{(II)} = \cup_{j\geq 0} B_j^{(I)}$, $U_0^{(II)} = \cap_{j\geq 0} U_j^{(I)}$, and for $t \geq 1$ set

$$B_t^{(II)} := \left\{ x \in U_{t-1}^{(II)} : \eta(x) \geq \deg_{U_{t-1}^{(II)}}(x) \right\}$$
$$U_t^{(II)} := U_{t-1}^{(II)} \setminus B_t^{(II)}.$$

It is not difficult to see that for all $\eta \in \mathcal{R}_G$, all vertices burn eventually.

Now build a bijection $\varphi_o : \mathcal{R}_G \to \mathcal{T}_G$ based on the above burning rule, similarly to what we did in Section 4.1. That is, if $x \in B_t^{(I)}$ for some $t \geq 1$, draw an oriented edge from x to one of its neighbours in $B_{t-1}^{(I)}$. If $x \in B_t^{(II)}$ for some $t \geq 1$, draw an edge from x to one of its neighbours in $B_{t-1}^{(II)}$. In both cases, break ties as in the usual bijection, if necessary.

The following two claims are not difficult to verify, and are left as exercises.

Exercise 5.6. Put

$$W_n = W_n(\eta) = \{\text{vertices that did not burn in Phase I}\} = U_0^{(II)}.$$

Then $W_n = \{\text{descendants of } o \text{ in } \tau = \varphi_o(\eta)\}$. Here a vertex w is called a descendant of the vertex v, if v lies on the unique path between w and s in the tree τ. See [43].

Exercise 5.7. Under the measure ν_n and conditional on the event $\deg_{W_n}(o) = i$, the random variable $\eta(o)$ is uniformly distributed on $\{i, i+1, \ldots, 2d-1\}$. *Hint:* Condition further on the entire set W_n, and consider the possible values of $\eta(o)$ in relation to the outgoing edge from o in $\varphi_o(\eta)$. See [55, Lemma 4].

The $p(i)$'s can be rephrased in terms of the quantities

$$q(i) := \lim_{n\to\infty} \nu_n[\deg_{W_n}(0) = i], \quad i = 0, 1, \ldots, 2d-1.$$

Existence of the limit follows, for example, from results presented in Section 6. Due to Exercise 5.7, we have

$$p(i) = \sum_{j=0}^{i} \frac{1}{2d-j} q(j). \tag{33}$$

Linearity of expectation and Wilson's algorithm yield

$$\sum_{i=0}^{2d-1} iq(i) = \lim_{n\to\infty} \sum_{x\sim o} \nu_n[x \in W_n]$$
$$= \lim_{n\to\infty} \sum_{x\sim o} \mathbf{P}[\text{LERW from } x \text{ to } s \text{ in } G_n \text{ visits } o].$$

By the definition of the infinite LERW and translation invariance the right hand side equals

$$\sum_{x\sim o} \mathbf{P}[\text{infinite LERW started from } x \text{ visits } o]$$
$$= \sum_{x\sim o} \mathbf{P}[\text{infinite LERW started from } o \text{ visits } -x]$$
$$= \xi.$$

The relation (33) now yields $\zeta = d + \frac{\xi-1}{2}$.

We are now ready to present the main ideas of Priezzhev's computation of $p(1)$ and $p(2)$. (We have seen in Section 5.2 that $p(0) = \frac{1}{4}q(0) = \frac{2}{\pi^2} - \frac{4}{\pi^3}$.) For the remainder of Section 5.5, we restrict to $d = 2$. One of our concerns will be to supply explicit error bounds that allow one to pass to the limit $n \to \infty$ in the computations. These are not provided in the physics literature, and we believe that such estimates may be useful in further work on related questions, and therefore would benefit a reader who is not yet familiar with the details of the work of physicists. Attention is also due to the fact that Priezzhev's integrals [75, Eqn. (6)] include logarithmically divergent singularities, and do not exist as Lebesgue integrals; see our Remark 5.14. Implicit in Priezzhev's formula is to apply a regularization that allows divergent singularities to cancel. We provide a suitable regularization in Propositions 5.12 and 5.13.

5.5.3. *Decomposition of $q(1)$ into three terms*

Due to (33), it is enough to find $q(1)$ and $q(2)$. We restrict to the computation of $q(1)$, as the computations for $q(2)$ follow similar ideas; see [44, 75].

Let $q_n(1) = \nu_n[\deg_{W_n}(o) = 1]$. We will work in the (large) finite graph G_n. Let j_1, j_2, j_3, j_4 be the south, west, north, east neighbours of the origin o, respectively.[2] Due to symmetry, we have

$$q_n(1) = 4\,\nu_n[\deg_{W_n}(o) = 1,\ j_1 \in W_n,\ j_2, j_3, j_4 \notin W_n]. \tag{34}$$

It will be useful to regard spanning trees of G_n as being oriented towards s. Then specifying a spanning tree is equivalent to specifying an acyclic rotor configuration ρ on V_n, and we are required to count certain acyclic rotor configurations.[3]

[2] We have tried to keep the notation consistent with that of [75] as much as possible.

[3] *Note:* Here we are using the bijection anchored at o introduced in Section 5.5.2, and *not* the sandpile group action on acyclic rotors of Section 4.2.

The event on the right hand side of (34) is equivalent to the event that the rotor at j_1 is pointing to o, and there is no directed path from any of j_2, j_3, j_4 to j_1. Using the idea of Exercise 5.7, we can fix the rotor at o to be pointing to j_2, say, and introduce a factor 3. That is:

$$q_n(1) = \frac{12}{\det(\Delta'_n)} \left| \left\{ \rho : \begin{array}{l} \rho \text{ acyclic, } \rho(j_1) = [j_1, o],\ \rho(o) = [o, j_2], \\ \text{head}(\rho(j_3)) \neq o,\ \text{head}(\rho(j_4)) \neq o, \text{ no} \\ \text{oriented path from } j_3 \text{ and } j_4 \text{ to } j_1 \end{array} \right\} \right|.$$

Due to planarity, it is in fact enough to require that there be no oriented path from j_4 to j_1. This is because if j_3 had such a path, so would j_4, due to the fact that j_2 has an oriented path to the sink. Hence

$$q_n(1) = \frac{12}{\det(\Delta'_n)} \left| \left\{ \rho : \begin{array}{l} \rho \text{ acyclic, } \rho(j_1) = [j_1, o],\ \rho(o) = [o, j_2], \\ \text{head}(\rho(j_3)) \neq o,\ \text{head}(\rho(j_4)) \neq o, \text{ no} \\ \text{oriented path from } j_4 \text{ to } j_1 \end{array} \right\} \right|. \tag{35}$$

The non-local constraint that there be no oriented path from j_4 to j_1 amounts to requiring that if a *second* rotor were introduced at o, pointing to j_4, then the resulting configuration would still be acyclic. For short let $e = [o, j_2]$, $f = [o, j_4]$, $h = [j_1, o]$, and put

$$\mathcal{T}_0 = \left\{ \rho_0 : \begin{array}{l} \rho_0 \text{ an acyclic rotor configuration on } V_n \setminus \{o\},\ \rho_0(j_1) = h, \\ \text{head}(\rho_0(j_2)) \neq o, \text{head}(\rho_0(j_3)) \neq o, \text{head}(\rho_0(j_4)) \neq o \end{array} \right\}.$$

Put

$$\mathcal{T}_e := \{\rho_0 \in \mathcal{T}_0 : \rho_0 \cup \{e\} \text{ is acyclic}\}$$
$$\mathcal{T}_f := \{\rho_0 \in \mathcal{T}_0 : \rho_0 \cup \{f\} \text{ is acyclic}\}.$$

Then $|\mathcal{T}_e \cap \mathcal{T}_f|$ counts the number of elements of the set in the right hand side of (35), that we write as:

$$|\mathcal{T}_e \cap \mathcal{T}_f| = |\mathcal{T}_e| - |\mathcal{T}_f^c| + |\mathcal{T}_e^c \cap \mathcal{T}_f^c|. \tag{36}$$

5.5.4. *An extension of the Matrix-tree theorem*

In order to get formulas for the three terms in (36), we are going to use the theorem below that states variations on the Matrix-Tree Theorem for directed graphs [7, Theorem II.14]. Let $G = (V \cup \{s\}, E)$ be a *directed* graph, with $-\Delta_{xy} = a_{xy}$ the number of directed edges from x to y, and $\Delta_{xx} = \text{outdeg}(x) := \sum_{y \in V \cup \{s\}} a_{xy}$. We assume that $\Delta_{sx} = 0$ for all $x \in V$. From now on, but in this section only, we are going to call an oriented cycle of G an *oriented loop* (to distinguish from permutation cycles in the proof below). In order to state the theorem, we need some notation. Let N_0 denote the number of rotor configurations on V with no oriented loop (acyclic rotor configurations). Given a directed edge h, let $N_1(h)$ denote the number of rotor configurations that contain precisely one oriented loop, with the edge h contained in this loop. Let

$$\widetilde{\Delta}^h_{xy} = \begin{cases} \Delta_{xy} & \text{if } [x,y] \neq h; \\ -\omega & \text{if } [x,y] = h; \end{cases}$$

where ω is a real parameter. Similarly, given oriented egdes f_1, f_2, f_3 of G, let $N_i(f_1, f_2, f_3)$, $i = 1, 2, 3$ respectively, denote the number of rotor configurations that contain precisely i oriented loops, respectively, in such a way that each loop contains at least one of f_1, f_2, f_3, and each of f_1, f_2, f_3 is contained in at least one of the loops. Let

$$\widetilde{\Delta}^{f_1,f_2,f_3}_{xy} = \begin{cases} \Delta_{xy} & \text{if } [x,y] \neq f_1, f_2, f_3; \\ -\omega & \text{if } [x,y] \in \{f_1, f_2, f_3\}. \end{cases}$$

As before, let Δ' denote the matrix obtained from Δ by restricting the indices to $V \times V$.

Theorem 5.8. *(Priezzhev [75]) We have:*

$$\begin{aligned} \det(\Delta') &= N_0 \\ \lim_{\omega\to\infty} \frac{1}{\omega} \det((\widetilde{\Delta}^h)') &= -N_1(h) \\ \lim_{\omega\to\infty} \frac{1}{\omega^3} \det((\widetilde{\Delta}^{f_1,f_2,f_3})') &= -N_1(f_1,f_2,f_3) + N_2(f_1,f_2,f_3) - N_3(f_1,f_2,f_3). \end{aligned}$$

Proof. Expand $\det(\Delta')$ as a sum over permutations of V, and for each permutation in the sum, consider its decomposition into cyclic permutations. Define the *weight* of a permutation cycle $(x_1, \dots, x_k)$ of length $k \geq 2$ to be $\prod_{i=1}^k a_{x_i,x_{i+1}}$, and the weight of a "trivial" permutation cycle (x) of length 1 to be Δ_{xx}. Hence we have:

$$\det(\Delta') = \sum_{\text{permutations}} (-1)^{\#\text{non-trivial perm. cycles}} \prod_{\text{perm. cycles}} \text{weight(perm. cycle)}.$$

Note that a non-trivial permutation cycle of k edges, $k \geq 2$, brings a sign $(-1)^k$ due to k factors of $-a_{x_i,x_{i+1}}$, and therefore the factor $(-1)^{\#\text{non-trivial perm. cycles}}$ ensures the correct sign for the signature of the permutation. The weight of a non-trivial cycle counts the number of oriented loops with the same vertex set. Let us group terms according to the number of non-trivial loops and write Γ for the set of all oriented loops in G. This yields:

$$\begin{aligned} \det(\Delta') = \prod_{x\in V} \text{outdeg}(x) &- \sum_{\gamma_1\in\Gamma} \prod_{x\in V\setminus\gamma_1} \text{outdeg}(x) \\ &+ \sum_{\substack{\gamma_1,\gamma_2\in\Gamma \\ \gamma_1\neq\gamma_2}} \prod_{x\in V\setminus(\gamma_1\cup\gamma_2)} \text{outdeg}(x) - \dots. \end{aligned}$$

The first term counts the number of all rotor configurations on V. The summand in the second term is the number of all rotor configurations that contain the oriented loop γ_1. The summand in the third term counts the number of rotor configurations that contain both loops γ_1 and γ_2; and so on. It follows from the inclusion-exclusion principle that the alternating sum counts precisely the number of rotor configurations with no loops. This proves the first statement.

Consider now the same expansion for the modified matrix $(\widetilde{\Delta}^h)'$. Due to the factor $\frac{1}{\omega}$, the only terms that remain are the ones where one of the oriented loops contains the edge h. Note that for each term there is at most one such loop. Grouping terms according to what this loop is:

$$\lim_{\omega\to\infty} \frac{1}{\omega} \det((\widetilde{\Delta}^h)') = - \sum_{\substack{\gamma_1 \in \Gamma: \\ h \in \gamma_1}} \Bigg[\prod_{x \in V \setminus \gamma_1} \mathrm{outdeg}(x) - \sum_{\substack{\gamma_2 \in \Gamma: \\ \gamma_2 \neq \gamma_1}} \prod_{x \in V \setminus (\gamma_1 \cup \gamma_2)} \mathrm{outdeg}(x) + \sum_{\substack{\gamma_2 \neq \gamma_3 \in \Gamma: \\ \gamma_2, \gamma_3 \neq \gamma_1}} \prod_{x \in V \setminus (\gamma_1 \cup \gamma_2 \cup \gamma_3)} \mathrm{outdeg}(x) - \dots \Bigg].$$

For each fixed $\gamma_1 \ni h$, the expression inside the square brackets is an inclusion-exclusion formula for the number of rotor configurations that contain γ_1 but no other oriented loop. Hence the second statement follows.

The third statement can be proved similarly to the second. This time the only terms that remain are the ones where there is a set of one, two or three loops that together contain f_1, f_2, f_3. Grouping terms according to what these loops are, we get a sum of inclusion-exclusion formulas yielding the terms $(-1)^i N_i(f_1, f_2, f_3)$, $i = 1, 2, 3$. $\square$

5.5.5. *The term* $|\mathcal{T}_e|$

Let us return to the three terms in (36). The first term $|\mathcal{T}_e|$ only involves local restrictions: certain rotor directions are forced or forbidden. Let us denote by j_1'', j_2'', j_4'' the south, west, east neighbours of j_1, respectively. The rotor $[o, j_2] = e$ can be forced by deleting the oriented edges $[o, j_1]$, $[o, j_3]$, $[o, j_4]$ from the graph. The rotor $[j_1, o] = h$ can be forced by deleting the oriented edges $[j_1, j_1'']$, $[j_1, j_2'']$, $[j_1, j_4'']$ from the graph. The requirements $\mathrm{head}(\rho_0(j_3)) \neq o$ and $\mathrm{head}(\rho_0(j_4)) \neq o$ can be achieved by deleting the oriented edges $[j_3, o]$ and $[j_4, o]$ from the graph (and note that the requirement $\mathrm{head}(\rho_0(j_2)) \neq o$ becomes superfluous due to acyclicity). It follows that we can apply the first statement of Theorem 5.8 to the matrix $(\Delta_n^{(1)})' = \Delta_n' + \delta^{(1)}$, where the only nonzero entries of $\delta^{(1)}$ are:

$$\delta^{(1)} = \begin{array}{c} \begin{array}{ccccccc} o & j_1 & j_3 & j_4 & j_1'' & j_2'' & j_4'' \end{array} \\ \begin{pmatrix} -3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ o \\ j_1 \\ j_3 \\ j_4 \end{array}$$

Using explicit values of the potential kernel (see (23)), we get

$$\lim_{n\to\infty} \frac{12|\mathcal{T}_e|}{\det(\Delta'_n)} = \lim_{n\to\infty} 12\,\det(I+\delta^{(1)}G_n) = \frac{6}{\pi} - \frac{30}{\pi^2} + \frac{48}{\pi^3}.$$

5.5.6. *The term* $|\mathcal{T}_f^c|$

The second term $|\mathcal{T}_f^c|$ in (36) involves the non-local restriction that f is contained in a loop. Necessarily, this loop ends with the edge h. We are going to force the loop by giving h the weight $-\omega$, and deleting the oriented edges $[o,j_1],[o,j_2],[o,j_3]$. We also delete $[j_2,o]$, $[j_3,o]$. (Note that this time the requirement $\mathrm{head}(\rho_0(j_4)) \neq o$ is superfluous.) Since $\omega\to\infty$, the rest of row j_1 of the matrix is immaterial. Hence we apply the second statement of Theorem 5.8 to the matrix $(\Delta_n^{(2)})' = (\widetilde{\Delta}^f)' = \Delta'_n + \delta^{(2)}$, where now

$$\delta^{(2)} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccc} o & j_1 & j_2 & j_3 \end{array} & \\ \begin{pmatrix} -3 & 1 & 1 & 1 \\ -\omega & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} & \begin{array}{c} o \\ j_1 \\ j_2 \\ j_3 \end{array} \end{array}$$

Since this time row j_1 of $\delta^{(2)}$ does not sum to 0, a divergent term of order $\log n$ arises. (This reflects the fact that the number of configurations containing a cycle is much larger than the number of acyclic ones.) Since we are evaluating a probability, the divergence will have to be cancelled by a term we get for $|\mathcal{T}_e^c \cap \mathcal{T}_f^c|$. In order to deal with the divergent terms, we need the following lemma.

Lemma 5.9. *For $K\geq 1$ and $z,w\in\mathbb{Z}^2$ with $|z|,|w|\leq K$, as $n\to\infty$, we have*

$$G_n(z,w) = G_n(o,o) - A(w-z) + O_K\left(\frac{\log n}{n}\right), \tag{37}$$

where the constant implied by O_K depends on K.

We do not prove this, as this is incorporated in Lemma 5.17 to come.

Replacing each term $G_n(z,w)$ in the matrix $I+\delta^{(2)}G_n$ by the expression on the right hand side of (37) and taking into account that $G_n(o,o)=O(\log n)$ (see Lemma 5.17) we get:

$$\begin{aligned} \frac{-12|\mathcal{T}_f^c|}{\det(\Delta'_n)} &= 12 \lim_{\omega\to\infty} \frac{1}{\omega} \det\left(I+\delta^{(2)}G_n\right) \\ &= \frac{3}{\pi^2} - G_n(o,o)\,12\left(\frac{2}{\pi^2}-\frac{4}{\pi^3}\right) + O\left(\frac{\log^2 n}{n}\right). \end{aligned} \tag{38}$$

5.5.7. *Priezzhev's "bridge trick" for the term* $|\mathcal{T}_e^c \cap \mathcal{T}_f^c|$

We are left to calculate the term $|\mathcal{T}_e^c\cap\mathcal{T}_f^c|$. Let $\rho_0\in\mathcal{T}_e^c\cap\mathcal{T}_f^c$, and let ρ stand for the set of edges $\rho=\rho_0\cup\{e\}\cup\{f\}$. There is a unique vertex $i_1\in V_n\setminus\{o\}$

such that ρ contains three oriented paths:
(i) a path from o to i_1 starting with e;
(ii) a path from o to i_1 starting with f;
(iii) a path from i_1 to o ending with h.

Moreover, the three paths are vertex-disjoint apart from the vertices o and i_1. We will call i_1 the *meeting point*. We are going to count configurations in $\mathcal{T}_e^c \cap \mathcal{T}_f^c$ separately for each fixed value i_1 of the meeting point.

The idea is to add three "bridge" edges between j_1, j_2, j_4 and three neighbours of i_1, and force the bridge edges to be in loops, via the third statement of Theorem 5.8. Then the existence of the loops gives the required paths, apart from possible flips in orientation of some of the paths. We would need to sum over all possible locations of i_1 and all possible choices of three neighbours of i_1. It turns out that symmetry considerations allow one to reduce the amount of calculations, and to count only two types of configurations according to the pattern of edges near i_1. In order to define these patterns, let w_1, w_3, w_4 be the south, north, east neighbours of i_1, respectively. Let G_n^{L,i_1} be the graph obtained from G_n by removing $[j_3, o]$ and adding three "bridges":

$$f_1^L = [j_1, i_1], \qquad f_2^L = [j_2, w_4], \qquad f_3^L = [j_4, w_3].$$

Following Priezzhev's notation [75], the symbol L indicates that the vertices w_3, i_1, w_4 form an L-shape. Let G_n^{Γ,i_1} be the graph obtained from G_n by removing $[j_3, o]$ and adding the three bridges:

$$f_1^\Gamma = [j_1, i_1], \qquad f_2^\Gamma = [j_2, w_1], \qquad f_3^\Gamma = [j_4, w_4].$$

The symbol Γ indicates the Γ-shape formed by the vertices w_1, i_1, w_4. If $i_1 = (k, l)$, we denote $i_1' = (-k, l)$. The computation is based on the following lemma.

Lemma 5.10. *There is a finite (explicit) set $P \subset \mathbb{Z}^2$, such that whenever $i_1, i_1' \in V_{n-1} \setminus P$, the following holds.*
(i) We have $N_2(f_1^L, f_2^L, f_3^L; z) = 0 = N_2(f_1^\Gamma, f_2^\Gamma, f_3^\Gamma; z)$ for $z = i_1, i_1'$.
(ii) We have

$$\sum_{z \in \{i_1, i_1'\}} \sum_{i=1,3} \left[N_i(f_1^L, f_2^L, f_3^L; z) + N_i(f_1^\Gamma, f_2^\Gamma, f_3^\Gamma; z) \right] = |\mathcal{T}_e^c \cap \mathcal{T}_f^c \cap \{\textit{meeting point equals } i_1 \textit{ or } i_1'\}|.$$

Proof. (i) This follows from planarity, as can be checked case-by-case. (This is the point in the computation where $d = 2$ is used in a crucial way.)

(ii) Consider first a configuration counted in $N_3(f_1^L, f_2^L, f_3^L; i_1)$. Remove the bridges. There are three vertex-disjoint oriented paths $i_1 \to o$, $w_4 \to j_2$ and $w_3 \to j_4$. Reverse the orientations of the paths arriving at j_2 and j_4, respectively. This leaves a rotor configuration with no rotor specified at o, w_3 and w_4. Adding the rotors $[w_3, i_1], [w_4, i_1]$ we get a configuration in $\mathcal{T}_e^c \cap \mathcal{T}_f^c$ such that the meeting point is i_1. The operations we performed are one-to-one between $N_3(f_1^L, f_2^L, f_3^L; i_1)$ and the configurations in the image.

We can perform similar steps for $N_1(f_1^L, f_2^L, f_3^L; i_1)$: remove the bridges, and reverse the orientation of the paths arriving at j_2 and j_4. The paths can occur in two distinct ways. One is obtained when we started with $i_1 \to j_2$, $w_4 \to j_4$, $w_3 \to o$, in which case, after we reversed orientations, there is no rotor specified at i_1 and w_4. We set these rotors as $[i_1, w_3]$ and $[w_4, i_1]$, yielding a configuration in $\mathcal{T}_e^c \cap \mathcal{T}_f^c$. The other possibility is that we started with $i_1 \to j_4$, $w_3 \to j_2$, $w_4 \to o$, in which case rotors will be missing at i_1 and w_3. We set these to be $[i_1, w_4]$ and $[w_3, i_1]$ to get a configuration in $\mathcal{T}_e^c \cap \mathcal{T}_f^c$.

Observe that the configurations constructed from N_1 are distinct from the ones arising from N_3. Let us add now the 'Γ-configurations', that is those arising from $N_3(f_1^\Gamma, f_2^\Gamma, f_3^\Gamma; i_1)$ and $N_1(f_1^\Gamma, f_2^\Gamma, f_3^\Gamma; i_1)$. There are four possibilities for the pattern of three edges incident with i_1 involved in the three paths. Configurations where the three edges only contain either the L- or the Γ-shape have been counted exactly once. Configurations where the three edges contain both the L- and the Γ-shape have been counted twice. The mirror images of these configurations (under $i_1 \leftrightarrow i_1'$) on the other hand, whose number is the same, are *not* counted in the corresponding terms $N_1(\cdot, \cdot, \cdot; i_1')$, $N_3(\cdot, \cdot, \cdot; i_1')$. Hence adding together the contributions for i_1 and i_1' restores the balance, and implies the statement.

The symmetry argument breaks down if $i_1 \in V_n \setminus V_{n-1}$. The path arguments may break down if i_1 or i_1' is too close to o, so there is a set of exceptions P. $\square$

Remark 5.11. When we take the limit $n \to \infty$, we are going to sum over all $i_1 \in \mathbb{Z}^2$ to have a workable expression as a Fourier integral. Then one needs to correct for the exceptions P individually. These can be handled with the ideas we used for $|\mathcal{T}_e|$ and $|\mathcal{T}_f^c|$; see [75]. The divergent term in (38) is cancelled by similar divergent terms in the contributions of the exceptional points in P. It also follows from the estimates in Lemma 5.15 that the contribution of the boundary terms $i_1, i_1' \in V_n \setminus V_{n-1}$ is negligible in the limit.

5.5.8. *Summation formulas for the N_1 and N_3 terms*

In evaluating the N_1 and N_3 terms, the bridges receive weight $-\omega$. The necessary modifications of the graph are encoded in the matrices:

$$\Delta_{i_1}(L) = \Delta'_n + \delta_{i_1}(L), \qquad \Delta_{i_1}(\Gamma) = \Delta'_n + \delta_{i_1}(\Gamma),$$

where

$$\delta_{i_1}(L) = \begin{array}{c} \begin{array}{ccccc} j_3 & o & i_1 & a = w_4 & b = w_3 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & 0 & -\omega \end{pmatrix} \end{array} \begin{array}{c} \\ j_3 \\ o \\ j_2 \\ j_4 \end{array} \tag{39}$$

$$\delta_{i_1}(\Gamma) = \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} j_3 & o & i_1 & a = w_1 & b = w_4 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & 0 & -\omega \end{pmatrix} \end{array} \begin{array}{c} \\ j_3 \\ o \\ j_2 \\ j_4 \end{array} \tag{40}$$

Due to Theorem 5.8 and Lemma 5.10(i) we have:

$$N_1^{L,i_1} + N_3^{L,i_1} = \lim_{\omega\to\infty} \frac{-1}{\omega^3} \frac{\det(\Delta_{i_1}(L))}{\det(\Delta'_n)} = \lim_{\omega\to\infty} \frac{-1}{\omega^3} \det(I + \delta_{i_1}(L) G_n)$$
$$= \begin{vmatrix} 1 + G_n(o,j_3) - G_n(j_3,j_3) & G_n(o,o) - G_n(j_3,o) & G_n(o,j_2) - G_n(j_3,j_2) & G_n(o,j_4) - G_n(j_3,j_4) \\ G_n(i_1,j_3) & G_n(i_1,o) & G_n(i_1,j_2) & G_n(i_1,j_4) \\ G_n(w_4,j_3) & G_n(w_4,o) & G_n(w_4,j_2) & G_n(w_4,j_4) \\ G_n(w_3,j_3) & G_n(w_3,o) & G_n(w_3,j_2) & G_n(w_3,j_4) \end{vmatrix} \tag{41}$$
$$=: \det(M_L).$$

We similarly get

$$N_1^{\Gamma,i_1} + N_3^{\Gamma,i_1}$$
$$= \begin{vmatrix} 1 + G_n(o,j_3) - G_n(j_3,j_3) & G_n(o,o) - G_n(j_3,o) & G_n(o,j_2) - G_n(j_3,j_2) & G_n(o,j_4) - G_n(j_3,j_4) \\ G_n(i_1,j_3) & G_n(i_1,o) & G_n(i_1,j_2) & G_n(i_1,j_4) \\ G_n(w_4,j_3) & G_n(w_4,o) & G_n(w_4,j_2) & G_n(w_4,j_4) \\ -G_n(w_1,j_3) & -G_n(w_1,o) & -G_n(w_1,j_2) & -G_n(w_1,j_4) \end{vmatrix} \tag{42}$$
$$=: -\det(M_\Gamma).$$

In order to deal with divergences as $n \to \infty$, we regularize by replacing G_n in rows 2–4 of the matrices M_L and M_Γ with the Green's function of the geometrically killed random walk:

$$G_n(z, w; r) := \frac{1}{4} \sum_{m=0}^{\infty} r^m \mathbf{P}^z[S(m) = w,\ \tau_{V_n^c} > m], \quad 0 < r \le 1,$$

where $\tau_{V_n^c}$ is the hitting time of V_n. Let $M_{L,r}$ and $M_{\Gamma,r}$ be the matrices obtained this way. We also let

$$A(z, w; r) := \frac{1}{4} \lim_{N\to\infty} \sum_{m=0}^{N} r^m \left(\mathbf{P}^z[S(m) = w] - \mathbf{P}^z[S(m) = z] \right),$$
$$z, w \in \mathbb{Z}^2,\ 0 < r \le 1,$$

and

$$G_{k,l}(r) := \frac{1}{4} \sum_{m=0}^{\infty} r^m \mathbf{P}^o[S(m) = (k,l)], \quad (k,l) \in \mathbb{Z}^2,\ 0 < r < 1.$$

The following two propositions, that we prove in Sections 5.5.9–5.5.10, state summation formulas for the N_1 and N_3 terms in the limit $n \to \infty$. These two propositions form the remaining part of the computation of $|\mathcal{T}_e^c \cap \mathcal{T}_f^c|$.

Proposition 5.12. *We have*

$$\lim_{n\to\infty} \sum_{i_1 \in V_n} [\det(M_L) - \det(M_\Gamma)] = \lim_{r \uparrow 1} \sum_{(k,l)\in\mathbb{Z}^2} \det(C_{k,l}(r)),$$

where

$$C_{k,l}(r) := \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{\pi} - \frac{1}{4} & \frac{1}{\pi} - \frac{1}{4} \\ G_{k,l-1} & G_{k,l} & G_{k+1,l} & G_{k-1,l} \\ G_{k+1,l-1} & G_{k+1,l} & G_{k+2,l} & G_{k,l} \\ G_{k,l} - G_{k,l-2} & G_{k,l+1} - G_{k,l-1} & G_{k+1,l+1} - G_{k+1,l-1} & G_{k-1,l+1} - G_{k-1,l-1} \end{pmatrix}, \tag{43}$$

with each G-entry evaluated at r.

Proposition 5.13. *We have*

$$\sum_{(k,l)\in\mathbb{Z}^2} \det(C_{k,l}(r)) = \frac{1}{32\pi^4} \iint \iint \frac{i \sin\beta_1 \, \det(M_1)\, d\alpha_1\, d\beta_1\, d\alpha_2\, d\beta_2}{D_r(\alpha_1,\beta_1) D_r(\alpha_2,\beta_2) D_r(\alpha_1+\alpha_2, \beta_1+\beta_2)},$$

where

$$M_1 := \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{\pi} - \frac{1}{4} & \frac{1}{\pi} - \frac{1}{4} \\ e^{i(\beta_1+\beta_2)} & 1 & e^{-i(\alpha_1+\alpha_2)} & e^{i(\alpha_1+\alpha_2)} \\ e^{i(\alpha_2-\beta_2)} & e^{i\alpha_2} & e^{2i\alpha_2} & 1 \\ e^{-i\beta_1} & 1 & e^{i\alpha_1} & e^{-i\alpha_1} \end{pmatrix},$$

and $D_r(\alpha,\beta) = 2 - r(\cos\alpha + \cos\beta)$.

Remark 5.14. When $r = 1$, the integral does not exist as a Lebesgue integral. In order to see this, consider the region of integration where (α_2, β_2) is in a small neighbourhood of $(0,0)$, and (α_1, β_1) is in a small neighbourhood of $(\pi/4, \pi/4)$, say. Subtract the last column from the other columns, and expand the determinant along the third row. The first three terms each contain a factor that vanishes as $(\alpha_2, \beta_2) \to (0,0)$, making the singularity due to $D_1(\alpha_2, \beta_2)$ integrable. But the last term is proportional to $(D_1(\alpha_2,\beta_2))^{-1}$, which is not integrable. Proposition 5.13 exhibits a delicate cancellation taking place.

5.5.9. Green function estimates

For the proofs of Propositions 5.12 and 5.13, we are going to need some estimates on Green functions.

Lemma 5.15.
(i) There exists a constant K such that

$$A(z, w; 1) = \frac{1}{2\pi} \log |w - z| + K + O\left(\frac{1}{|w - z|^2}\right), \quad |w - z| \geq 1.$$

(ii) Uniformly in $0 < r \leq 1$ and $w \in \mathbb{Z}^2$, we have

$$A(z, w; r) - A(o, w; r) = O(\log |z|), \quad |z| \geq 2.$$

(iii) Uniformly in $0 < r \leq 1$ and for $|f| = 1 = |h|$ we have

$$\partial_f^{(1)} A(z, o; r) = O\left(\frac{1}{|z|}\right), \quad |z| \geq 1;$$

$$\partial_h^{(2)} A(z, o; r) = O\left(\frac{1}{|z|}\right), \quad |z| \geq 1;$$

$$\partial_f^{(1)} \partial_h^{(2)} A(z, o; r) = O\left(\frac{1}{|z|^2}\right), \quad |z| \geq 1.$$

Proof. Statement (i) is [52, Theorem 4.4.4]. When $r = 1$, statements (ii) and (iii) follow immediately from (i). For the case $0 < r < 1$, the proof of [52, Theorem 4.4.4] can be adapted, and we sketch how this can be done. By considering a "lazy" random walk (that holds in place with probability ε on each step), we may replace the simple random walk by an aperiodic one. (Indeed, for the lazy walk $G_\varepsilon(z, w; r) = (1 - \varepsilon r)^{-1} G(z, w; r)$; see [52, (4.17)].)

Following the proof of [52, Theorem 4.4.4], write

$$\begin{aligned} A(o, z; r) = \sum_{m \leq |z|^2} r^m \mathbf{p}^m(o, o) - \sum_{m \leq |z|^2} r^m \mathbf{p}^m(o, z) \\ + \sum_{m > |z|^2} r^m (\mathbf{p}^m(o, o) - \mathbf{p}^m(o, z)). \end{aligned}$$

Let

$$B(z; r) = \sum_{1 \leq m \leq |z|^2} \frac{r^m}{m}.$$

Then the computations in [52] show that

$$\sum_{m \leq |z|^2} r^m \mathbf{p}^m(o, o) = c_1 \, B(z; r) + C + O(|z|^{-2}),$$

with c_1 and C independent of z and r. We also have that $\sum_{m \leq |z|} r^m \mathbf{p}^m(o, z)$ decays faster than any power of $|z|$ (uniformly in r). An application of the local central limit theorem yields that

$$\sum_{|z| < m \leq |z|^2} \left[r^m \mathbf{p}^m(o, z) - r^m \frac{c_1}{m} e^{-|z|^2/m} \right] = O(|z|^{-2}).$$

Also, the proof of [52, Lemma 4.3.2] shows that

$$\begin{aligned} \sum_{|z| < m \leq |z|^2} \frac{r^m}{m} e^{-|z|^2/m} &= \int_1^\infty \frac{1}{y} \exp\left(-\frac{y}{2} - \frac{\beta |z|^2}{y}\right) + O(|z|^{-2}) \\ &=: I_1(z; r) + O(|z|^{-2}), \end{aligned}$$

where $-\beta = \log r \in (-\infty, 0)$. Therefore,

$$\sum_{m \le |z|^2} r^m \left(\mathbf{p}^m(o,o) - \mathbf{p}^m(o,z)\right) = c_1 B(z;r) + C + c_1 I_1(z;r) + O(|z|^{-2}).$$

A similar computation yields

$$\begin{aligned}\sum_{m > |z|^2} r^m \left(\mathbf{p}^m(o,o) - \mathbf{p}^m(o,z)\right) &= c_1 \int_0^1 \frac{1}{y}(1 - e^{-y/2}) e^{-\beta|z|^2/y} + O(|z|^{-2}) \\ &=: c_1 I_2(z;r) + O(|z|^{-2}).\end{aligned}$$

Note that $I_1(z;r) = O(1)$ and $I_2(z;r) = O(1)$, uniformly in z and r. Statement (ii) now follows from

$$\begin{aligned}|A(z,w;r) - A(o,w;r)| &= c_1|B(w-z;r) - B(w;r)| + O(1) \\ &\le c_1|B(w-z;1) - B(w;1)| + O(1) \\ &= O(\log|z|).\end{aligned}$$

In order to prove the statements in (iii), write

$$\begin{aligned}\partial_f^{(1)} A(z,o;r) = c_1[B(z+f;r) - B(z;r)] &+ c_1[I_1(z+f;r) - I_1(z;r)] \\ + c_1[I_2(z+f;r) - I_2(z;r)] &+ O(|z|^{-2}).\end{aligned} \tag{44}$$

Using that $|z+f|^2 - |z|^2 = 2\langle z,f\rangle + 1 = O(|z|)$, the first term is $O(|z|^{-1})$. In order to estimate the second term, write

$$I_1(z+f;r) - I_1(z;r) = \int_1^\infty \frac{1}{y} e^{-y/2}\left(e^{-\beta|z+f|^2/y} - e^{-\beta|z|^2/y}\right).$$

We treat the cases $\beta/y \le |z|^{-1}$ and $\beta/y > |z|^{-1}$ separately. When $\beta/y \le |z|^{-1}$, we have

$$\begin{aligned}\left|e^{-\beta|z+f|^2/y} - e^{-\beta|z|^2/y}\right| &= e^{-\beta|z|^2/y}\left|e^{-\beta(2\langle z,f\rangle+1)} - 1\right| \\ &\le e^{-\beta|z|^2/y}\frac{C\beta|z|}{y} \\ &= \frac{C'}{|z|}\frac{\beta|z|^2}{y}e^{-\beta|z|^2/y} \\ &= O(|z|^{-1}).\end{aligned} \tag{45}$$

When $\beta/y > |z|^{-1}$, we have

$$e^{-\beta|z+f|^2/y},\ e^{-\beta|z|^2/y} = O\left(e^{-c|z|}\right). \tag{46}$$

This shows that the second term in (44) is $O(|z|^{-1})$. Similar considerations apply to the third term in (44). The argument for $\partial_h^{(2)} A(z,o;r)$ is identical.

Finally, for the last statement of (iii) we write

$$\begin{aligned} &\partial_f^{(1)} \partial_h^{(2)} A(z, o; r) \\ &\qquad = c_1 [B(z+f-h; r) - B(z+f; r) - B(z-h; r) + B(z; r)] \\ &\qquad\quad + c_1 [I_1(z+f-h; r) - I_1(z+f; r) - I_1(z-h; r) + I_1(z; r)] \\ &\qquad\quad + c_1 [I_2(z+f-h; r) - I_2(z+f; r) - I_2(z-h; r) + I_2(z; r)] \\ &\qquad\quad + O(|z|^{-2}). \end{aligned} \tag{47}$$

In the first term, cancellations take place between the four summations. The net result is that apart from $O(1)$ terms (that are each $O(|z|^{-2})$), there are $O(|z|)$ pairs of terms that come with opposite signs. For each pair (treating the cases $r \geq 1 - |z|^{-1}$ and $r < 1 - |z|^{-1}$ separately), we have the estimate

$$\frac{r^{m_1}}{m_1} - \frac{r^{m_2}}{m_2} = O(|z|^{-3}), \quad \text{if } m_1 = |z|^2 + O(|z|) \text{ and } m_2 = |z|^2 + O(|z|).$$

Summing these we get that the first term in (47) is $O(|z|^{-2})$. For the second term in (47), we argue similarly to (45)–(46) (treating the cases $\beta/y \leq |z|^{-1}$ and $\beta/y > |z|^{-1}$ separately). This gives

$$e^{-\beta|z+f-h|^2/y} - e^{-\beta|z+f|^2/y} - e^{-\beta|z-h|^2/y} + e^{-\beta|z|^2/y} = O(|z|^{-2}),$$

and it follows that the second term in (47) is $O(|z|^{-2})$. The argument for the third term is similar. This completes the proof. □

Let $\{S_r(m)\}_{m\geq 0}$ be the random walk killed at a $\mathsf{Geometric}(1-r)$ time that is independent of the walk. Below we interpret $A(S_r(m), w; r)$ as 0 after the killing time.

Lemma 5.16. *For all $0 < r \leq 1$, $z, w \in V_n$ we have*

$$G_n(z, w; r) = \mathbf{E}^z \left[A(S_r(\tau_{V_n^c}), w; r) \right] - A(z, w; r).$$

Proof. The case $r = 1$ is [52, Lemma 4.6.2(b)]. The proof is similar when $0 < r < 1$. Note that $M_m := A(S_r(m), w; r) - \frac{1}{4} \sum_{j=0}^{m-1} \mathbf{1}_{S_r(j)=w}$ is a martingale. This gives

$$A(z, w; r) = \mathbf{E}^z [A(S_r(N \wedge \tau_{V_n^c}), w; r)] - \mathbf{E}^z \left[\sum_{0 \leq j < N \wedge \tau_{V_n^c}} \mathbf{1}_{S_r(j)=w} \right].$$

Letting $N \to \infty$ and using bounded and monotone convergence, respectively, for the two terms we get the statement of the Lemma. □

Lemma 5.17. *Uniformly in $0 < r \leq 1$, $z \in V_n$, $n \geq 1$ and for $|f| = 1 = |h|$, we have*

$$G_n(o, o; r) = O(\log n) \tag{48}$$

$$G_n(z,o;r) - G_n(o,o;r) = -A(z,o;r) + O\left(|z|\frac{\log n}{n}\right) = O\left(\log|z|\right) \tag{49}$$

$$\partial_f^{(1)} G_n(z,o;r) = \partial_f^{(1)} A(z,o;r) + O\left(\frac{\log n}{\operatorname{dist}(z,V_n^c)}\right) \tag{50}$$

$$= O\left(\frac{1}{|z|}\right) + O\left(\frac{\log n}{\operatorname{dist}(z,V_n^c)}\right) \tag{51}$$

$$\partial_h^{(2)} G_n(z,o;r) = \partial_h^{(2)} A(z,o;r) + O\left(\frac{1}{n}\right) = O\left(\frac{1}{|z|}\right) \tag{52}$$

$$\partial_f^{(1)}\partial_h^{(2)} G_n(z,o;r) = \partial_f^{(1)}\partial_h^{(2)} A(z,o;r) + O\left(\frac{1}{n\operatorname{dist}(z,V_n^c)}\right) \tag{53}$$

$$= O\left(\frac{1}{|z|^2}\right) + O\left(\frac{1}{n\operatorname{dist}(z,V_n^c)}\right). \tag{54}$$

Proof of Lemma 5.17. The estimate (48) follows from

$$G_n(o,o;r) \le G_n(o,o;1) = \mathbf{E}^o[A(S(\tau_{V_n^c}))] = O(\log n).$$

In order to prove (49), we use Lemma 5.16 to write

$$\begin{aligned} &G_n(z,o;r) - G_n(o,o;r) \\ &\quad = -A(z,o;r) + \mathbf{E}^z[A(S_r(\tau_{V_n^c}),o;r)] - \mathbf{E}^o[A(S_r(\tau_{V_n^c}),o;r)]. \end{aligned}$$

Due to Lemma 5.15(ii), the first term is $O(\log|z|)$. By the same lemma, the random variable inside the expectations is $O(\log n)$. Due to a difference estimate for harmonic functions [52, Theorem 6.3.8], the total variation distance between the exit distributions of the random walk (without killing) started from z and o, respectively, is $O(|z|/n)$. This implies the same for the killed random walk, and the statement follows.

The proofs of (51) and (52) are similar to the proof of (54), so we only give the latter. We write

$$\begin{aligned} \partial_f^{(1)}\partial_h^{(2)} G_n(z,o;r) = -\partial_f^{(1)}\partial_h^{(2)} A(z,o;r) &+ \mathbf{E}^{z+f}\left[A(S_{\bar\tau_n},h;r) - A(S_{\bar\tau_n},o;r)\right] \\ &- \mathbf{E}^{z}\left[A(S_{\bar\tau_n},h;r) - A(S_{\bar\tau_n},o;r)\right]. \end{aligned}$$

Due to Lemma 5.15(iii), the first term is $O(|z|^{-2})$. The random variable inside the expectations is $O(\frac{1}{n})$, again due to Lemma 5.15(iii). Again due to the difference estimate for harmonic functions [52, Theorem 6.3.8], the total variation distance between exit distributions of the random walk (without killing) started from z and $z+f$, respectively, is $O(1/\operatorname{dist}(z,V_n^c))$. This implies the claim. □

5.5.10. Proof of the summation formulas

Proof of Proposition 5.12. In the first row of $M_L - M_\Gamma$ we can take the limit $n \to \infty$ directly and we obtain the first row of M_r. In order to deal with the divergent entries, we use row and column operations to exhibit cancellations in the determinant that allow us to take the limit $n \to \infty$. Subtracting the second column from the other columns and then subtracting the second row from the third and fourth rows we have:

$$\det(M_L - M_\Gamma) = \begin{vmatrix} \frac{2}{4} + o(1) & \frac{1}{4} + o(1) & \frac{1}{\pi} - \frac{2}{4} + o(1) & \frac{1}{\pi} - \frac{2}{4} + o(1) \\ \partial^{(2)}_{e_2} G_n & G_n & \partial^{(2)}_{-e_1} G_n & \partial^{(2)}_{e_1} G_n \\ \partial^{(1)}_{e_1} \partial^{(2)}_{e_2} G_n & \partial^{(1)}_{e_1} G_n & \partial^{(1)}_{e_1} \partial^{(2)}_{-e_1} G_n & \partial^{(1)}_{e_1} \partial^{(2)}_{e_1} G_n \\ (\partial^{(1)}_{-e_2} - \partial^{(1)}_{e_2}) & (\partial^{(1)}_{-e_2} - \partial^{(1)}_{e_2}) G_n & (\partial^{(1)}_{-e_2} - \partial^{(1)}_{e_2}) & (\partial^{(1)}_{-e_2} - \partial^{(1)}_{e_2}) \\ \times \partial^{(2)}_{e_2} G_n & & \times \partial^{(2)}_{-e_1} G_n & \times \partial^{(2)}_{e_1} G_n \end{vmatrix},$$

where each entry in rows 2–4 is evaluated at $(i_1, o; r = 1)$.

We split the determinant into two terms by writing $G_n(i_1, o; 1)$ (the second entry in the second row) as

$$G_n(i_1, o; 1) = G_n(o, o; 1) + (G_n(i_1, o; 1) - G_n(o, o; 1)).$$

This gives the terms:

$$\det(M_L - M_\Gamma) = G_n(o, o; 1) \det(\tilde{M}^o) + \det(\tilde{M}^{\text{diff}}),$$

where $\tilde{M}^o$ is the minor of $M_L - M_\Gamma$ obtained by removing the second row and second column, and $\tilde{M}^{\text{diff}}$ is obtained by replacing the entry G_n by $G_n(i_1, o; 1) - G_n(o, o; 1)$.

The estimates of Lemma 5.17 with $z = i_1$ show that

$$\tilde{M}^o = \begin{pmatrix} \frac{2}{4} + o(1) & \frac{1}{\pi} - \frac{2}{4} + o(1) & \frac{1}{\pi} - \frac{2}{4} + o(1) \\ O(|z|^{-2}) + & O(|z|^{-2}) + & O(|z|^{-2}) + \\ O(n^{-1} \text{dist}(z, V_n^c)^{-1}) & O(n^{-1} \text{dist}(z, V_n^c)^{-1}) & O(n^{-1} \text{dist}(z, V_n^c)) \\ O(|z|^{-2}) + & O(|z|^{-2}) + & O(|z|^{-2}) + \\ O(n^{-1} \text{dist}(z, V_n^c)^{-1}) & O(n^{-1} \text{dist}(z, V_n^c)^{-1}) & O(n^{-1} \text{dist}(z, V_n^c)) \end{pmatrix}.$$

This implies that

$$G_n(o, o; 1) \sum_{i_1 \in V_n} \det(\tilde{M}^o) = G_n(o, o; 1) \sum_{z \in \mathbb{Z}^2} \det(M^o) + O\left(\frac{\log n}{n}\right),$$

where

$$M^o := \begin{pmatrix} \frac{2}{4} & \frac{1}{\pi} - \frac{2}{4} & \frac{1}{\pi} - \frac{2}{4} \\ \partial^{(1)}_{e_1} \partial^{(2)}_{e_2} A & \partial^{(1)}_{e_1} \partial^{(2)}_{-e_1} A & \partial^{(1)}_{e_1} \partial^{(2)}_{e_1} A \\ (\partial^{(1)}_{-e_2} - \partial^{(1)}_{e_2}) \partial^{(2)}_{e_2} A & (\partial^{(1)}_{-e_2} - \partial^{(1)}_{e_2}) \partial^{(2)}_{-e_1} A & (\partial^{(1)}_{-e_2} - \partial_{e_2}) \partial^{(2)}_{e_1} A \end{pmatrix},$$

with each entry evaluated at $(z, o; 1)$.

Lemma 5.18. *We have* $\sum_{z\in\mathbb{Z}^2} \det(M^o) = 0$.

Proof. We are going to use that $A(z,o;1) = \lim_{r\uparrow 1} A(z,o;r)$. (We note that it is not strictly necessary here to regularize, and we could argue directly at $r = 1$. But this helps to avoid some delicate integrability issues, and we will need regularization anyway when we consider $\tilde{M}^{\mathrm{diff}}$.)

Due to the uniformity in r of the bounds in Lemma 5.15(iii), by dominated convergence we get

$$\sum_{z\in\mathbb{Z}^2} \det(M^o) = \lim_{r\uparrow 1} \sum_{z\in\mathbb{Z}^2} \det(M^o_r),$$

where M^o_r is defined the same way as M^o, but each entry evaluated at $(z,o;r)$. We are going to use the Fourier formula (see [52, Proposition 4.2.3]):

$$A(z,o;r) = \frac{1}{8\pi^2} \iint \frac{1 - e^{i(\alpha k+\beta l)}}{2 - r(\cos\alpha + \cos\beta)}\, d\beta\, d\alpha, \quad 0 < r < 1,$$

where $z = (k,l)$, and both integrals are over $[-\pi,\pi]$. This implies

$$\partial^{(1)}_{e_1}\partial^{(2)}_{e_2} A(z,o;r) = \frac{1}{8\pi^2} \iint e^{i(\alpha k+\beta l)} \frac{(e^{i\alpha}-1)(e^{-i\beta}-1)}{2 - r(\cos\alpha + \cos\beta)}\, d\beta\, d\alpha,$$

and similar formulas hold for the other entries in rows 2–3. It follows that

$$\det(M^o_r) = \frac{1}{64\pi^4} \iint\iint \frac{e^{i(\alpha_1+\alpha_2)k + i(\beta_1+\beta_2)l} \det(C^o)\, d\beta_1\, d\alpha_1\, d\beta_2\, d\alpha_2}{(2 - r(\cos\alpha_1 + \cos\beta_1))(2 - r(\cos\alpha_2+\cos\beta_2))}, \tag{55}$$

where

$$\begin{aligned}\det(C^o) &= \begin{vmatrix} \frac{2}{4} & \frac{1}{\pi} - \frac{2}{4} & \frac{1}{\pi} - \frac{2}{4} \\ (e^{i\alpha_1}-1)(e^{-i\beta_1}-1) & (e^{i\alpha_1}-1)(e^{i\alpha_1}-1) & (e^{i\alpha_1}-1)(e^{-i\alpha_1}-1) \\ -2i\sin\beta_2(e^{-i\beta_2}-1) & -2i\sin\beta_2(e^{i\alpha_2}-1) & -2i\sin\beta_2(e^{-i\alpha_2}-1) \end{vmatrix} \\ &= -2i\sin\beta_2(e^{i\alpha_1}-1) \begin{vmatrix} \frac{2}{4} & \frac{1}{\pi} - \frac{2}{4} & \frac{1}{\pi} - \frac{2}{4} \\ e^{-i\beta_1}-1 & e^{i\alpha_1}-1 & e^{-i\alpha_1}-1 \\ e^{-i\beta_2}-1 & e^{i\alpha_2}-1 & e^{-i\alpha_2}-1 \end{vmatrix}.\end{aligned}$$

Since the integrand in (55) is smooth, summation over $(k,l) = z \in \mathbb{Z}^2$ amounts to setting $(\alpha_2,\beta_2) = (-\alpha_1,-\beta_1)$ and keeping only the integrals over α_1,β_1. Therefore,

$$\begin{aligned}\sum_{(k,l)\in\mathbb{Z}^2} \det(M^o_r) = \frac{1}{16\pi^2} \iint & e^{i(\alpha_1 k+\beta_1 l)}(2i\sin\beta_1)(e^{i\alpha_1}-1) \\ & \times \frac{\begin{vmatrix} \frac{2}{4} & \frac{1}{\pi} - \frac{2}{4} & \frac{1}{\pi} - \frac{2}{4} \\ e^{-i\beta_1}-1 & e^{i\alpha_1}-1 & e^{-i\alpha_1}-1 \\ e^{i\beta_1}-1 & e^{-i\alpha_1}-1 & e^{i\alpha_1}-1 \end{vmatrix}}{(2 - 2r(\cos\alpha_1+\cos\beta_1))^2}.\end{aligned}$$

Write the factor in front of the determinant as $e^{i\alpha_1} - 1 = (\cos\alpha_1 - 1) + (i\sin\alpha_1)$, and split the intergal into a sum of two terms. Then the first term is anti-symmetric under $\alpha_1 \leftrightarrow -\alpha_1$ (since this exchanges the second and third columns in the determinant). The second term is anti-symmetric under the exchange $(\alpha_1, \beta_1) \leftrightarrow (-\alpha_1, -\beta_1)$ (since this exchanges the second and third rows). Hence both terms contribute 0 to the integral and this completes the proof of the lemma. □

We return to the proof of Proposition 5.12. Applying the estimates of Lemma 5.17 now to the entries of $\tilde{M}^{\text{diff}}$, we get

$$\sum_{i_1 \in V_n} \det(\tilde{M}^{\text{diff}}) = \sum_{z \in \mathbb{Z}^2} \det(M^{\text{diff}}) + O\left(\frac{\log n}{n}\right),$$

where

$$M^{\text{diff}} = -\begin{vmatrix} \frac{2}{4} & \frac{1}{4} & \frac{1}{\pi} - \frac{2}{4} & \frac{1}{\pi} - \frac{2}{4} \\ \partial^{(2)}_{e_2} A & A & \partial^{(2)}_{-e_1} A & \partial^{(2)}_{e_1} A \\ \partial^{(1)}_{e_1}\partial^{(2)}_{e_2} A & \partial^{(1)}_{e_1} A & \partial^{(1)}_{e_1}\partial^{(2)}_{-e_1} A & \partial^{(1)}_{e_1}\partial^{(2)}_{e_1} A \\ \begin{matrix}(\partial^{(1)}_{-e_2} - \partial^{(1)}_{e_2}) \\ \times\partial^{(2)}_{e_2} A\end{matrix} & (\partial^{(1)}_{-e_2} - \partial^{(1)}_{e_2}) A & \begin{matrix}(\partial^{(1)}_{-e_2} - \partial^{(1)}_{e_2}) \\ \times\partial^{(2)}_{-e_1} A\end{matrix} & \begin{matrix}(\partial^{(1)}_{-e_2} - \partial^{(1)}_{e_2}) \\ \times\partial^{(2)}_{e_1} A\end{matrix} \end{vmatrix},$$

with each entry in rows 2–4 evaluated at $(z, o; 1)$. We use that $A(z, o; 1) = \lim_{r\uparrow 1} A(z, o; r)$. The bounds of Lemma 5.15(ii),(iii) and dominated convergence imply that

$$\sum_{z \in \mathbb{Z}^2} \det(M^{\text{diff}}) = \lim_{r\uparrow 1} \sum_{z \in \mathbb{Z}^2} \det(M^{\text{diff}}_r),$$

where M^{diff}_r is defined in the same way as M^{diff}, except the A-entries are evaluated at $(z, o; r)$.

Since we now have $0 < r < 1$, we can write

$$\begin{aligned} A(z, o; r) &= G(o, o; r) - G(z, o; r) \\ \partial^{(1)}_f A(z, o; r) &= -\partial^{(1)}_f G(z, o; r) \\ \partial^{(2)}_h A(z, o; r) &= -\partial^{(2)}_h G(z, o; r) \\ \partial^{(1)}_f \partial^{(2)}_h A(z, o; r) &= -\partial^{(1)}_f \partial^{(2)}_h G(z, o; r). \end{aligned}$$

Due Lemma 5.18, we can drop the term $G(o, o; r)$ from the A entry in M^{diff}_r. Now undoing the row and column operations brings the determinant to the form (43), as required. □

Proof of Proposition 5.13. We use the Fourier formula:

$$G_{k,l}(r) = \frac{1}{8\pi^2} \iint \frac{e^{i(\alpha k + \beta l)}}{D_r(\alpha, \beta)}\, d\alpha\, d\beta$$

with variables (α_1, β_1) in row 4, with (α_2, β_2) in row 3 and (α_3, β_3) in row 2. This writes $\det(C_{k,l}(r))$ as a 6-fold integral. Since $0 < r < 1$, the integrand is smooth, and therefore summation over $(k, l) \in \mathbb{Z}^2$ amounts to setting the Fourier variables $\alpha_3 = -(\alpha_1 + \alpha_2)$ and $\beta_3 = -(\beta_1 + \beta_2)$. This yields the formula in the statement. □

5.6. Further computations in 2D

Poghosyan, Grigorev, Priezzhev and Ruelle [72] extended Priezzhev's calculations of the height probabilities to the correlation function between height 0 and height 1, 2, 3. Their result is that for some explicit non-zero constants c_h, d_h one has

$$\nu[\eta(o) = 0,\, \eta(y) = h] - p(0)p(h) = \frac{c_h \log|y| + d_h}{|y|^4} + O\left(\frac{1}{|y|^5}\right), \quad \text{as } |y| \to \infty,\ h = 1, 2, 3.$$

The presence of the logarithmic term in the correlation function had been predicted by Piroux and Ruelle [71], who also predicted, based on conformal field theory calculations, that

$$\nu[\eta(o) = h_1,\, \eta(y) = h_2] - p(h_1)p(h_2) \sim c_{h_1,h_2} \frac{\log^2|y|}{|y|^4} \quad \text{as } |y| \to \infty,\ h_1, h_2 = 1, 2, 3.$$

Open Question 5.19. Show that the correlation between heights 1, 2, 3 is of order $(\log^2|y|)/|y|^4$.

The presence of logarithmic terms was brought to light by computing height probabilities near the boundary on the discrete upper half plane [44, 71]. Two natural boundary conditions are:

(i) **open**: at boundary sites $\Delta^{\text{open}}_{xx} = 4$, one particle leaves the system on toppling;
(ii) **closed**: at boundary sites $\Delta^{\text{closed}}_{xx} = 3$, no particle leaves the system on toppling.

The potential kernel on the discrete upper-half plane, with either boundary condition, is easily expressed in terms of the potential kernel on $\mathbb{Z}^2$, and Jeng, Piroux and Ruelle use this to extend Priezzhev's calculations to these cases. After lengthy computations they arrive at the following result. Let

$$p^{\text{open}}(i; m) = \nu^{\text{open}}[\eta((0, m)) = i];$$
$$p^{\text{closed}}(i; m) = \nu^{\text{closed}}[\eta((0, m)) = i].$$

Then, with explicit constants a_i, b_i,

$$p^{\text{open}}(i; m) = p(i) + \frac{1}{m^2}\left(a_i + \frac{b_i}{2} + b_i \log m\right) + o(m^{-2}),$$

$$\text{as } m \to \infty,\ i = 1, 2, 3;$$

$$p^{\text{closed}}(i; m) = p(i) - \frac{1}{m^2}(a_i + b_i \log m) + o(m^{-2}),$$

$$\text{as } m \to \infty,\ i = 1, 2, 3.$$

Jeng, Piroux and Ruelle make the remarkable observation that, up to terms of order $o(m^{-2})$, the probabilities of heights 2 and 3 are linear combinations of the probabilities of heights 0 and 1. That is, with either boundary condition $* = \text{open}, \text{closed}$ they get:

$$\begin{aligned}\frac{48 - 12\pi + 5\pi^2 - \pi^3}{2(\pi - 2)} p^*(0; m) + (\pi - 8) p^*(1; m) + 2(\pi - 2) p^*(2; m) \\ = \frac{(\pi - 2)(\pi - 1)}{\pi} + o(m^{-2}).\end{aligned}$$

They conjecture that the same relationship between the height probabilities will hold in all domains and with any boundary condition.

5.7. Minimal configurations

We close this section by giving a theorem that states the general type of events for which the determinantal computations sketched in Sections 5.2, 5.3 and 5.4 can be applied. At the same time, we give an alternative formulation that highlights the connection with the Transfer-Current Theorem. Let $G = (V \cup \{s\}, E)$ be a finite connected multigraph.

Definition 5.20. Let $W \subset V$, and let ξ be a particle configuration on W. We say that ξ is **minimal**, if the sandpile η^* defined by

$$\eta^*(x) := \begin{cases} \xi(x) & \text{if } x \in W; \\ \eta^{\max}(x) & \text{if } x \in V \setminus W; \end{cases}$$

is recurrent, but $\eta^* - \mathbf{1}_x \notin \mathcal{R}_G$ for any $x \in W$.

Theorem 5.21. *[43, 63] Let ξ be minimal on W. There exists a set of edges $\mathcal{E}_W$ touching W, such that*

$$\nu_G[\eta : \eta(x) = \xi(x),\ x \in W] = \det(K_G(e, f))_{e, f \in \mathcal{E}_W}.$$

The reason the theorem works is that minimal sandpile events can be expressed, via a particular version of the burning bijection, as the absence of a fixed set of edges from the uniform spanning tree.

6. Infinite graphs

In this section we look at whether the sandpile dynamics via topplings can be extended to infinite graphs. Let $G = (V, E)$ be a locally finite, connected,

infinite graph. A sequence $V_1 \subset V_2 \subset \cdots \subset V$ of finite sets of vertices such that $\cup_{n=1}^{\infty} V_n = V$ is called an **exhaustion** of G. We let $G_n = (V_n \cup \{s\}, E_n)$ denote the wired graph obtained from V_n. We write ν_n for the stationary distribution of the sandpile Markov chain on G_n. The questions we will be interested in are:
(i) Does $\nu_n \Rightarrow$ some limit ν?
(ii) If yes, are avalanches ν-a.s. finite, if we add a single particle to the infinite configuration?

A fruitful approach to the above questions turns out to be to translate them into questions about the uniform spanning tree via the burning bijection.

6.1. The wired uniform spanning forest

The following theorem, due to [25, 70], says that the measure UST_{G_n} converges to a unique limit, regardless of the exhaustion. In order to formulate this as weak convergence of probability measures, regard a spanning tree as a set of edges. Then UST_{G_n} can be be viewed as a probability measure on 2^E (note that edges in E_n, including the ones leading to s, are uniquely idenitified with elements of E).

Theorem 6.1. *There exists a measure* WSF *such that for any exhaustion we have* $\mathsf{UST}_{G_n} \Rightarrow \mathsf{WSF}$ *as* $n \to \infty$*, in the sense of weak convergence of probability measures. Under* WSF*, each connected component is an infinite tree, almost surely.*

The limit WSF is called the **wired uniform spanning forest** measure.

Definition 6.2. An infinite tree has **one end**, if any two infinite self-avoiding paths in the tree have a finite symmetric difference.

Theorem 6.3. *[4, 70] When* $G = \mathbb{Z}^d$*, we have the following.*
(i) If $2 \le d \le 4$ *then* WSF*-a.s. there is a single tree and it has one end.*
(ii) If $d \ge 5$ *then* WSF*-a.s. there are infinitely many trees and each one has one end.*

Given a spanning tree τ in a finite graph with a sink s, we say that vertex y is a **descendant** of vertex x, if x lies on the unique path from y to s in τ. This notion extends naturally to infinite one-ended trees: in this case y is called a **descendant** of x, if y lies on the unique infinite self-avoiding path starting at x.

It is usually not an easy problem to decide whether for a given infinite graph each tree has one end WSF-a.s. Nevertheless the one end property is known for a large class of graphs beyond $\mathbb{Z}^d$; see [4, 58]. Examples of graphs where the one end property fails are given by the direct product of $\mathbb{Z}$ with any finite connected graph. On such graphs, WSF-a.s. there is a single tree with two ends. See [39] for a study of sandpiles on these graphs.

6.2. One end property and the sandpile model

The usefulness of the one end property for the sandpile model is illustrated by the following theorem.

Theorem 6.4. *[43] Suppose that* WSF*-a.s each tree has one end. Then there exists a measure ν such that for any exhaustion $\nu_n \Rightarrow \nu$. That is, for any finite $Q \subset V$ and any particle configuration ξ on Q we have*

$$\lim_{n\to\infty} \nu_n \left[\eta : \eta_Q = \xi\right] = \nu \left[\eta : \eta_Q = \xi\right].$$

The main idea of the proof is a decomposition of the burning bijection of Majumdar and Dhar into two phases. Such a decomposition was used in [75] when $Q = \{o\}$ (see Section 5.5), and is also implicit in [63]. Fix a finite set $Q \subset V$, and suppose that our aim is to show that under ν_n, the marginal distribution of the sandpile in Q converges as $n \to \infty$.

Burning bijection anchored at Q. We split the burning process into two phases.

Phase I. Follow the usual burning process with the restriction that no vertex of Q is allowed to burn. Phase I ends when there are no more vertices that can be burnt this way.

Phase II. Now follow the usual burning process to burn the remaining vertices.

A formal definition can be given along the lines of the case $Q = \{o\}$ presented in Section 5.5.

Given $\eta \in \mathcal{R}_{G_n}$, let $W_n(\eta)$ denote the set of vertices that are burnt in Phase II (so that $Q \subset W_n(\eta) \subset V_n$). A bijection $\varphi_Q : \mathcal{R}_{G_n} \to \mathcal{T}_{G_n}$ can be constructed from the above burning rule similarly to Sections 4.1 and 5.5.

The proof of Theorem 6.4 is based on the following lemma that we state without proof; see [43].

Lemma 6.5. *Fix W such that $Q \subset W \subset V_n$. Under ν_n, the restrictions of sandpile to $V \setminus W$ and W, respectively, are conditionally independent, given the event $\{\eta : W_n(\eta) = W\}$.*

Sketch of the proof of Theorem 6.4.. Due to Lemma 6.5 we can write

$$\begin{aligned} \nu_n[\eta : \eta_Q = \xi] &= \sum_{Q \subset W \subset V_n} \nu_n[\eta : W_n(\eta) = W,\, \eta_Q = \xi] \\ &= \sum_{Q \subset W \subset V_n} \nu_n[\eta : W_n(\eta) = W]\, p_{W,\xi}, \end{aligned}$$

where the numbers $p_{W,\xi}$ do not depend on n. On the other hand, the event $\{W_n = W\}$ is spanning tree local, and hence $\nu_n[W_n = W] \to q_W$, for some numbers q_W. This suggests that

$$\nu[\eta_Q = \xi] = \sum_{\substack{Q \subset W \subset V \\ W \text{ finite}}} q_W\, p_{W,\xi}.$$

The proof can be completed by showing that the random sets W_n converge weakly to a limit W_∞ that is a.s. finite. For this, the following property is key: in the *first step* of Phase II, no vertex in $W_n \setminus Q$ can burn. Indeed, in Phase I we have examined such vertices, and they were found to be not burnable. This implies that the spanning tree will contain no edges between $W_n \setminus Q$ and $V \setminus W_n$. In fact, W_n equals the set of all descendants of Q under the bijection; see Exercise 5.6. Due to the one end hypothesis of the theorem, the set of descendants of Q is finite WSF-a.s. From this one can conclude the convergence $W_n \Rightarrow W_\infty$, where W_∞ is the set of all descendants of Q in the wired spanning forest. □

The following theorem shows that on transient graphs at least, the **sandpile measure** ν constructed in Theorem 6.4 is nicely behaved, in that it has finite avalanches. The theorem can be proved along the lines of [40, Theorem 3.11], although that proof was stated in $\mathbb{Z}^d$. Cases of graphs more general than $\mathbb{Z}^d$ but with more restrictive assumptions than those of Theorem 6.4 were considered in [38].

Theorem 6.6. *[40, Theorem 3.11] Assume the hypotheses of Theorem 6.4. If in addition G is transient, then for ν-a.e. η and all $x \in V$, the configuration $\eta + \mathbf{1}_x$ can be stabilized with finitely many topplings.*

Idea of the proof.. On transient graphs, $\mathbf{E}_\nu[n(x, y, \cdot)] = G(x, y) < \infty$. Hence ν-a.s., every site topples finitely many times, when a particle is added at x. However, this is not enough, since we may still have infinitely many vertices toppling (note that $\sum_{y \in V} G(x, y) = \infty$).

In order to show that only finitely many vertices topple, one can use a decomposition of the avalanche into **waves**, introduced by Ivashkevich, Ktitarev and Priezzhev [36]. Waves are defined as follows. After we added a particle at x, topple x, and all other vertices that can be toppled, but do not allow x to topple a second time. It is not difficult to see that each vertex topples at most once under this restriction. The set of vertices that toppled is called the **first wave**. After the first wave, if x is still unstable (this will be the case if and only if all of its neighbours were in the first wave), topple x a second time and topple all other vertices that can be toppled, not allowing x to topple a third time. This is called the **second wave**, etc. Ivashkevich, Ktitarev and Priezzhev show that the ensemble of all possible waves started at x is in bijection with the ensemble of all spanning forests of G_n with two components such that x and s are in distinct components.

The expected number of waves under ν is finite: $\nu[n(x, x, \cdot)] = G(x, x) < \infty$; see [40]. Therefore, it is sufficient to show that each wave is finite ν-a.s. The latter can be deduced from the one end assumption. □

On recurrent graphs, finiteness of avalanches is much more subtle, and this is largely open.

Open Question 6.7. Consider $\mathbb{Z}^2$. Is it true that for ν-a.e. η the configuration $\eta + \mathbf{1}_0$ can be stabilized with finitely many topplings? Note that for *all* $x \in \mathbb{Z}^2$

we have

$$\mathbf{E}_{\nu}[n(o,x;\cdot)] = \lim_{n\to\infty} \mathbf{E}_{\nu_n}[n(o,x;\cdot)] = \lim_{n\to\infty} G_n(o,x) = \infty.$$

Here the proof of the first equality is not immediate; see [5]. *On average* every vertex topples infinitely often.

On graphs of the form $\mathbb{Z}\times G_0$, where G_0 is a finite connected graph, avalanches are not finite in general. When G_0 consists of a single vertex, this follows from the fact that ν concentrates on the single configuration $\eta \equiv 1$ (and hence all vertices topple infinitely often); see Exercise 4.6 and [62]. When G_0 has at least two vertices, the stationary distributions do not have a unique weak limit point. Let $G_{-n,m}$ denote the wired graph constructed from $\{-n,-n+1,\dots,m-1,m\}\times G_0$, and let $\nu_{-n,m}$ denote the stationary distribution of the sandpile on $G_{-n,m}$. It can be shown [39] that there are two distinct ergodic weak limit points of $\nu_{-n,m}$, as $n,m\to\infty$. This arises from the fact that the burning process on $G_{-n,m}$ operates both from the left and the right end of the graph. A typical recurrent configuration has a "left-burnable" and a "right-burnable" region, and these give rise to two distinct ergodic sandpile measures ν^L and ν^R. With respect to either ν^L and ν^R, there is a strictly positive probability of both finite and infinite avalanches; see [39].

6.3. The sandpile group of infinite graphs

The sandpile measure ν is supported on the closed set

$$\mathcal{R} = \left\{ \eta \in \prod_{x\in V} \{0,\dots,\deg(x)-1\} : \begin{array}{l} \eta \text{ is ample for } F \text{ for every} \\ \text{finite } \emptyset \neq F \subset V \end{array} \right\}.$$

This follows from Exercise 4.5 and weak convergence of ν_n to ν. When avalanches are ν-a.s. finite, the addition operators E_x are defined ν-a.e. for all $x \in V$. It can be shown that they leave ν invariant and the Abelian property holds: $E_xE_y = E_yE_x$; see [40]. Hence the addition operators generate an Abelian group of measure-preserving transformations of $(\mathcal{R},\nu)$. [After the first version of these notes appeared on arXiv, E. Verbitsky (private communication) pointed out to us that in the case $V = \mathbb{Z}^d$, this group is isomorphic to $\mathbb{Z}[u_1^{\pm 1},\dots,u_d^{\pm 1}]/(f)$, where (f) is the ideal generated by the polynomial $2d - \sum_{j=1}^{d}(u_j + u_j^{-1})$.]

The case that is perhaps best understood is sandpiles that dissipate particles on every toppling. For $d \geq 1$ and an integer $\gamma \geq 1$ we define the **dissipative sandpile** with bulk dissipation γ as follows. Let $V_n \subset \mathbb{Z}^d$ be finite, and let $G_n^{(\gamma)} = (V_n \cup \{s\}, E_n^{(\gamma)})$ denote the graph obtained from the wired graph (V_n,E_n) by adding γ edges between each $x \in V_n$ and s. That is, a vertex x in configuration η will be stable when $\eta(x) < 2d+\gamma$, and when an unstable vertex is toppled, it sends γ particles to the sink, in addition to sending one particle to each of its neighbours. The effect of toppling can be formally written in terms of the graph

Laplacian of $G_n^{(\gamma)}$. This is

$$\Delta_{xy}^{(\gamma)} = \begin{cases} 2d+\gamma & \text{if } x = y; \\ -1 & \text{if } x \sim y; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$T_x\eta(y) = \eta(y) - \sum_{z \in V_n} \Delta_{xz}^{(\gamma)}, \quad x, y \in V_n.$$

Maes, Redig and Saada [60] show that Dhar's the formalism of the sandpile group carries through in the limit $V_n \uparrow \mathbb{Z}^d$, and the limiting sandpile measure ν can be identified with Haar measure on a compact Abelian group. More precisely, the following is proved in [60]. Let

$$\begin{aligned} \mathcal{R}^{(\gamma)} &= \left\{ \eta \in \{0, \dots, 2d+\gamma-1\}^{\mathbb{Z}^d} : \begin{array}{l} \eta \text{ is ample for } F \text{ for every} \\ \text{finite } \emptyset \neq F \subset \mathbb{Z}^d \end{array} \right\} \\ &= \left\{ \eta \in \{0, \dots, 2d+\gamma-1\}^{\mathbb{Z}^d} : \begin{array}{l} \text{for every finite } \emptyset \neq F \subset \mathbb{Z}^d \\ \text{there exists } x \in F \text{ such that} \\ \eta(x) \geq \deg_F(x) \end{array} \right\}. \end{aligned}$$

Introduce the following equivalence relation on $\mathcal{R}^{(\gamma)}$. We say that $\eta \sim \zeta$, if there exists $m : \mathbb{Z}^d \to \mathbb{Z}$ such that $\eta - \zeta = \Delta^{(\gamma)} m$. Let $[\eta]$ denote the equivalence class of η. It is shown that $\mathcal{R}/\sim = \{[\eta] : \eta \in \mathcal{R}\}$ is a compact Abelian group. It is also shown that ν_n converges weakly to a measure ν that concentrates on $\mathcal{R}^{(\gamma)}$, and that for ν-a.e. η one has $[\eta] = \{\eta\}$. Finally, it is shown that ν projected to $\mathcal{R}/\sim$ is the Haar measure. See [81] for interesting properties of $\mathcal{R}$ and $\mathcal{R}^{(\gamma)}$ related to the above.

7. Stabilizability of infinite configurations

In Theorem 6.6 we saw that under certain conditions we can add particles to a ν-typical configuration, and only finitely many topplings result a.s. A more general question that is interesting in its own right is: what infinite configurations can be stabilized (in some appropriate sense)? A more basic question that is still not fully understood is: what happens if we add a single column of n particles to a stable background configuration, and attempt to stabilize?

7.1. Relaxing a column of particles

A recent survey that covers the topics in this section is [56]. If we start with a large number of particles at the origin and stabilize, what will be the shape of the region visited by the particles? We collect some results on this question in three related models. Striking computer simluations of these questions are available: see for example [54, 56, 57, 69].

7.1.1. Three models

A. Sandpile. Start with n particles at $0 \in \mathbb{Z}^d$, and no particles elsewhere. Now stabilize via topplings. Let

$$S_n = \{x \in \mathbb{Z}^d : x \text{ was visited by a particle during stabilization}\}.$$

More generally: start with h particles at each $x \in \mathbb{Z}^d \setminus \{0\}$, where $h \leq 2d-2$ (the case $h = 2d-1$ being trivial). Here h is allowed to be *negative*, that is, we allow a "hole" of depth $|h|$ that first has to be "filled", before topplings can occur. Let $S_{n,h}$ denote the set of vertices visited.

B. Rotor-router. Start with n particles at the origin and arbitrary initial rotors everywhere on $\mathbb{Z}^d$. Each particle in turn follows rotor-router walk until it arrives at a vertex that has not been visited before, and there it stops. Let

$$A_n = \{\text{vertices occupied after all particles stopped}\}.$$

C. Divisible sandpile. Start with a non-negative real mass m at the origin and no mass anywhere else. If $x \in \mathbb{Z}^d$ has mass ≥ 1, distribute the mass in excess of 1 equally among the neighbours. In this model topplings do not commute, but the stabilization is still well-defined; see [54]. Let

$$D_m = \{x \in \mathbb{Z}^d : x \text{ has final mass} = 1\}.$$

Heuristically, this model corresponds to taking $n = m|h|$ in Model A and letting $h \to -\infty$.

7.1.2. Shape theorems / shape estimates

Let us write $B_r = \{x \in \mathbb{Z}^d : |x| < r\}$ and let ω_d = volume of the unit ball in $\mathbb{R}^d$. Levine and Peres [54, 56] show that the rotor-router model and the divisible sandpile satisfy spherical shape theorems in the strong sense that there exist $c, c' > 0$ such that if $m = \omega_d r^d$ then

$$B_{r - c\log r} \subset A_m \subset B_{r + c'\log r};$$

and there exist $c, c' > 0$ depending on d such that if $m = \omega_d r^d$ then

$$B_{r-c} \subset D_m \subset B_{r+c'}. \tag{56}$$

Simulations suggest that for the sandpile the asymptotic shape is *not* circular. Levine and Peres [54] prove that if $-h \geq 2 - 2d$, then

$$B_{c_1 r - c_2} \subset S_{n,h},$$

where $c_1 = (2d-1-h)^{-1/d}$ and c_2 only depends on d. Also, when $-h \geq 1-d$, then for every $\varepsilon > 0$ they get

$$S_{n,h} \subset B_{c_1' r + c_2'},$$

where $c'_1 = (d - \varepsilon - h)^{-1/d}$, and c'_2 depends only on d, h and ε. The inner and outer bounds approach each other as $h \downarrow -\infty$. This reinforces the idea that this limit corresponds to the divisible sandpile, for which the limit shape is circular in the strong sense (56).

For the values $d \le h \le 2d-2$, Fey, Levine and Peres [16] prove an outer bound of a cube of order $n^{1/d}$: for any $\varepsilon > 0$ they get

$$S_n \subset \{x \in \mathbb{Z}^d : \|x\|_\infty \le r\}, \tag{57}$$

where $r = \frac{d+\varepsilon}{2d-1-h}(n/\omega_d)^{1/d}$.

In the analysis of all three models, an important role is played by the **odometer** function. In the case of the sandpile model, this is the function $v_n(x) =$ number of topplings at x. Let $s_n(x) = (n\mathbf{1}_o)^\circ(x)$, $x \in \mathbb{Z}^d$, denote the stabilization of a pile of n particles at o (with no initial holes). Comparing the number of incoming and outgoing particles at $x \in \mathbb{Z}^d$, we have

$$s_n(x) = n\mathbf{1}_o(x) - 2dv_n(x) + \sum_{y:y\sim x} v_n(y) = n\mathbf{1}_o(x) - (\Delta v_n)(x),$$

where Δ is the discrete Laplacian on $\mathbb{Z}^d$. Rearranging, we have $\Delta v_n = n\mathbf{1}_o - s_n$, where s_n is between 0 and $2d-1$, and hence is bounded in n. Thus heuristically, v_n should be compared to the function Φ_n satisfying $\Delta\Phi_n = n\mathbf{1}_o$.

7.1.3. Scaling limit of the final configuration

In the sandpile model, simulations show intricate fractal patterns in the final configuration reached from a column of height n. Pegden and Smart [69] prove that this pattern has a scaling limit. In order to state their result, recall that $s_n(x)$ is the stabilized configuration:

$$s_n(x) = (n\mathbf{1}_0)^\circ(x), \quad x \in \mathbb{Z}^d.$$

Define the rescaled function

$$\bar{s}_n(x) = s_n(n^{1/d}x), \quad x \in n^{-1/d}\mathbb{Z}^d,$$

and extend $\bar{s}_n$ to all of $\mathbb{R}^d$, in such a way that it is constant on each cube of the form $y + [-\frac{1}{2}n^{-1/d}, \frac{1}{2}n^{-1/d})^d$, $y \in n^{-1/d}\mathbb{Z}^d$.

Theorem 7.1. *[69] There exists a unique $s \in L^\infty(\mathbb{R}^d)$, such that for all functions φ continuous with compact support we have*

$$\int_{\mathbb{R}^d} \bar{s}_n\,\varphi\,dx \overset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} s\,\varphi\,dx.$$

Moreover, $\int_{\mathbb{R}^d} s\,dx = 1$, $0 \le s \le 2d-1$, and s vanishes outside some ball.

7.2. Explosions

Given an unstable sandpile on $\mathbb{Z}^d$, we can attempt to stabilize it by carrying out (legal) topplings in such a way that if at any time some vertex x is unstable, then our procedure ensures that x is toppled at a later time. Let us call such a toppling sequence **exhaustive**.

Definition 7.2. We call a sandpile η on $\mathbb{Z}^d$ **stabilizable**, if there exists an exhaustive toppling sequence such that every vertex topples finitely often.

Definition 7.3. A stable background configuration η on $\mathbb{Z}^d$ is called **explosive**, if there exists $1 \le n < \infty$ such that in attempting to stabilize $\eta + n\mathbf{1}_0$ all of $\mathbb{Z}^d$ topples. The background is called **robust**, if there are finitely many topplings for all $n \ge 1$.

Note: Explosive implies that in fact all vertices topple infinitely many times.

Example 7.4. Write $\overline{k}$ for the configuration that equals the constant value k everywhere. It is easy to see that $\overline{2d-1}$ is explosive. On the other hand, $\overline{2d-2}$ is robust, due to (57).

The following two examples, due to Fey, Levine and Peres [16], show that there are robust configurations arbitrarily close to $\overline{2d-1}$, and explosive ones arbitrarily close to $\overline{2d-2}$. For the first example, let

$$\Lambda(m) = \{x \in \mathbb{Z}^d : m \not| x_i,\ 1 \le i \le d\},$$

that is, remove from $\mathbb{Z}^d$ all vertices that have a coordinate divisible by m. Then for any $m \ge 1$ the background $\overline{2d-2} + \mathbf{1}_{\Lambda(m)}$ is robust; see [16, Theorem 1.2]. For the second example, let

$$\beta(x) = \begin{cases} 1 & \text{with probability } \varepsilon; \\ 0 & \text{with probability } 1-\varepsilon. \end{cases}$$

Then for any $\varepsilon > 0$, with probability 1, the background $\overline{2d-2} + \beta$ is explosive; see [16, Proposition 1.4].

7.3. Ergodic configurations

Finally, we state some results on stabilizability of sandpiles that are random samples from a translation invariant ergodic measure on $\{0, 1, 2, \dots\}^{\mathbb{Z}^d}$. It is tempting to assume that the boundary for stabilizability would be given by whether the particle density is above or below the critical sandpile density $\rho_c = \mathbf{E}_\nu[\eta(0)]$. However, this is not so, even for product measures, as is demonstrated in various ways in [17].

The following theorem states some results proved by Fey and Redig [19] and Meester and Quant [66].

Theorem 7.5. *[19, 66] Let μ be a translation invariant ergodic measure on sandpiles on $\mathbb{Z}^d$.*
(a) If $\mathbf{E}_\mu[\eta(0)] < d$, then μ-a.e. η is stabilizable.
(b) If $\mathbf{E}_\mu[\eta(0)] > 2d - 1$, then μ-a.e. η is not stabilizable.

The picture of stabilizability is more complete in dimension $d = 1$, since the upper and lower bounds in Theorem 7.5(a),(b) coincide. Fey, Meester and Redig [18] determined what happens at the critical density.

Theorem 7.6. *[18, Theorem 3.2] Let μ be a product measure on sandpiles on $\mathbb{Z}$ such that $\mathbf{E}_\mu[\eta(0)] = 1$ and $\mu[\eta(0) = 0] > 0$. Then μ-a.e. η is not stabilizable.*

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