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A NATURAL TOPOLOGY FOR UPPER SEMICONTINUOUS FUNCTIONS AND A BAIRE CATEGORY DUAL FOR CONVERGENCE IN MEASURE

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Let X be a compact metric space. If we identify each upper semicontinuons function f with its hypograph $\{(x, \alpha): \alpha \leqq f(x)\}$ in $X \times R$, then the set $UC(X)$ of all u.s.c. functions can be viewed as a metric subspace of the hyperspace of $X \times R$. Convergence with respect to this topology is in some respects analagous to convergence in measure. For example if $\{f_n\}$ is a sequence of continuous functions convergent to an u.s.c. limit f, then there exists a dense G_δ set G such that for each x in G $f(x)$ is a subsequential limit of $\{f_n(x)\}$. Integral convergence theorems are also presented. However, the main results are as follows: (a) a characterization of this topology on $UC(X)$ in terms of the monotone functionals on $C(X)$ that are u.s.c. with respect to the uniform metric (b) several characterizations of sublattices of $UC(X)$ from which $UC(X)$ is retrievable via pointwise limits of monotone decreasing sequences, e.g., $C(X)$ or the sublattice of u.s.c. step functions.

1. Introduction. The analogies between Baire category and Lebesgue measure, so elegantly described by J. Oxtoby [7], have been from time to time the objects of study of some of the most eminent mathematicians of this century, including Banach, Ulam, Sierpinski, Erdos, and Kuratowski. Category duals of standard results in measure theory abound, although frequently the results may be stronger, weaker, or otherwise modified. Here is a typical example:

MEASURE THEOREM. *A set is in the σ-algebra generated by the Borel sets and the sets of measure zero if and only if it can be represented as a F_σ set plus a set of measure zero.*

CATEGORY DUAL. *A set is in the σ-algebra generated by the Borel sets and the sets of first category if and only if it can be represented as a F_σ set minus a set of first category.*

A notion of importance in measure theory is that of convergence in measure, for convergence almost everywhere can be replaced by this notion in the standard integral convergence theorems: Fatou's Lemma, the Monotone Convergence Theorem, and the Dominated Convergence Theorem. It is one purpose of this article to exhibit a category dual for convergence in measure for upper semicontinuous functions defined on a compact metric space X. We remark that if μ is a regular Borel measure defined on such a space, then convergence in measure can be described as convergence with respect to a certain metric ρ (identifying functions equal almost everywhere) [3]:

$$\rho(f, g) = \int_X \frac{|f - g|}{1 + |f - g|} d\mu .$$

Our category dual for the topology of convergence in measure also admits a metric description: the upper semicontinuous functions are viewed as a subspace of the hyperspace of $X \times R$.

Although the relation of this topology to measure and integration is of interest, the primary purpose of this paper is to show that this topology for the upper semicontinuous functions is a natural one, both in terms of approximation theory and with respect to the extension of monotone functionals defined on the continuous functions on the underlying space (e.g., Radon measures).

2. Preliminaries. Let X be a compact metric space with metric d. One of a number of ways to make $X \times R$ a metric space in a manner compatible with the product uniformity is to define the distance between (x_1, α_1) and (x_2, α_2) to be $\max\{d(x_1, x_2), |\alpha_2 - \alpha_1|\}$. Since no confusion results we will symbolize this distance in $X \times R$ by d, too. We can now make the closed subsets of $X \times R$ a uniform space, called the *hyperspace* of $X \times R$ [5], by defining the *Hausdorff distance* D between closed sets C and K to be

$$D(C, K) = \inf\{\lambda > 0 : B_\lambda[C] \supseteq K \text{ and } B_\lambda[K] \supseteq C\}$$

where $B_\lambda[C]$ (resp. $B_\lambda[K]$) denotes the union of closed λ-balls whose centers run over C. We note that each metric uniformly equivalent to d induces the same hyperspace topology; a metric that is merely topologically equivalent might induce a different one. We also note that distance so defined might not yield a finite number, whence the hyperspace is not in general a metric space.

It is well known that $f: X \to R$ is upper semicontinuous (u.s.c.) on X if and only if its *hypograph*, the set $\text{hypo} f = \{(x, \alpha): x \in X \text{ and } \alpha \leqq f(x)\}$ is a closed subset of $X \times R$. Thus, if we identify an u.s.c. function with its hypograph, then the set of all u.s.c. functions $UC(X)$ on X can be viewed as a subspace of hyperspace of $X \times R$. If f and g are in $UC(X)$, let us write $D(f, g)$ for $D(\text{hypo} f, \text{hypo} g)$. We list some basic facts about $UC(X)$ equipped with the metric D, all of which are established in [1].

THEOREM A. *If f and g are in $UC(X)$ then $D(f, g) < \infty$ so* *that D is a bona fide metric on $UC(X)$. Moreover, $D(f, g) \leqq \sup_{x \in X} |f(x) - g(x)|$.*

THEOREM B. *For each f in $UC(X)$ and for each $\lambda > 0$ define $f_\lambda^+ : X \to R$ by*

$$f_\lambda^+(x) = \sup \{\alpha: (x, \alpha) \in B_\lambda[\text{hypo } f]\} .$$

Then f_λ^+ is both bounded and u.s.c., *and $D(f_\lambda^+, f) = \lambda$. Moreover, $D(f, g) \leqq \lambda$ if and only if $f \leqq g_\lambda^+$ and $g \leqq f_\lambda^+$.*

THEOREM C. *Let $\{f_n\}$ be a sequence in $UC(X)$. Then $\{f_n\}$ is D-convergent to f in $UC(X)$ if and only if at each point x in X*

(i) *whenever $\{x_n\} \to x$ then $\limsup_{n\to\infty} f_n(x_n) \leqq f(x)$.*

(ii) *there exists a sequence $\{x_n\}$ convergent to x for which $\liminf_{n\to\infty} f_n(x_n) = f(x)$.*

3. Category analogues for convergence in measure theorems. A basic result of F. Riesz that we must seek to dualize asserts that if a sequence of functions converges in measure, then the sequence has a subsequence that converges almost everywhere. A category dual of equal strength would state that if a sequence of u.s.c. functions D-converges, then a subsequence converges on a dense G_δ set pointwise. We have obtained a somewhat weaker result.

THEOREM 1. *Let X be a compact metric space. Let $\{f_n\}$ be a sequence of continuous functions convergent in the metric D to an* u.s.c. *function f. Then there exists a dense G_δ set G such that for each x in G the number $f(x)$ is a subsequential limit of $\{f_n(x)\}$.*

Proof. Let $r > 0$ be arbitrary. For each $k \in Z^+$ we claim that the closed set

$$A_{k,r} = \{x: f(x) - r \geqq f_n(x) \text{ for each } n \geqq k\}$$

is nowhere dense. To see this suppose instead that $x \in \text{int}(A_{k,r})$. By Theorem C there exists $\{x_n\}$ convergent to x for which $f(x) = \liminf_{n\to\infty} f_n(x_n)$. Thus there exists $N \in Z^+$ such that for all $n > N$ both $f_n(x_n) > f(x) - r/2$ and $f(x_n) < f(x) + r/2$. However, this contradicts $\{x_n\}$ in $A_{k,r}$ eventually because for all $n > N$

$$f_n(x_n) > f(x) - \frac{r}{2} > \left[f(x_n) - \frac{r}{2}\right] - \frac{r}{2}$$
$$= f(x_n) - r .$$

Next for each two positive integers k and n let $E_{k,n} = A_{k,1/n}$.

Since X is a complete metric space

$$G = \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} (E_{k,n})^c$$

is a dense G_δ set. If x is in G, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that for each k $f_{n_k}(x) > f(x) - 1/k$. Moreover, by Theorem C again we have $\limsup_{k\to\infty} f_{n_k}(x) \leqq f(x)$ for each x in X. In particular this inequality holds in G and the theorem is proved.

No such G_δ set need exist if the terms of the sequence are merely u.s.c. functions. To see this on $X = [0, 1]$ for each $n \in Z^+$ let $E_n = \{k2^{-n}: k = 0, 1, \cdots, 2^n\}$ and define f_n to be χ_{E_n}. Then $\{f_n\}$ D-converges to $f = \chi_{[0,1]}$, but $f(x)$ is a subsequential limit of $\{f_n(x)\}$ if and only if x is a dyadic rational. By the Baire Category Theorem the set of dyadic rationals is not a dense G_δ set.

Can we find a single subsequence in Theorem 1 that serves for *all* points except for a set of first category? The answer is negative. It suffices to construct a sequence of continuous functions $\{f_n\}$ on $X = [0, 1]$ for which both $\{f_n\}$ and $\{1 - f_n\}$ are D-convergent to $\chi_{[0,1]}$. For the moment assume their existence and denote $1 - f_n$ by g_n. Now suppose that in Theorem 1 the subsequence can be always chosen the same for each point of some dense G_δ set. By twice passing to a subsequence we can guarantee that $\{f_n\}$ and $\{g_n\}$ both converge pointwise to $\chi_{[0,1]}$ on dense G_δ sets G_1 and G_2, respectively. Since $[0, 1]$ is a complete metric space, $G_1 \cap G_2$ is also a dense G_δ set, and if $x \in G_1 \cap G_2$ then

$$\lim_{n\to\infty} f_n(x) + g_n(x) = 2 .$$

This contradicts $f_n(x) + g_n(x) = 1$ for each n. We next produce such a sequence $\{f_n\}$. For each $n \in Z^+$ again let $E_n = \{k2^{-n}: k = 0, 1, \cdots, 2^n\}$ and let V_n denote the 4^{-n} neighborhood of E_n in X. Define $f_n: [0, 1] \to [0, 1]$ by $f_n(x) = 4^n d(x, V_n^c)$. Notice that each f_n is piecewise affine, equal to one on E_n, and equal to zero on V_n^c. Clearly, both f_n and $g_n = 1 - f_n$ are continuous. For each $x \in [0, 1]$ and $n \in Z^+$ we can choose x_n in E_n such that $|x - x_n| < 2^{-n}$. Now $f_n(x_n) = 1$ and $g_n(x_n + 4^{-n}) = 1$. Since both $\{x_n\}$ and $\{x_n + 4^{-n}\}$ are convergent to x it follows from Theorem C that both $\{f_n\}$ and $\{g_n\}$ are D-convergent to $\chi_{[0,1]}$.

We next point out that a sequence in $UC(X)$ can D-converge yet converge nowhere pointwise in a manner totally analogous to the usual construction of a sequence convergent in measure yet nowhere pointwise. Consider the following sequence of subintervals of $[0, 1]$:

$$\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right], \left[0, \frac{1}{8}\right], \left[\frac{1}{8}, \frac{1}{4}\right], \cdots .$$

For each such subinterval take the negative of its characteristic function and approximate it by a piecewise affine continuous function that agrees with the step function except on the subintervals adjacent to the one inducing the function. For example, the function corresponding to the subinterval [3/8, 1/2] would have a graph consisting of line segments connecting the following points in succession: (0, 0), (0, 1/4), (3/8, −1), (1/2, −1), (5/8, 0), and (1, 0). The sequence of functions so constructed converges to the zero function on [0, 1] in the metric D but converges nowhere pointwise.

We now produce an analogue of Fatou's Lemma.

THEOREM 2. *Let X be a compact metric space and let μ be a regular Borel measure on X. Then the map $I: UC(X) \to [-\infty, \infty)$ defined by $I(f) = \int_X f d\mu$ is upper semicontinuous with respect to the D metric.*

Proof. Let $\{f_n\}$ be a sequence in $UC(X)$ D-convergent to an u.s.c. function f. For each $k \in Z^+$ let $h_k = f^+_{1/k}$. Each h_k is u.s.c., and since X is metric, each h_k is the pointwise limit of decreasing sequence of continuous functions. Since $\{h_k\}$ is a decreasing sequence of such limits that is pointwise convergent to f, Proposition 4.24 of [9] concerning the first Daniell extension of a Radon measure asserts that $\lim_{k\to\infty} I(h_k) = I(f)$. Since $\mu(X) < \infty$ and f is bounded above, $I(f) < \infty$. If $I(f)$ is finite let $\varepsilon > 0$ be fixed. Choose k satisfying $I(h_k) < I(f) + \varepsilon$. Since $\{f_n\}$ D-converges to f there exists $N \in Z^+$ such that whenever $n > N$ we have $D(f_n, f) \leqq 1/k$. By Theorem B $f_n \leqq h_k$ so that

$$I(f_n) \leqq I(h_k) < I(f) + \varepsilon .$$

Upper semicontinuity of I at f if $I(f) = -\infty$ is established in the same way.

As a consequence of this theorem we see that if $\{f_n\} \to f$ in measure and "in category", then $\lim_{n\to\infty} I(f_n) = I(f)$ (provided that $I(\inf f_n) > -\infty$ so that Fatou's lemma holds). We of course also get a monotone convergence theorem.

COROLLARY. *Let X be a compact metric space and let μ be a regular Borel measure on X. Suppose $\{f_n\}$ is a sequence in $UC(X)$ D-convergent to f and for each n $f_n \geqq f$. Then* $\lim_{n\to\infty} I(f_n) = I(f)$.

If we consider the sequence $\{\chi_{E_n}\}$ mentioned after Theorem 1 that is D-convergent to $\chi_{[0,1]}$ and μ denotes ordinary Lebesgue measure on [0, 1], then for each n

$$\int_0^1 \chi_{E_n} d\mu = 0$$

whereas

$$\int_0^1 \chi_{[0,1]} d\mu = 1 .$$

This shows that there is no hope for a dominated convergence theorem with respect to "convergence in category". Actually, we get something that is in some sense just as nice: a characterization of convergent sequences $\{f_n\}$ of summable u.s.c. functions for which the integral of the limit is the limit of the integrals.

THEOREM 3. *Let X be compact metric space and let μ be a regular Borel measure on X. Let I denote the integration functional induced by μ. Suppose $\{f_n\}$ is a sequence of summable* u.s.c. *functions D-convergent to a summable* u.s.c. *function f. For each $k \in Z^+$ define $I_k: \{f_n: n \in Z^+\} \to R$ by $I_k(f_n) = I[(f_n)^+_{1/k}]$. Then $\lim_{n\to\infty} I(f_n) = I(f)$ if and only if $\{I_k\}$ converges uniformly to I on $\{f_n: n \in Z^+\}$.*

Proof. Let $\Omega = \{f\} \cup \{f_n: n \in Z^+\}$. Since $\{f_n\}$ D-converges to f, Ω is D-compact. If $\lim_{n\to\infty} I(f_n) = I(f)$ then I is continuous on Ω because f is its only limit point. Set $I_k(f)$ equal to $I(f^+_{1/k})$. By Theorem 2 I is u.s.c. and since it is monotone, for each $g \in \Omega$ we have $I_1(g) \geqq I_2(g) \geqq \cdots \geqq I(g)$ and $\lim_{k\to\infty} I_k(g) = I(g)$. By Dini's theorem, $\{I_k\}$ converges uniformly to I on Ω and therefore uniformly on the subset $\{f_n: n \in Z^+\}$.

Conversely suppose $\lim_{n\to\infty} I(f_n) \neq I(f)$. Since I is u.s.c. at f, by passing to a subsequence we can assume that for some $\varepsilon > 0$ and for all n both $D(f_n, f) \leqq 1/n$ and $I(f_n) < I(f) - \varepsilon$. Since $(f_n)^+_{1/n} \geqq f$ it follows that

$$I_n(f_n) \geqq I(f) > I(f_n) + \varepsilon .$$

Hence $\{I_k\}$ can't converge uniformly to I on $\{f_n: n \in Z^+\}$.

4. On the extension of monotone functionals. If μ is a regular Borel measure on a compact metric space, then the integration functional I induced by μ when restricted to $UC(X)$ is a monotone functional u.s.c. with respect to the D-metric. This functional is the extension of one on $C(X)$, the continuous functions on X, that is continuous with respect to the uniform metric on $C(X)$. In this section we shall show that any monotone functional on $C(X)$ continuous with respect to the uniform metric admits a unique D-u.s.c.

monotone extension to $UC(X)$. Actually we shall obtain much more inclusive results.

The main theorem of [1] is a characterization of sublattices Ω of $UC(X)$ that are *upper dense* with respect to the D-metric, i.e., sublattices Ω of $UC(X)$ for which each f in $UC(X)$ is in the closure of $\{g: g \in \Omega \text{ and } g \geqq f\}$. Thus if Ω is upper dense then for each f in $UC(X)$ there exists a sequence $\{g_n\}$ in Ω such that $g_1 \geqq g_2 \geqq \cdots \geqq f$ and $\{g_n\}$ D-converges to f. Precisely a sublattice Ω of $UC(X)$ is D-upper dense if and only if whenever (x_1, α_1) and (x_2, α_2) are points in $X \times R$ such that either $x_1 \neq x_2$ or $x_1 = x_2$ and $\alpha_1 < \alpha_2$, then there exists f in Ω such that

$$(x_1, \alpha_1) \in \operatorname{int}(\operatorname{hypo} f) \quad \text{and} \quad (x_2, \alpha_2) \notin \operatorname{hypo} f \ .$$

In the terminology of [1] Ω is said to *isolate points* in $X \times R$. The sublattice $C(X)$ certainly meets this criterion as does the sublattice of $UC(X)$ consisting of those u.s.c. functions with finite range.

Our first result concerns the extension of certain functionals whose domoins are filters in upper dense sublattices. A family of u.s.c. functions θ on X is called a *filter* if $\{\operatorname{hypo} f: f \in \theta\}$ is a filter of sets. Thus θ is a filter if (i) whenever $f \in \theta$ then $f \leqq g$ and $g \in UC(X)$ imply $g \in \theta$ (ii) $f \in \theta$ and $g \in \theta$ imply $f \wedge g \in \theta$.

THEOREM 4. *Let X be a compact metric space and let Ω be a sublattice of $UC(X)$ that is upper dense. Let θ be a filter in $UC(X)$ and let $\phi: \Omega \cap \theta \to [-\infty, \infty]$ be monotone and* u.s.c. *with respect to the D-metric. Then there exists a unique monotone D-*u.s.c. *extension ϕ^* of ϕ to θ.*

Proof. Let $f \in \theta$ be arbitrary. Since Ω is upper dense there exists a sequence $\{g_n\}$ in Ω such that $g_1 \geqq g_2 \geqq \cdots \geqq f$ and $\{g_n\}$ D-converges to f. Since θ is a filter each function g_n is in $\theta \cap \Omega$ whence $\phi(g_n)$ is defined. If ϕ^* were an extension of ϕ the monotonicity of ϕ^* would require that $\phi^*(f) \leqq \inf\{\phi(g): g \in \theta \cap \Omega \text{ and } g \geqq f\}$. However, the upper semicontinuity of ϕ^* forces

$$\phi^*(f) \geqq \lim_{n\to\infty} \phi(g_n) \geqq \inf\{\phi(g): g \in \theta \cap \Omega \text{ and } g \geqq f\} \ .$$

Thus the extension if it exists must be defined by

$$\phi^*(f) = \inf\{\phi(g): g \in \theta \cap \Omega \text{ and } g \geqq f\} \ .$$

We note that this infimum is approached on any sequence in $\theta \cap \Omega$ D-convergent to f from above because D makes $UC(X)$ a topological join semilattice.

Clearly ϕ^* so defined is monotone and is an extension of ϕ. To show ϕ^* is u.s.c. at each f in θ there are three cases to consider: (i) $\phi^*(f) = \infty$ (ii) $\phi^*(f)$ is finite (iii) $\phi^*(f) = -\infty$. Case (i) is trivial. In case (ii) for each $n \in Z^+$ Theorem B allows us to choose g_n in $\Omega \cap \theta$ for which $g_n \geqq f^+_{2^{-n-1}}$ and $D(g_n, f^+_{2^{-n-1}}) \leqq 2^{-n-1}$. Since $D(g_n, f) \leqq 2^{-n}$ $\{g_n\}$ D-converges to f from above whence $\lim_{n\to\infty} \phi(g_n) = \phi^*(f)$. Pick n so large that $\phi(g_n) < \phi^*(f) + \varepsilon$. If $D(h, f) < 2^{-n-1}$ we have

$$\phi^*(h) \leqq \phi^*(f^+_{2^{-n-1}}) \leqq \phi(g_n) < \phi^*(f) + \varepsilon .$$

This establishes the upper semicontinuity of ϕ^* at f if $\phi^*(f)$ is finite. Case (iii) is similar and is left to the reader.

We remark that in applications the filter θ in Theorem 4 is likely to be either $UC(X)$ itself or a principal filter, in particular $\{f: f \geqq 0\}$. The next result might seem surprising in that by Theorem A the topology of uniform convergence on $C(X)$ is stronger than the one induced by the D-metric; as we have seen $\{f_n\}$ can D-converge to f yet converge nowhere pointwise to f.

THEOREM 5. *Let $\phi: C(X) \to [-\infty, \infty]$ be monotone and* u.s.c. *with respect to the topology of uniform convergence. Then ϕ is* u.s.c. *with respect to the D-metric on $C(X)$.*

Proof. Let f be in $C(X)$. Since $C(X)$ is upper dense in $UC(X)$ we can as in the proof of the last theorem choose for each $n \in Z^+$ a continuous g_n for which $g_n \geqq f^+_{2^{-n-1}}$ and $D(g_n, f^+_{2^{-n-1}}) \leqq 2^{-n-1}$. By Theorem C $\{g_n\}$ converges pointwise to f and since $g_1 \geqq g_2 \geqq \cdots \geqq f$, Dini's theorem ensures that the convergence is uniform. Thus $\lim_{n\to\infty} \phi(g_n) = \phi(f)$. To verify upper semicontinuity at f we again dispense with the three cases (i) $\phi(f) = \infty$ (ii) $\phi(f)$ is finite (iii) $\phi(f) = -\infty$ exactly as in the proof of the last theorem.

We remark that if $\phi: C(X) \to [-\infty, \infty]$ is monotone and continuous with respect to the uniform metric, then ϕ need not be continuous with respect to the D-metric. A simple example: let $\phi(f) = \inf \{f(x): x \in X\}$.

Since the topology of uniform convergence on $C(X)$ is stronger than the one induced by the D-metric, the last two theorems combined yield the following useful result.

THEOREM 6. *Let $\phi: C(X) \to [-\infty, \infty]$ be monotone and* u.s.c. *with respect to the topology of uniform convergence. Then ϕ admits a unique monotone extension to $UC(X)$ that is* u.s.c. *with respect to the D-metric. Conversely, if $\phi: UC(X) \to [-\infty, \infty]$ is monotone and* u.s.c. *with respect to the D-metric, then $\phi|C(X)$ is* u.s.c. *with respect to* *the topology of uniform convergence.*

One might think that the uniqueness of the extensions as described in Theorem 6 is a triviality and has nothing to do with the particular topology we have placed on $UC(X)$. Surely if $f \in UC(X)$ we must define $\phi(f)$ to be $\inf\{\phi(g): g \in C(X) \text{ and } g \geqq f\}$! Such reasoning is fallacious. For instance if we give $UC(X)$ the topology of uniform convergence, then a monotone u.s.c. functional ϕ on $C(X)$ can have more than one extension even if ϕ is continuous and finite valued. For example define $\phi: C(X) \to R$ by $\phi(f) = (f \vee 0)(x_0)$ where x_0 is a nonisolated point of X. Then ϕ can be extended in two different ways to $UC(X)$ equipped with the topology of uniform convergence so that it remains monotone and u.s.c.:

$$\phi_1(f) = (f \vee 0)(x_0)$$
$$\phi_2(f) = \liminf_{x \to x_0} (f \vee 0)(x) .$$

These extensions are different because $\phi_1(\chi_{\{x_0\}}) = 1$ whereas $\phi_2(\chi_{\{x_0\}}) = 0$. Only ϕ_1 is upper semicontinuous with respect to the D-metric. To see that ϕ_2 is not u.s.c. let E_n be the closed ball in X of radius $1/n$ with center x_0. Then $\{\chi_{E_n}\}$ D-converges to $\chi_{\{x_0\}}$ but for each n $\phi_2(\chi_{E_n}) = 1$.

In what sense is the D-metric topology on $UC(X)$ "determined" by the monotone functionals on $C(X)$ that are u.s.c. with respect to the uniform metric? To answer this question let Δ denote the collection of all monotone functionals $\phi: UC(X) \to [-\infty, \infty]$ that are u.s.c. with respect to the D-metric. By Theorem 6 Δ is totally determined by the monotone functionals ϕ on $C(X)$ that are u.s.c. with respect to the uniform metric. More concretely if $\phi \in \Delta$ then for each $f \in UC(X)$

$$\phi(f) = \inf\{\phi(h): h \in C(X) \text{ and } h \geqq f\}$$

where $\phi | C(X)$ is u.s.c. with respect to the uniform metric. Consider the weakest topology on $UC(X)$ with respect to which each member of Δ is u.s.c. Although it is weaker than the topology induced by the D-metric, it is closely related to it.

THEOREM 7. *Let X be a compact metric space. For each $f \in UC(X)$ let $\theta[f; \lambda] = \{g: g \leqq f_\lambda^+\}$. Then $\{\theta[f; \lambda]: \lambda > 0\}$ forms a local base at f for a topology σ on $UC(X)$ that is weakest with respect to which each member of Δ is* u.s.c.

Proof. Using a standard criterion [6, II.2.E] it is easy to show that the sets so described do form a local base at f for a topology

σ on $UC(X)$. We first show that $\tau \subseteqq \sigma$ where τ denotes the weakest topology with respect to which member of Δ is u.s.c.

Let $\phi \in \Delta$ and let $f \in UC(X)$. If $\phi(f) > -\infty$ and $\varepsilon > 0$ there exists $\lambda > 0$ such that if $D(f, g) \leqq \lambda$ then $\phi(g) < \phi(f) + \varepsilon$. If $g \in \theta[f; \lambda]$ then $D(f \vee g, f) \leqq \lambda$ and we have

$$\phi(g) \leqq \phi(f \vee g) < \phi(f) + \varepsilon .$$

Thus ϕ is σ-u.s.c. at f when $\phi(f) > -\infty$. A similar argument applies if $\phi(f) = -\infty$. This proves that $\tau \subseteqq \sigma$.

To show that $\sigma \subseteqq \tau$ it suffices to show that if $f \in UC(X)$ and $\lambda > 0$ there exists $\phi \in \Delta$ and $\alpha > 0$ such that $f \in \phi^{-1}([-\infty, \alpha)) \subseteqq \theta[f; \lambda]$. Choose $g \in C(X)$ such that $g \geqq f$ and $D(f, g) \leqq \lambda/2$. Since g is continuous, for each h in $UC(X)$ the function $h \vee g - g$ is u.s.c. and thus attains a maximum value on the compact set X. Define ϕ: $UC(X) \rightarrow R$ by

$$\phi(h) = \max \{(h \vee g)(x) - g(x): x \in X\} .$$

We claim that the functional ϕ is u.s.c. with respect to the D-metric. First note that the operator $h \rightarrow h \vee g$ is continuous because

$$D(h_1 \vee g, h_2 \vee g) \leqq D(h_1, h_2)$$

whenever h_1 and h_2 are in $UC(X)$. It remains to show that σ: $UC(X) \rightarrow R$ defined by $\sigma(h) = \max_{x \in X} h(x) - g(x)$ is D-u.s.c. To see this let $\{h_n\}$ be a sequence of u.s.c. functions D-convergent to h, and choose for each n in N an x_n in X such that $h_n(x_n) - g(x_n)$ is maximal. Set $\alpha = \limsup_{n \rightarrow \infty} h_n(x_n) - g(x_n)$. By passing twice to a subsequence we can assume both $\{x_n\}$ converges to some point x in X and $\{h_n(x_n) - g(x_n)\}$ converges to α. By Theorem C $\limsup_{n \rightarrow \infty} h_n(x_n) \leqq h(x)$; so, by the continuity of g at x

$$\begin{aligned} \alpha &= \limsup_{n \rightarrow \infty} \sigma(h_n) = \limsup_{n \rightarrow \infty} h_n(x_n) - \lim_{n \rightarrow \infty} g(x_n) \\ &\leqq h(x) - g(x) = \sigma(h) . \end{aligned}$$

Since ϕ is also monotone we conclude that $\phi \in \Delta$.

Now if $h \in \phi^{-1}([-\infty, \lambda/2))$ then $\max_{x \in X} (h \vee g)(x) - g(x) < \lambda/2$. Thus the uniform distance of $h \vee g$ from g is less than $\lambda/2$, and by Theorem A $D(h \vee g, g) < \lambda/2$. Hence if $h \in \phi^{-1}([-\infty, \lambda/2)) = \phi^{-1}([0, \lambda/2))$, then

$$h \leqq h \vee g \leqq g^+_{\lambda/2} \leqq f^+_\lambda .$$

This proves that $h \in \theta[f; \lambda]$ and $\sigma \subseteqq \tau$ is established.

Let Y be a nonempty set. A nonempty collection $\mathscr{D}$ of subsets of $Y \times Y$ is called a *quasi-uniformity* on Y [4] if $\mathscr{D}$ is a filter of sets such that for each $U \in \mathscr{D}$

(i) $\{(y, y): y \in Y\} \subseteqq U$.

(ii) there exists $V \in \mathscr{D}$ such that $V \circ V \subseteqq U$.

As with uniformities the sets $\{U[y]: U \in \mathscr{D}\}$ for each y in Y form a local base at y for a topology on Y. Unlike uniformities, each topological space is "quasi-uniformizable", not just the completely regular ones ([2], [8]). If $\mathscr{D}$ is a quasi-uniformity on Y then it is clear that

$$\{U \cap V^{-1}: U \in \mathscr{D} \quad \text{and} \quad V \in \mathscr{D}\}$$

describes the smallest uniformity on Y containing $\mathscr{D}$. Moreover, if $\mathscr{D}'$ and $\mathscr{D}$ are equivalent quasi-uniformities (in the sense that they determine the same topology for Y), then the uniformities that they generate are equivalent. Thus it makes sense to speak of the completely regular topology generated by a given topology.

These concepts allow us to see precisely how the monotone functionals on $C(X)$ u.s.c. with respect to the topology of uniform convergence determine the D-metric topology on $UC(X)$. There is a weakest topology σ on $UC(X)$ with respect to which passage via monotone limits from functionals on $C(X)$ monotone and u.s.c. with respect to the topology of uniform convergence results in u.s.c. functionals on $UC(X)$. The D-metric topology is simply the completely regular topology generated by σ, for σ is determined by a quasi-uniformity on $UC(X)$ with base $\{U_\lambda: \lambda > 0\}$ where for each λ $U_\lambda = \{(f, g): g \leqq f_\lambda^+\}$.

We close with a list of characterizations of D-upper dense sublattices of $UC(X)$ which are, once again, sublattices Ω of $UC(X)$ for which each f in $UC(X)$ is in the closure of $\{g: g \in \Omega$ and $g \geqq f\}$.

THEOREM 8. *Let Ω be a sublattice of $UC(X)$. The following are equivalent*:

1. *Ω is D-upper dense.*
2. *Whenever $\phi: \Omega \to [-\infty, \infty]$ is monotone and D-*u.s.c., *there exists a unique monotone D-*u.s.c. *extension ϕ^* of ϕ to $UC(X)$.*
3. *Whenever $\phi: \Omega \to R$ is monotone and D-*u.s.c., *there exists a unique monotone extended valued D-*u.s.c. *extension ϕ^* of ϕ to $UC(X)$.*
4. *For each $f \in UC(X)$ there exists a sequence $\{g_n\}$ in Ω convergent pointwise to f for which $g_1 \geqq g_2 \geqq \cdots \geqq f$.*
5. *Ω isolates points in $X \times R$.*

Proof.

$(1 \to 2)$ This is a special case of Theorem 4.

$(2 \to 3)$ Trivial.

$(3 \to 1)$ Suppose Ω fails to be D-upper dense. Since $C(X)$ is upper dense in $UC(X)$ there must be $f \in C(X)$ that cannot be D-approximated from above by members of Ω. The proof of Theorem 7 showed that the functional $\sigma: UC(X) \to R$ defined by $\sigma(h) = \max\{h(x) - f(x): x \in X\}$ is D-u.s.c. Since Theorem C ensures that $\Sigma = \{h: h \in UC(X)$ and $h \geqq f\}$ is a closed set, the functional $\phi_1: UC(X) \to R$ defined by

$$\phi_1(h) = \begin{cases} \sigma(h) & \text{if } h \in \Sigma \\ 0 & \text{otherwise} \end{cases}$$

is D-u.s.c. and monotone. Let ϕ denote the restriction of ϕ_1 to Ω. We claim that ϕ has an additional extension to $UC(X)$. We distinguish two cases: (1) no function in Ω majorizes f (2) there exists g in Ω such that $g \geqq f$. In the first case ϕ extends to both ϕ_1 and the zero functional; these are distinct because $\phi_1(f + 1) = 1$. In the second case there must exist $\lambda > 0$ for which no g in Ω satisfies $f \leqq g \leqq f_\lambda^+$. Define $\Gamma \subseteqq UC(X)$ as follows:

$$\Gamma = \{h: \text{there exists } x \in X \text{ such that } h(x) \geqq f(x) + \lambda\} .$$

Notice that $\Omega \cap \Sigma \subseteqq \Gamma$ because for each x in X $f(x) + \lambda \leqq f_\lambda^+(x)$. We claim that Γ is also a closed set. As expected Theorem C will again be invoked. Let $\{h_n\} \subseteqq \Gamma$ be D-convergent to h. Choose for each n a point x_n in X such that $h_n(x_n) \geqq f(x_n) + \lambda$. By passing to a subsequence we can assume that $\{x_n\}$ converges to some x in X. From the continuity of f at x and Theorem C

$$h(x) \geqq \limsup_{n\to\infty} h_n(x_n) \geqq \limsup_{n\to\infty} f(x_n) + \lambda = f(x) + \lambda .$$

In addition to the extension ϕ_1 of ϕ, since $\Omega \cap \Sigma \subseteqq \Gamma$ and Γ is closed we get an additional extension ϕ_2:

$$\phi_2(h) = \begin{cases} \sigma(h) & \text{if } h \in \Sigma \cap \Gamma \\ 0 & \text{otherwise} . \end{cases}$$

The extensions are distinct because $\phi_2(f + \lambda/2) = 0$ whereas $\phi_1(f + \lambda/2) = \lambda/2$.

($1 \to 4$) Since Ω is D-upper dense and is a lattice there is a decreasing sequence $\{g_n\}$ in Ω that D-converges to f from above. By Theorem C the convergence must also be pointwise.

($4 \to 5$) First suppose (x_1, α_1) and (x_2, α_2) in $X \times R$ satisfy $x_1 = x_2$ and $\alpha_1 < \alpha_2$. The constant function $f = (\alpha_1 + \alpha_2)/2$ is in $UC(X)$. Choose $\{g_n\} \subseteqq \Omega$ convergent pointwise to f for which $g_1 \geqq g_2 \geqq \cdots \geqq f$. There exists a subscript n for which $g_n(x_1) < \alpha_2$. We have

$$(x_1, \alpha_1) \in \text{int}(\text{hypo } f) \subseteqq \text{int}(\text{hypo } g_n)$$

and

$$(x_2, \alpha_2) = (x_1, \alpha_2) \notin \text{hypo}\, g_n .$$

Thus, g_n isolates (x_1, α_1) from (x_2, α_2). Next suppose $x_1 \neq x_2$. Let B be a closed ball with center x_1 that does not contain x_2. Define $f: X \to R$ by

$$f(x) = \begin{cases} |\alpha_1| + |\alpha_2| + 1 & \text{if} \quad x \in B \\ \alpha_2 - 1 & \text{if} \quad x \notin B . \end{cases}$$

Since B is a closed set, f is u.s.c. As in the first case we can approximate f from above by a member of Ω closely at x_1 and at x_2 thus isolating (x_1, α_1) from (x_2, α_2).

$(5 \to 1)$ This is the main result of [1] alluded to earlier.

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