ON THE SIMILARITY TRANSFORMATION BETWEEN A MATRIX AND ITS TRANSPOSE

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It was observed by one of the authors that a matrix transforming a companion matrix into its transpose is symmetric. The following two questions arise:

I. Does there exist for every square matrix with coefficients in a field a non-singular symmetric matrix transforming it into its transpose?

II. Under which conditions is every matrix transforming a square matrix into its transpose symmetric?

The answer is provided by

THEOREM 1. *For every $n \times n$ matrix $A = (\alpha_{ik})$ with coefficients in a field F there is a non-singular symmetric matrix transforming A into its transpose A^T.*

THEOREM 2. *Every non-singular matrix transforming A into its transpose is symmetric if and only if the minimal polynomial of A is equal to its characteristic polynomial* i.e. *if A is similar to a companion matrix.*

Proof. Let $T = (t_{ik})$ be a solution matrix of the system $\sum(A)$ of the linear homogeneous equations.

$$TA - A^T T = 0 \tag{1}$$

$$T - T^T = 0\,. \tag{2}$$

The system $\sum(A)$ is equivalent to the system

$$TA - A^T T^T = 0 \tag{3}$$

$$T - T^T = 0 \tag{4}$$

which states that T and TA are symmetric. This system involves $n^2 - n$ equations and hence is of rank $n^2 - n$ at most. Thus there are at least n linearly independent solutions of $\sum(A)$.[1]

On the other hand it is well known that there is a non-singular matrix T_0 satisfying

$$T_0 A T_0^{-1} = A^T\,,$$

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[1] This part of the proof was provided by the referee. Our own argument was more lengthy.

From (1) we derive

(1a)
$$T_0^{-1}TA = AT_0^{-1}T$$

and conversely, (1a) implies (1) so that there is the linear isomorphism

$$T \to T_0^{-1}T$$

of the solution space of (1) onto the centralizer ring of the matrix A.

If the minimal polynomial of A is equal to the characteristic polynomial then the centralizer of A consists only of the polynomials in A with coefficients in F. In this case the solution space of (1) is of dimension n. A fortiori the solution space of $\sum(A)$ is at most of dimension n since the corresponding system involves more equations. Together with the inequality in the other direction it follows that the dimension of the solution space of $\sum(A)$ is exactly n. This implies that every solution matrix of (1) is symmetric.

If the square matrix A is arbitrary then we apply first a similarity (in the field F) which transforms it to the form

$$B = \begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & A_r \end{pmatrix}$$

where A_i is a square matrix of the form

$$\begin{pmatrix} {}_pA & & & & & \\ L & {}_pA & & & & \\ & L & {}_pA & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & \cdot & \\ & & & & L & {}_pA \end{pmatrix}$$

Here ${}_pA$ is the companion matrix of the irreducible polynomial p which is a factor of the characteristic polynomial of A and L is the matrix with 1 in the bottom left corner and 0 elsewhere, of appropriate size (Reference 1, p. 94). The matrix A is derogatory if two blocks A_i corresponding to the same p appear in B. Let A_1 and A_2 be two such blocks.

There is a non-singular matrix Y satisfying

$$Y{}_pA = {}_pA^T Y .$$

The matrix of matrices V that has Y in the top left corner and 0 elsewhere, of appropriate size, satisfies

$$VA_2 = A_1^T V .$$

Consider then the matrix

$$\begin{pmatrix} S_1 & V & & & \\ & S_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & S_r \end{pmatrix}$$

where S_i is a non-singular matrix transforming A_i into A_i^T. It is a non-singular non-symmetric matrix which transform B into its transpose. Thus Theorem 2 is proved.

REMARK. M. Newman pointed out to us that the product of two non-singular skew symmetric matrices B, C can always be transformed into its transpose by a non-symmmtric matrix, namely

$$B^{-1}BCB = (BC)^T = CB .$$

Theorem 2 shows that such a product BC must be derogatory.[2] This can also be shown directly in the following way:

Let λ be a characteristic root of BC and x a corresponding characteristic vector, then

$$BCx = \lambda x .$$

Since B is non-singular this implies

$$Cx = \lambda B^{-1}x$$

or

$$(C - \lambda B^{-1})x = 0 .$$

Since B is a non-singular skew symmetric matrix, it follows that the degree of B and hence the degree of $C - \lambda B^{-1}$ is even. Moreover, the skew symmetric matrix $C - \lambda B^{-1}$ has even rank.

[2] Although Newman's comment is only significant for fields of characteristic $\neq 2$ the remainder of this section holds generally if skew symmetric is understood to mean $T = -T^T$ and vanishing of the diagonal elements. We observe that this definition is invariant under the transformation $T \to X^TTX$. This is the transformation T undergoes when the matrix A in (1), (2) undergoes the similarity transformation $A \to X^{-1}AX$. Since this transformation preserves linear independence, we are permitted to apply it for the purpose of finding a non 'skew symmetric' solution of (1), (2). We now extend the field of reference to include the eigenvalues of A (from the theory of homogeneous linear equations it follows that the maximal number of linear independent solutions will remain the same). It can then be observed that for a block of the Jordan canonical form of a matrix any matrix with all coefficients zero excepting the first diagonal coefficient satisfies (1), (2). Therefore

It follows that another vector y exists such that also

$$(C - \lambda B^{-1})y = 0$$

and hence also

$$BCy = \lambda y \ .$$

This implies that λ is a characteristic root of multiplicity at least two and with at least two corresponding vectors. The product of two general non-singular skew symmetric matrices B, C has every characteristic root of multiplicity exactly 2. For, specialize to the case $B = C$. Then BC is a symmetric matrix whose characteristic roots are the squares of the roots of B, hence all exactly double for a general B. This shows that the general BC has all its characteristic roots double with two independent characteristic vectors. Such a matrix is derogatory and its characteristic polynomial is the square of its minimum polynomial.

References

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for any matrix A we can find solutions of (1), (2) that are non 'skew-symmetric'.

[3] This paper which is related to our investigation was pointed out to us by the referee to whom we are indebted for other useful comments.