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MOD p DECOMPOSITIONS OF H-SPACES; ANOTHER APPROACH

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Let M and M' be unstable modules over the mod p Steenrod algebra such that there are spaces Y and Y' with $H^*(Y;Z_p)=U(M)$ and $H^*(Y';Z_p)=U(M')$. Here $U(\ )$ is the free-associative-graded-commutative-unstable algebra functor introduced by Steenrod. Suppose $g\colon M'\to M$ is a morphism of unstable modules. We develop an obstruction theory which decides when g can be realized by a map G: $Y_{(p)}\to Y'_{(p)}$, that is, $g=H^*(G,Z_p)|_{M'}$. We then apply this obstruction theory to obtain p-equivalences of certain H-spaces with products of spheres and sphere bundles over spheres which are determined by the cohomology structure of the H-space.

The decomposition of H-spaces into products of simpler spaces has been extensively studied by various authors [5, 7, 8, 9, 12, 15, 16, 17]. The problem is to obtain conditions on an arbitrary H-space and a prime p for which $H^*(Y;Z_p)$ completely determines the mod p homotopy type of Y. In [7] Hopf showed that a finite-dimensional H-space is rationally equivalent to a product of odd-dimensional spheres. For a simply-connected Lie group, Serre [15], Kumpel [8] and later Mimura and Toda [14] have provided conditions for which a group is p-equivalent to a product of odd-dimensional spheres and spaces, $B_n(p)$, which are sphere bundles over spheres.

The main thrust of this paper is to describe an obstruction theory, based on techniques of Massey and Peterson [10], which is used to prove

THEOREM A. ([9]). *Let Y be a* mod p *H-space where*
(1) *$H^*(Y;Z_p)$ is primitively generated,*
(2) *$H^*(Y;Z_p)=\Lambda(x_{2n_1+1},\cdots,x_{2n_l+1})$ where $n_1\leqq n_2\leqq\cdots\leqq n_l$, and*
(3) *$p\geqq n_l-n_1+2$,*
then $Y_{(p)}$ is homotopy equivalent to $S^{2n_1+1}_{(p)}\times S^{2n_2+1}_{(p)}\times\cdots\times S^{2n_l+1}_{(p)}$.

THEOREM B. *Let Y be a* mod p *H-space where*
(1) *$H^*(Y;Z_p)$ is primitively generated,*
(2) *$H^*(Y;Z_p)=\Lambda(x_{2n_1+1},\cdots,x_{2n_l+1})$ where $n_1\leqq n_2\leqq\cdots\leqq n_l$, and*
(3) *$2p>n_l-n_1+2$ and $p\geqq 5$,*
then $Y_{(p)}$ is homotopy equivalent to the product $\prod_s B_{m_s}(p)_{(p)}\times\prod_t S^{2m_t+1}_{(p)}$ with the numbers m_s and m_t determined by the action of $\mathscr{P}^1$ on $H)*Y;Z_p)$.

Theorem B includes most cases of theorems proved by Harper [5] and Wilkerson and Zabrodsky [16]. The condition $p \geqq 5$ is technical and can be eliminated by other means. We will concentrate on the obstruction theory which arises as follows.

DEFINITION. Let M be a module over the mod p Steenrod algebra $\mathscr{A}(p)$. We say that M is an *unstable module* if for $p = 2$, $\mathscr{S}q^i x = 0$ when $\dim x < i$ and for p odd, $\mathscr{P}^i x = 0$ when $\dim x < 2i$ and $\beta\mathscr{P}^i x = 0$ when $\dim x \leqq 2i$. An algebra over $\mathscr{A}(p)$ is unstable if it is an unstable module and for $p = 2$, $\mathscr{S}q^i x = x^2$ when $\dim x = i$ and for p odd, $\mathscr{P}^i x = x^p$ when $\dim x = 2i$.

Let $\mathscr{U}\mathscr{M}$ and $\mathscr{U}\mathscr{A}$ denote the categories of unstable modules and unstable algebras with degree-preserving maps. The definitions have been chosen so that $H^*(\;; Z_p)$ is a contravariant functor: $\mathscr{T}\mathscr{O}\mathscr{P} \to \mathscr{U}\mathscr{A}$.

The forgetful functor $\mathscr{F}\colon \mathscr{U}\mathscr{A} \to \mathscr{U}\mathscr{M}$ has an adjoint $U\colon \mathscr{U}\mathscr{M} \to \mathscr{U}\mathscr{A}$ defined by $U(M) = T(M)/D$ where $T(M)$ is the tensor algebra generated by M and D is the ideal generated by elements of the form $x \otimes y - (-1)^{\dim x \dim y} y \otimes x$ and for $p = 2$, $\mathscr{S}q^i x - x \otimes x$ when $\dim x = i$, for p odd $\mathscr{P}^i x - x \otimes x \otimes \cdots \otimes x$ (p times) when $\dim x = 2i$. We will call a space *very nice* (following [2]) if $H^*(Y; Z_p) = U(M_Y)$ for some unstable module M_Y. Examples of such spaces include $K(\pi, n)$'s for π finitely generated, odd-dimensional spheres, most H-spaces and a few projective spaces.

Suppose Y and Y' are very nice spaces and $g\colon M_{Y'} \to M_Y$ is a morphism of unstable modules. We ask whether there is a continuous function $G\colon W \to W'$ such that $H^*(W; Z_p) = H^*(Y; Z_p)$, $H^*(W'; Z_p) = H^*(Y'; Z_p)$ and $G^*|_{M_{Y'}} = g$? If such a function G exists we say that g is *realizable* by G. The obstruction theory provides a series of obstruction sets, $\mathscr{O}_n(g)$, inductively defined and lying in computable groups such that

THEOREM. *There exists a function $G\colon Y_{(p)} \to Y'_{(p)}$ realizing g if and only if $0 \in \mathscr{O}_n(g)$ for all n.*

This result has been obtained independently by John Harper using the unstable Adams spectral sequence where the obstructions are not as explicitly identified.

In the first section we will provide a thumbnail sketch of the Massey-Peterson theory providing details where they will be of later use. The second section is a presentation of the obstruction theory and in the third section we give the proofs of Theorems A and B.

The results in this paper include part of my Temple University

doctoral dissertation written under the direction of Dr. James Stasheff. I am grateful to him for his encouragement and guidance.

1. The Massey-Peterson theory. Let $M \in \mathscr{U}\mathscr{M}$. We define an endomorphism $\lambda: M \to M$ by $\lambda|_{M^n} = \mathscr{S}q^n$ when $p = 2$ and $\lambda|_{M^{2n}} = \mathscr{P}^n$ and $\lambda|_{M^{2n+1}} = \beta\mathscr{P}^n$ when p is odd. Since λ is an endomorphism this induces an action of $Z_p[\lambda]$ on M. We say that M is a *free λ-module* if M has a homogeneous basis over $Z_p[\lambda]$ or equivalently if for all $x \in M$, $\lambda x = 0$ if and only if $x = 0$. The fact that M is a module over the polynomial algebra $Z_p[\lambda]$ implies that submodules of free λ-modules are also free λ-modules.

The important examples of free λ-modules are $MK(Z, n)$ and $MK(Z_r, n)$ where $r = p^k$ for $k \geqq 1$ and $H^*(K(\pi, n); Z_p) = U(MK(\pi, n))$ for $n > 1$.

Using the map λ, we introduce a functor $\Omega: \mathscr{U}\mathscr{M} \to \mathscr{U}\mathscr{M}$ defined by the rule $(\Omega M)_k = (M/\lambda M)_{k+1}$. For $f: M \to N$, a morphism in $\mathscr{U}\mathscr{M}$, f commutes with the action of $\mathscr{A}(p)$ and so $f(\lambda M) \subset \lambda N$. Thus $\Omega f: \Omega M \to \Omega N$ is well-defined. When π is finitely generated, by considering the Cartan basis one can show that $\Omega MK(\pi, n) = MK(\pi, n-1)$. In the topological category, $\Omega K(\pi, n) \cong K(\pi, n-1)$; this motivates the choice of notation.

PROPOSITION 1.1. *If $P \xrightarrow{f} Q \xrightarrow{g} R \to 0$ is exact in $\mathscr{U}\mathscr{M}$, then $\Omega P \xrightarrow{\Omega f} \Omega Q \xrightarrow{\Omega g} \Omega R \to 0$ is also exact. In addition, if f is a monomorphism and R is a free λ-module then Ωf is also a monomorphism.*

The theorem recorded below is due to Massey and Peterson [10] for the case $p = 2$ and to Barcus [1] for p odd.

Let $\xi_0 = (E_0, p_0, B_0, F)$ be a fibration satisfying

(a) The system of local coefficients of the fibration is trivial,

(b) $H^*(F; Z_p) = U(A)$ where $A \subset H^*(F; Z_p)$ consists of transgressive elements.

(c) E_0 is acyclic and the ideal generated by the extended image of A in $H^*(B_0; Z_p)$ under transgression contains all elements of positive dimension.

By the *extended image* of A we mean the set $\{y_i\} \cup \{\nu y_i\}$ in $H^*(B_0; Z_p)$ where $\nu: A \to A$ is defined $\nu|_{A^{2n}} = 0$ and $\nu|_{A^{2n}} = \beta\mathscr{P}^n$ and $\{y_i\}$ projects to a basis for the image of the trangression τ in $H^*(B_0; Z_p)/Q$; Q denotes the indeterminacy of τ.

Let $f: B \to B_0$ be a map and $\xi = (E, p, B, F)$ the induced fibration. Suppose

(d) $H^*(B_0; Z_p) = U(R)$ and R is a free λ-module,

(e) $H^*(B; Z_p) = U(Z)$ and $Z = Z_0 \oplus Z_1$ in $\mathscr{U}\mathscr{M}$ and Z_0 is a free λ-module, and

(f) $f^*: H^*(B_0; Z_p) \to H^*(B; Z_p)$ is such that $f^*(R) \subset Z_0$.

THEOREM 1.2. (*Massey-Peterson-Barcus*). *Given* ξ, ξ_0 *and* $f: B \to B_0$ *satisfying* (a) *through* (f), *let* $Z' = \operatorname{coker} f_R^*: R \to Z$ *and* $R' = \ker f^*|_R$, *then as algebras over* Z_p, $H^*(E; Z_p) = U(Z') \otimes U(\Omega R')$ *and as algebras over* $\mathscr{A}(p)$, $H^*(E; Z_p)$ *is determined by the short exact sequence in* $\mathscr{U}\mathscr{M}$,

$$0 \longrightarrow U(Z') \xrightarrow[p^*]{} N \xrightarrow[i^*]{} \Omega R' \longrightarrow 0$$

called the fundamental sequence for ξ, *where* $i: F \to E$ *is the inclusion and* N *is an* $\mathscr{A}(p)$*-submodule that generates* $H^*(E; Z_p)$.

For a proof we refer the reader to [10] and [1]. The theorem gives a clear picture of the mod p cohomology of certain fiber spaces. This result will allow us to make certain topological constructions that carry useful algebraic information.

It is an easy consequence of a theorem of Cartan [3] that the module $MK(Z_p, n)$ is the free unstable module on one generator of dimension n. We also have that $MK(Z_p, n)$ is projective in $\mathscr{U}\mathscr{M}$ and so we can talk of resolutions of a module in $\mathscr{U}\mathscr{M}$. Suppose Y is a very nice space with $H^*(Y; Z_p) = U(M_Y)$ and $\mathscr{X}(M_Y): 0 \leftarrow M_Y \xleftarrow[\varepsilon]{} X_0 \xleftarrow[d_0]{} X_1 \xleftarrow[d_1]{} \cdots$ is a (not necessarily projective) resolution of M_Y by modules which are direct sums of $MK(\pi, n)$'s for $\pi = Z$, or Z_p. Using Theorem 1.2 we construct a tower of fibrations that carries the algebraic information contained in $\mathscr{X}(M_Y)$.

By a *realization*, $\mathscr{E}(\mathscr{X}(M_Y))$, of $\mathscr{X}(M_Y)$ we will mean a system of principal fibrations:

$$\begin{array}{ccccccccccccc}
 & & F_{s-1} & & & & & & F_3 & & F_2 & & F_1 \\
 & & \uparrow{\scriptstyle f_s} & & & & & & \uparrow{\scriptstyle f_3} & & \uparrow{\scriptstyle f_2} & & \uparrow{\scriptstyle f_1} \\
Y & \xrightarrow[p_s]{} & E_s & \xrightarrow[p_s^{s-1}]{} & E_{s-1} & \xrightarrow[p_{s-1}^{s-2}]{} & \cdots & \xrightarrow[p_3^2]{} & E_2 & \xrightarrow[p_2^1]{} & E_1 & \xrightarrow[p_1^0]{} & E_0 \\
 & & \uparrow{\scriptstyle j_s} & & \uparrow{\scriptstyle j_{s-1}} & & & & \uparrow{\scriptstyle j_2} & & \uparrow{\scriptstyle j_1} & & \\
 & & \Omega F_{s-1} & & \Omega F_{s-2} & & & & \Omega F_2 & & \Omega F_1 & &
\end{array}$$

that satisfies:

(1) E_0 and F_i are products of $K(\pi, n)$'s that is, generalized Eilenberg-MacLane spaces (*gEMs*)$\cdot$

(2) $H^*(E_0; Z_p) = U(X_0)$, $H^*(F_1; Z_p) = U(X_1)$ and $H^*(F_s; Z_p) = U(\Omega^{s-1} X_s)$.

(3) $f_1^* = d_0$, $j_s^* \circ f_{s+1}^*: \Omega^s X_{s+1} \to \Omega^s X_s$ is $\Omega^s d_s$.

(4) The fibration p_s^{s-1} is induced by the path-loop fibration

over f_s.

(5) $p_i\colon Y \to E_i$ is the composition $p^i_{i+1} \circ p^{i+1}_{i+2} \circ \cdots \circ p^{s-1}_s \circ p_s$.

(6) $p_0^*|_{X_0}\colon X_0 \to M_Y$ is ε.

By using Theorem 1.2 in the construction below we also obtain

(7) $H^*(E_s; Z_p) \cong U(M_Y) \otimes U(\Omega^s \ker d_{s-1})$ as algebras over $\mathscr{A}(p)$.

THEOREM 1.3. *Given* Y, M_Y *and* $\mathscr{X}(M_Y)$ *as above, there exists a realization of* $\mathscr{X}(M_Y)$.

Proof. We construct $\mathscr{E}(\mathscr{X}(M_Y)) = \{E_i, p_i^{i-1}, F_j, j_k, p_i; Y\}$ by induction. $Y \underset{p_0}{\to} E_0 \underset{f_1}{\to} F_1$ comes for free because E_0 and F_1 are the appropriate $gEMs$ and maps between spaces and products of $K(Z_p, m)$'s and $K(Z, n)$'s are determined by morphisms in $\mathscr{U}\mathscr{M}$. Construct $\Omega F_1 \underset{j_1}{\to} E_0 \underset{p_1^0}{\to} F_1$ by pulling back the path-loop fibration $\Omega F_1 \to PF_1 \to F_1$. Clearly p_1^0 satisfies (a) through (f) of Theorem 1.2 and so we can conclude that $H^*(E_1; Z_p) = U(\operatorname{coker} f_1^*|_{X_1}) \otimes U(\Omega \ker f_1^*|_{X_1})$. However $f_1^* = d_0$ on X_1 and $\operatorname{coker} d_0 = M_Y$. Hence $H^*(E_1; Z_p) = U(M_Y) \otimes U(\Omega \ker d_0)$ as an algebra over Z_p. Construct $p_1\colon Y \to E_1$ as a lifting of p_0 to the fibration; p_1 exists since $(f_1 \circ p_0)^* = \varepsilon \circ d_0 = 0$. To obtain the $\mathscr{A}(p)$-algebra structure of $H^*(E_1; Z_p)$ we observe that the fundamental sequence for p_1^0 splits by the map p_1^*.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U(M_Y) & \xrightarrow{(p_1^0)^*} & N_1 & \xrightarrow{j_1^*} & \Omega \ker d_0 & \longrightarrow & 0 \\ & & & \diagdown\!\!\diagdown & \big\downarrow{\scriptstyle p_1^*} & & & & \\ & & & & U(M_Y) & & & & \end{array}$$

Thus $H^*(E_1; Z_p) = U(M_Y) \otimes U(\Omega \ker d_0)$ as an algebra over $\mathscr{A}(p)$.

Now $0 \to \ker d_1 \to X_2 \to \ker d_0 \to 0$ is exact from the resolution. Since everything in sight is a free λ-module, by Proposition 1.1, $0 \to \Omega \ker d_1 \to \Omega X_2 \to \Omega \ker d_0 \to 0$ is also exact. Using the splitting of the fundamental sequence and the fact that F_2 is a $gEMs$, we can choose $f_2\colon E_1 \to F_2$ such that $(f_2 \circ j_1)^* = \Omega d_1$.

The inductive step simply repeats this procedure for f_n to obtain E_{n+1} and f_{n+1}.

The role of the space Y in this construction is vital since the splitting of the fundamental sequence depends on the map $p_s\colon Y \to E_s$. This splitting will play a crucial role in the obstruction theory.

Recall that a graded module is *n-connected* if $M_k = 0$ for $k \leqq n$. Let M be in $\mathscr{U}\mathscr{M}$ and $\mathscr{X}(M)\colon 0 \leftarrow M \underset{\varepsilon}{\leftarrow} X_0 \underset{d_0}{\leftarrow} X_0 \underset{d_1}{\leftarrow} X_2 \leftarrow \cdots$ a resolution of M in $\mathscr{U}\mathscr{M}$. We will call $\mathscr{X}(M)$ *convergent* if $\Omega^s X_s$ is $f(s)$-connected for all s and $f(s) \to \infty$ as $s \to \infty$. Using minimal resolutions and allowing modules $MK(Z, n)$ in the construction of resolutions we can guarantee the existence of convergent resolutions for most $M \in \mathscr{U}\mathscr{M}$.

Now suppose Y and M_Y are as above and $\mathscr{X}(M_Y)$ is a convergent resolution of M_Y. Note $\varinjlim_s \Omega^s \ker d_{s-1} \subset \varinjlim_s \Omega^s X_s = 0$. Hence $\varinjlim_s H^*(E_s; Z_p) = \varinjlim_s [U(M_Y) \otimes U(\Omega^s \ker d_{s-1})] = U(M_Y)$. If we let $p_\infty = \varprojlim_s p_s \colon Y \to \varprojlim_s E_s$ be the inverse limit of the realization of $\mathscr{X}(M_Y)$, then $p_\infty^* \colon H^*(\varprojlim_s E_s; Z_p) \to H^*(Y; Z_p)$ is an isomorphism. Thus p_∞ induces a homotopy equivalence $(\lim_s E_s)_{(p)} \cong Y_{(p)}$ where $W_{(p)}$ is the mod p localization of the space W. In this way we can think of a realization of a convergent resolution as a successive approximation to the space Y at the prime p.

2. The obstruction theory. In this section we will assume that Y and Y' are two very nice spaces with modules M_Y and $M_{Y'}$ in $\mathscr{U}\mathscr{M}$ such that $H^*(Y; Z_p) = U(M_Y)$ and $H^*(Y'; Z_p) = U(M_{Y'})$. Let $\mathscr{X}(M_Y)\colon 0 \leftarrow M_Y \underset{\varepsilon}{\leftarrow} X_0 \underset{d_0}{\leftarrow} X_1 \underset{d_1}{\leftarrow} \cdots$ and $\mathscr{X}(M_{Y'})\colon 0 \leftarrow M_{Y'} \underset{\varepsilon}{\leftarrow} X_0' \underset{d_0'}{\leftarrow} X_1' \underset{d_1'}{\leftarrow} \cdots$ denote resolutions of M_Y and $M_{Y'}$ in $\mathscr{U}\mathscr{M}$. Because we have been liberal in our choices of modules to use in the construction of resolutions we need a definition that provides the analogue of the defining property of projective resolutions. Suppose we have a morphism $g\colon M_{Y'} \to M_Y$ in $\mathscr{U}\mathscr{M}$. We will say that g *lifts through the resolutions* $\mathscr{X}(M_{Y'})$ and $\mathscr{X}(M_Y)$ if there exist maps $g_i\colon X_i' \to X_i$ in $\mathscr{U}\mathscr{M}$ such that the following ladder commutes:

$$\begin{array}{ccccccccc} 0 & \longleftarrow & M_{Y'} & \overset{\varepsilon}{\longleftarrow} & X_0' & \overset{d_0'}{\longleftarrow} & X_1' & \overset{d_1'}{\longleftarrow} & \cdots \\ & & \downarrow g & & \downarrow g_0 & & \downarrow g_1 & & \\ 0 & \longleftarrow & M_Y & \underset{\varepsilon}{\longleftarrow} & X_0 & \underset{d_0}{\longleftarrow} & X_1 & \underset{d_1}{\longleftarrow} & \cdots \end{array}$$

If $\mathscr{X}(M_Y)$ is already a projective resolution, then any map can be lifted.

The focus of this section will be on the realizability of morphisms in $\mathscr{U}\mathscr{M}$. The following theorem indicates the effect of a realizable map on the realizations $\mathscr{E}(\mathscr{X}(M_Y))$ and $\mathscr{E}(\mathscr{X}(M_{Y'}))$.

THEOREM 2.1. ([10]). *Let $k\colon Y \to Y'$ be a map such that $k^*(M_{Y'}) \subset M_Y$ and k^* lifts through the resolutions. Let $\{k_j\}\colon \mathscr{X}(M_{Y'}) \to \mathscr{X}(M_Y)$ be such a lift. Then there exists a map $\Phi\colon \mathscr{E}(\mathscr{X}(M_Y)) \to \mathscr{E}(\mathscr{X}(M_{Y'}))$ realizing the lift of k^*, that is, Φ is a collection $\{\phi_i\colon E_i \to E_i', \psi_j\colon F_j \to F_j'\}$ satisfying the following*:

(2.1A) $\psi_j^* = U(\Omega^{j-1}k_j)\colon U(\Omega^{j-1}X_j') \to U(\Omega^{j-1}X_j)$. And the following diagrams commute up to homotopy:

$$\text{(2.1B)}\quad \begin{array}{ccc} E_i & \xrightarrow{\phi_i} & E_i' \\ {\scriptstyle p_i^{i-1}}\downarrow & & \downarrow{\scriptstyle 'p_i^{i-1}} \\ E_{i-1} & \xrightarrow[\phi_{i-i}]{} & E_{i-1}' \end{array} \qquad \text{(2.1C)}\quad \begin{array}{ccc} \Omega F_i & \xrightarrow{\Omega\psi_i} & \Omega F_i' \\ {\scriptstyle j_i}\downarrow & & \downarrow{\scriptstyle j_i'} \\ E_i & \xrightarrow[\phi_i]{} & E_i' \end{array}$$

$$\text{(2.1D)}\quad \begin{array}{ccc} E_i & \xrightarrow{\phi_i} & E_i' \\ {\scriptstyle f_{i+1}}\downarrow & & \downarrow{\scriptstyle f_{i+1}'} \\ F_{i+1} & \xrightarrow[\psi_{+1}]{} & F_{i+1}' \end{array} \qquad \text{(2.1E)}\quad \begin{array}{ccc} Y & \xrightarrow{k} & Y' \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p_i'} \\ E_i & \xrightarrow[\phi_i]{} & E_i' \end{array}$$

This theorem illustrates the naturality (up to homotopy) of the constructions we have introduced thus far. We record two corollaries to this theorem.

The maps $\phi_n\colon E_n \to E_n'$ induce morphisms $\phi_n^*\colon N_n' \to N_n$ of the extensions in the fundamental sequences for the fibrations $'p_n^{n-1}$ and p_n^{n-1}. In the proof of Theorem 1.3 we observed that N_n' and N_n are split extensions. We ask then whether the morphisms ϕ_n^* respect this splitting. Combining 2.1D) and 2.1E) we get that $[f_{n+1}'\circ\phi_n\circ p_n] = [\psi_{n+1}\circ f_{n+1}\circ p_n] = 0$ in $[Y, F_{n+1}']$. Thus $p_n^*\circ\phi_n^*\circ(f_{n+1}')^* = 0$ which implies that $\phi_n^*(\operatorname{Im}(f_{n+1}')^*) \subset \ker p_n^*$. By construction $\operatorname{Im}(f_{n+1}')^* = \Omega^n\ker d_{n-1}'$ and $\ker p_n^* = \Omega^n\ker d_{n-1}$. Thus $\phi_n^*\colon \Omega^n\ker d_{n-1}' \to \Omega^n\ker d_{n-1}$. From 2.1B) we obtain the following commutative diagram which implies $\phi_n^*\colon U(M_{Y'}) \to U(M_Y)$.

$$\begin{array}{ccccc} 0 & \longrightarrow & U(M_{Y'}) & \xrightarrow{('p_n^{n-1})^*} & N_n' \\ & & \downarrow{\scriptstyle \phi_{n-1}^*} & & \downarrow{\scriptstyle \phi_n^*} \\ 0 & \longrightarrow & U(M_Y) & \xrightarrow[(p_n^{n-1})^*]{} & N_n \end{array}$$

COROLLARY 2.2. *The mappings $\phi_n\colon E_n \to E_n'$ induce morphisms of split extension $\phi_n^*\colon N_n' \to N_n$.*

Now suppose that Y is a primitively generated mod p H-space. The multiplication $m\colon Y\times Y \to Y$ induces $m^*\colon U(M_Y) \to U(M_Y \oplus M_Y)$ such that $m^*(M_Y) \subset M_Y \oplus M_Y$. From Theorem 2.1 and the primitivity we have

COROLLARY 2.3. *For Y a primitively generated* mod p *H-space, the spaces E_n are* mod p *H-spaces and the maps $f_n\colon E_{n-1} \to F_n$ are H-maps.*

The next theorem obtains a partial converse to Theorem 2.1 and provides the basis for the obstruction theory.

THEOREM 2.4. *Let* $g\colon M_{Y'} \to M_Y$ *be given such that* g *lifts through the resolutions* $\mathscr{X}(M_{Y'})$ *and* $\mathscr{X}(M_Y)$ *and let* $\{g_i\colon X_i' \to X_i\}$ *be such a lift. Suppose* $\mathscr{X}(M_{Y'})$ *and* $\mathscr{X}(M_Y)$ *are convergent resolutions and* $\Phi = \{\phi_i\colon E_i \to E_i', \psi_j\colon F_j \to F_j'\}\colon \mathscr{E}(\mathscr{X}(M_Y)) \to \mathscr{E}(\mathscr{X}(M_{Y'}))$ *is a map of realizations satisfying* 2.1A, B, C *and* D. *Then there exists a map* $G\colon Y_{(p)} \to Y'_{(p)}$ *such that* $G^*|_{M_{Y'}} = g$.

Proof. Let $E_\infty = \varprojlim \{E_i, p_i^{i-1}\}$, $E'_\infty = \varprojlim \{E'_i, 'p_i^{i-1}\}$. Applying a theorem of J. Cohen [4] to the inverse systems of homotopy commutative squares

$$\begin{array}{ccc} Y & \xrightarrow{p_{i+1}} & E_{i+1} \\ \| & & \downarrow{\scriptstyle p^{i+1}_{i+1}} \\ Y & \xrightarrow[p_i]{} & E_i \end{array} \qquad \begin{array}{ccc} E_{i+1} & \xrightarrow{\phi_{i+1}} & E'_{i+1} \\ {\scriptstyle p^i_{i+1}}\downarrow & & \downarrow{\scriptstyle 'p^{i}_{i+1}} \\ E_i & \xrightarrow[\phi_i]{} & E'_i \end{array} \qquad \begin{array}{ccc} Y' & \xrightarrow{p'_{i+1}} & E'_{i+1} \\ \| & & \downarrow{\scriptstyle 'p^i_{i+1}} \\ Y' & \xrightarrow[p'_i]{} & E'_i \end{array}$$

we may choose maps $p_\infty\colon Y \to E_\infty$, $p'_\infty\colon Y' \to E'_\infty$ and $\phi_\infty\colon E_\infty \to E'_\infty$ such that the following diagram commutes up to homotopy

$$\begin{array}{ccccccc} Y & \xrightarrow{p_\infty} & E_\infty & \xrightarrow{\phi_\infty} & E'_\infty & \xrightarrow{p'_\infty} & Y' \\ & {\scriptstyle p_0}\searrow & \downarrow & & \downarrow & \swarrow{\scriptstyle p'_0} & \\ & & E_0 & \xrightarrow[\phi_0]{} & E'_0 & & \end{array} .$$

If we localize everything in sight at the prime p we get

$$\begin{array}{ccccccc} Y_{(p)} & \xrightarrow{p_\infty} & E_{\infty(p)} & \xrightarrow{\phi_\infty} & E'_{\infty(p)} & \xleftarrow{p'_\infty} & Y'_{(p)} \\ & {\scriptstyle p_0}\searrow & \downarrow & & \downarrow & \swarrow{\scriptstyle p'_0} & \\ & & E_{0(p)} & \xrightarrow[\phi_0]{} & E'_{0(p)} & & \end{array}$$

where the maps are understood to be localized. By the assumption that $\mathscr{X}(M_{Y'})$ and $\mathscr{X}(M_Y)$ are convergent, $p_\infty\colon Y_{(p)} \cong E_{\infty(p)}$ and $p'_\infty\colon Y'_{(p)} \cong E_{\infty(p)}$. Let q'_∞ denote a homotopy inverse of p'_∞ and define $G = q'_\infty \circ \phi_\infty \circ p_\infty$. This gives the diagram

$$\begin{array}{ccc} Y_{(p)} & \xrightarrow{G} & Y'_{(p)} \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p'_0} \\ E_{0(p)} & \xrightarrow[\phi_0]{} & E'_{0(p)} \end{array} .$$

Now apply $H^*(\ ;Z_p)$. From the properties of the mod p localization we get the following commutative diagram in $\mathscr{U}\mathscr{M}$ after restriction.

$$\begin{array}{ccc} X_0' & \xrightarrow{g_0} & X_0 \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon'} \\ M_{Y'} & \xrightarrow[G^*]{} & M_Y \end{array}$$

Since ε and ε' are epimorphisms, by cancellation we have $G^*|_{M_{Y'}} = g$.

Now fix a morphism $g: M_{Y'} \to M_Y$ in $\mathscr{U}\mathscr{M}$. We will assume that g can be lifted thorugh $\mathscr{X}(M_{Y'})$ and $\mathscr{X}(M_Y)$ and that the resolutions are convergent. Because we have taken the F_i and F_i' to be *gEMs* the lifting $\{g_i: X_i' \to X_i\}$ gives rise to a collection of maps $\{\psi_i: F_i \to F_i'\}$ such that $\psi_i^* = U(\Omega^{i-1}g_i)$. Theorem 2.1 motivates the following

DEFINITION 2.5. Let $\gamma: E_n \to E_n'$. We will say that γ is an *n-realizer* for g if

2.5a$_n$ for $0 \leqq i < n$ there exists $\phi_i: E_i \to E_i'$ such that ϕ_i is an i-realizer and (2.1B) holds. Also the following diagrams homotopy commute:

2.5b$_n$
$$\begin{array}{ccc} E_n & \xrightarrow{\gamma} & E_{n-1}' \\ {\scriptstyle p_n^{n-1}}\downarrow & & \downarrow{\scriptstyle p_n^{n-1}} \\ E_{n-1} & \xrightarrow[\phi_{n-1}]{} & E_{n-1}' \end{array}$$

2.5c$_n$
$$\begin{array}{ccc} \Omega F_n & \xrightarrow{\Omega\psi_n} & \Omega F_n' \\ {\scriptstyle j_n}\downarrow & & \downarrow{\scriptstyle j_n'} \\ E_n & \xrightarrow[\gamma]{} & E_n' \end{array}$$

2.5d$_n$
$$\begin{array}{ccc} E_n & \xrightarrow{\gamma} & E_n' \\ {\scriptstyle f_{n+1}}\downarrow & & \downarrow{\scriptstyle f_{n+1}'} \\ F_{n+1} & \xrightarrow[\psi_{n+1}]{} & F_{n+1}' \end{array}$$

From the definition of a realization of a resolution, everything at the 0-level is a *gEMs* and so the existence of a 0-realizer comes for free. Suppose we have an $(n-1)$-realizer ϕ_{n-1}. We now construct a particular candidate for γ an n-realizer. By 2.5d$_n$ there is a homotopy $H: E_{n-1} \times I \to F_n'$ such that $H(x, 0) = f_n' \circ \phi_{n-1}(x)$ and $H(x, 1) = \psi_n \circ f_n(x)$. Recall that $E_n = \{(\lambda, x) \mid \lambda \in PF_n,\ x \in E_{n-1}$ and $\lambda(1) = f_n(x)\}$ and E_n' is the analogous subset of $PF_n' \times E_{n-1}'$. Define $\gamma: E_n \to E_n'$ by $\gamma(\lambda, x) = (\lambda_H, \phi_{n-1}(x))$ where λ_H is the path

$$\lambda_H(t) = \begin{cases} \psi_n \circ \lambda(2t), & 0 \leqq t \leqq 1/2 \\ H(x, 2-2t), & 1/2 \leqq t \leqq 1\,. \end{cases}$$

Since $\lambda_H(1) = H(x, 0) = f'_n(\phi_{n-1}(x))$, $(\lambda_H, \phi_{n-1}(x))$ is in E'_n and so γ is well-defined. It is easy to show that γ is continuous and satisfies 2.5a_n, b_n and c_n. The fundamental sequences of p_n^{n-1} and $'p_n^{n-1}$ give us the key to condition 2.5d_n.

THEOREM 2.6. *The obstruction to* γ *being an n-realizer is the class* $[f'_{n+1}\circ\gamma\circ p_n]$ *in* $[Y, F'_{n+1}]$.

Proof. Consider the diagram of spaces

$$\begin{array}{ccccc} Y & \xrightarrow{p_n} & E_n & \xrightarrow{\gamma} & E'_n \\ & & {\scriptstyle f_{n+1}}\downarrow & & \downarrow{\scriptstyle f'_{n+1}} \\ & & F_{n+1} & \xrightarrow[\psi_{n+1}]{} & F'_{n+1} . \end{array}$$

From the construction of a realization $f_{n+1}\circ p_n \simeq *$; if $[f'_{n+1}\circ\gamma\circ p_n] \neq 0$, 2.5$d_n$ has no chance of being satisfied. Suppose $[f'_{n+1}\circ\gamma\circ p_n] = 0$. Then $p_n^*\circ\gamma^*\circ(f'_{n+1})^* = 0$ which implies $\gamma^*((f'_{n+1})^*(\Omega^n X'_{n+1}))$ is contained $\ker p_n^*|_{N_n} = \Omega^n\ker d_{n-1}$. Since $(f'_{n+1})^*(\Omega^n X'_{n+1}) = \Omega^n\ker d'_{n-1}$, it follows that $\gamma^*(\Omega^n\ker d'_{n-1}) \subset \Omega^n\ker d_{n-1}$. By 2.5$c_n$ and the naturality of the fundamental sequence we get the following commutative diagram:

$$\begin{array}{ccc} \Omega^n X'_{n+1} & \xrightarrow{(f'_{n+1})^*} & \Omega^n\ker d'_{n-1} \\ {\scriptstyle \Omega^n g_{n+1}}\downarrow & & \downarrow{\scriptstyle \gamma^*} \\ \Omega^n X_{n+1} & \xrightarrow[(f_{n+1})^*]{} & \Omega^n\ker d_{n-1} . \end{array}$$

Since F_{n+1} and F'_{n+1} are *gEMs* the commutativity of this square implies 2.5d_n and hence γ is an n-realizer.

Observe that $[Y, F'_{n+1}] = H^*(Y; \pi_*(F'_{n+1}))$; this with Theorem 2.1 gives

THEOREM 2.7. γ *is an n-realizer if and only if* $[f'_{n+1}\circ\gamma\circ p_n] = 0$ *in* $H^*(Y; \pi_*(F'_{n+1}))$.

The map γ as constructed above was a single candidate for an n-realizer. Since $'p_n^{n-1}\colon E'_n \to E'_{n-1}$ is a principal fibration we can vary γ by the principal action $\mu\colon \Omega F'_n \times E'_n \to E'_n$. That is, if $\zeta \in [E_n, E'_n]$ and $['p_n^{n-1}\circ\zeta] = ['p_n^{n-1}\circ\gamma] = [\phi_{n-1}\circ p_n^{n-1}]$ then there exists a $w \in [E_n, \Omega F'_n]$ such that $[\mu\circ(w\times\gamma)\circ\Delta] = [\zeta]$ in $[E_n, E'_n]$. If ζ is a map obtained in this manner from γ and the principal action, then ζ satisfies 2.5a_n, b_n and c_n and hence Theorems 2.6 and 2.7 hold when γ is replaced by ζ.

Suppose we are given an $(n-1)$-realizer. Define $\Gamma_n\colon [E_n, \Omega F'_n] \to$

$[Y, F'_{n+1}]$ to be the composite $[E_n, \Omega F'_n] \xrightarrow[\mu\#(-,\gamma)]{} [E_n, E'_n] \xrightarrow[(f'_{n+1})\#]{} [E_n, F'_{n+1}] \xrightarrow[p_n^\#]{} [Y, F'_{n+1}]$ where $F_\#[q] = [q \circ F]$ and $F^\#[q] = [F \circ q]$. By the previous paragraph the obstructions determined by all possible candidates for an n-realizer for g lie in the image of Γ_n in $[Y, F'_{n+1}]$. Let $\mathcal{O}_n(g)$ denote the image of Γ_n.

THEOREM 2.8. *Given an $(n-1)$-realizer for g, it extends to an n-realizer for g if and only if $0 \in \mathcal{O}_n(g) \subset H^*(Y, \pi_*(F'_{n+1}))$.*

If an n-realizer exists for all n, then by Theorem 2.4 we have that g is realizable. From this and Theorem 2.1 we conclude

THEOREM 2.9. *g is realizable if and only if, for all n, $0 \in \mathcal{O}_n(g)$.*

In [6] Harper proves that the principal action, $\mu: \Omega F'_n \times E'_n \to E'_n$ is primitive in the following sense: If $H^*(E'_n; Z_p) = U(N'_n)$ and $y \in N'_n$ then $\mu^*(y) = 1 \otimes y + (j'_n)^*(y) \otimes 1$ in $H^*(\Omega F'_n; Z_p) \otimes H^*(E'_n; Z_p)$. From the definition of a realization of a resolution, the map $f'_{n+1} \circ j'_n: \Omega F'_n \to F'_{n+1}$ is determined by $\Omega^n d_n: \Omega^n X_{n+1} \to \Omega^n X_n$. Since $\Omega F'_n$ and F'_{n+1} are $gEMs$, the map $f'_{n+1} \circ j'_n$ determines a primary operation $\Xi_n: H^*(\ ; \pi_*(\Omega F'_n)) \to H^*(\ ; \pi_*(F'_{n+1}))$. Utilizing Harper's result we obtain

THEOREM 2.10. *$\mathcal{O}_n(g)$ is the coset $[f'_{n+1} \circ \gamma \circ p_n] + \Xi_n H^*(Y; \pi_*(\Omega F'_n))$ in $H^*(Y; \pi_*(F'_{n+1}))$.*

Proof. Without loss of generality we will assume $F'_{n+1} = K(Z_p, m)$ and so take $[f'_{n+1}] = v$, a homogeneous class in N'_n. An arbitrary class ξ in $\mathcal{O}_n(g)$ may be written as the composite

$$Y \xrightarrow[p_n]{} E_n \xrightarrow[\Delta]{} E_n \times E_n \xrightarrow[w \times \gamma]{} \Omega F'_n \times E'_n \xrightarrow[\mu]{} E'_n \xrightarrow[f'_{n+1}]{} E'_{n+1}$$

where w is in $[E_n, F'_n]$. Thus $\xi = [f'_{n+1} \circ \mu(w, \gamma) \circ \Delta \circ p_n] = p_n^* \circ \Delta^* \circ (w^* \otimes \gamma^*) \circ \mu^*(v)$. By Harper's result we have

$$\begin{aligned}
\xi &= p_n^* \circ \Delta^* \circ (w^* \otimes \gamma^*)(1 \otimes v + (j'_n)^*(v) \otimes 1) \\
&= p_n^* \circ \Delta^*(1 \otimes \gamma^*(v) + w^* \circ (j'_n)^*(v) \otimes 1) \\
&= p_n^*(\gamma^*(v) + w^* \circ (j'_n)^*(v)) \\
&= p_n^* \circ \gamma^*(v) + p_n^* \circ w^* \circ (j'_n)^*(v) \\
&= [f'_{n+1} \circ \gamma \circ p_n] + [f'_{n+1} \circ j'_n \circ w \circ p_n] \\
&= [f'_{n+1} \circ \gamma \circ p_n] + \Xi_n[w \circ p_n]\,.
\end{aligned}$$

If we let w vary over $[E_n, \Omega F'_n] = H^*(E_n; \pi_*(\Omega F'_n))$ we obtain all of the set $\mathcal{O}_n(g)$. Hence we can write $\mathcal{O}_n(g) = [f'_{n+1} \circ \gamma \circ p_n] + p_n^* \Xi_n H^*(E_n; \pi_*(\Omega F'_n))$. Now observe that $p_n^* \circ \Xi_n = \Xi_n \circ p_n^*$ because primary cohomology operations are natural. Furthermore p_n^* takes $H^*(E_n; \pi_*(\Omega F'_n))$ onto $H^*(Y; \pi_*(\Omega F'_n))$. Thus we can write $\mathscr{O}_n(g) = [f'_{n+1}\circ\gamma\circ p_n^*] + \Xi_n H^*(Y; \pi_*(\Omega F'_n))$.

Observe that if Ξ_n is trivial on $H^*(Y; \pi_*(\Omega F'_n))$, then the only obstruction to the existence of an n-realizer for g is the class $[f'_{n+1}\circ\gamma\circ p_n^*]$.

3. **Applications.** It is a consequence of Borel's structure theorem for Hopf algebras that if Y is an H-space without p-torsion in its integral cohomology then $H^*(Y; Z_p) = \Lambda(x_{2n_1+1}, \cdots, x_{2n_l+1})$ where dim $x_r = r$. For those primes for which $\mathscr{P}^1$ acts trivially on $H^*(Y; Z_p)$, Y shares the same cohomology as the space $S_p(Y) = S^{2n_1+1}\times\cdots\times S^{2n_l+1}$. If there is a map $S_p(Y)\to Y$ inducing an isomorphism in mod p cohomology then, from the theory of localization, $S_p(Y)_{(p)}$ and $Y_{(p)}$ are homotopy-equivalent and the mod p homotopy information about Y is determined by the product space $S_p(Y)_{(p)}$. If such a map exists, we call the prime p *regular* for Y.

Now consider those primes for which $\mathscr{P}^1$ is the only element of $\mathscr{A}(p)$ to act nontrivially on $H^*(Y; Z_p)$. Mimura and Toda [14] have introduced complexes, $B_m(p)$, which are sphere bundles over spheres with cohomology $H^*(B_m(p); Z_p) = \Lambda(x_{2m+1}, \mathscr{P}^1x_{2m+1})$. If $\mathscr{P}^1$ acts nontrivially we can ask whether or not Y "looks like" a product of spheres and $B_m(p)$'s at the prime p. More precisely, if $H^*(Y; Z_p) = \Lambda(x_{2m_1+1}, \mathscr{P}^1x_{2m_1+1}, \cdots, x_{2m_k+1}, \mathscr{P}^1x_{2m_k+1}, x_{2m_{k+1}+1}, \cdots, x_{2m_s+1})$, then we wish a map $K_p(Y)\to Y$ which induces an isomorphism in mod p cohomology where $K_p(Y) = \prod_{i=1}^k B_{m_i}(p)\times\prod_{j=k+1}^s S^{2m_j+1}$. If such a map exists, $K_p(Y)_{(p)}\cong Y_{(p)}$ and we say that p is *quasi-regular* for Y.

We can translate these questions of regularity and quasi-regularity into questions about the realizability of morphisms in $\mathscr{U}\mathscr{M}$ by observing that $H^*(Y; Z_p) = \Lambda(x_{2n_1+1}, \cdots, x_{2n_l+1}) = U(M_Y)$ where M_Y is a direct sum of modules $\mathrm{Tr}(2m_j+1) = \{x_{2m_j+1}\}$ and $MB_{m_i}(p) = \{x_{2m_i+1}, \mathscr{P}^1x_{2m_i+1}\}$. As unstable algebras, $H^*(Y, Z_p)\cong H^*(K_p(Y); Z_p)\cong U(M_Y)$ so we can ask if there is a map $R_p\colon K_p(Y)_{(p)}\to Y_{(p)}$ which realizes the map of modules id: $M_Y\to M_Y$. The existence of such a map implies that $K_p(Y)_{(p)}\cong Y_{(p)}$ as desired.

The strategy of the proofs of Theorems A and B will be to employ the obstruction theory to realize each projection from the direct sum, $M_Y\to\mathrm{Tr}(2m_j+1)$ or $M_Y\to MB_{m_i}(p)$ by a map $r_j\colon S_{(p)}^{2m_j+1}\to Y_{(p)}$ or $r_i\colon B_{m_i}(p)_{(p)}\to Y_{(p)}$. We then consider the composite map

$$R_p\colon B_{m_1}(p)_{(p)}\times\cdots\times B_{m_k}(p)_{(p)}\times S_{(p)}^{2m_{k+1}+1}\times\cdots\times S_{(p)}^{2m_s+1} \xrightarrow[r_1\times\cdots\times r_k\times r_{k+1}\times\cdots\times r_s]{} Y_{(p)}\times Y_{(p)}\times\cdots\times Y_{(p)}\xrightarrow[\xi_s]{} Y_{(p)}$$

where $\xi_s(y_1, y_2, y_3, \cdots, y_s) = (\cdots((y_1\cdot y_2)\cdot y_3)\cdots)\cdot y_s$ is induced by the multiplication on $Y_{(p)}$. To see that R_p^* is an isomorphism it suffices to check $R_p^*: H^*(Y_{(p)}; Z_p) \to H^*(K_p(Y); Z_p)$ on the indecomposables (=the primitives in this case) to determine that R_p^* gives the obvious isomorphism. Let u be an indecomposable in $H^*(Y_{(p)}; Z_p)$.

$$\begin{aligned} R_p^*(u) &= r_1^* \otimes r_2^* \otimes \cdots \otimes r_s^*(\xi_s^*(u)) \\ &= r_1^* \otimes r_2^* \otimes \cdots \otimes r_s^* \Big(\sum_{i=1}^{s} 1 \otimes 1 \otimes \cdots \otimes \underset{\substack{| \\ i\text{th place}}}{u} \otimes \cdots \otimes 1\Big) \\ &= \sum_{i=1}^{s} (1 \otimes 1 \otimes \cdots \otimes r_i^*(u) \otimes \cdots \otimes 1)\,. \end{aligned}$$

Now observe that u is an indecomposable implies $u \in M_Y$ and without loss of generality we may assume u is in the jth direct summand of M_Y. Since

$$r_j^* = \mathrm{proj}_j\colon M_Y \longrightarrow \mathrm{Tr}(2m_j + 1) \text{ or } MB_{m_j}(p) \text{ then } r_i^*(u) = \begin{cases} u, & \text{if } i = j\,, \\ 0, & \text{if } i \neq j\,. \end{cases}$$

Thus $R_p^*(u) = u$, the corresponding class in $H^*(K_p(Y)_{(p)}; Z_p)$ and we will obtain the desired homotopy equivalence if we can realize each projection $M_Y \to \mathrm{Tr}(2m_j + 1)$ or $MB_{m_i}(p)$.

Now suppose we want a map, $W_{r(p)} \to Y_{(p)}$, to realize each projection $M_Y \to N_r$ where $W_r = S^{2m_r+1}$ or $B_{m_r}(p)$ and $N_r = \mathrm{Tr}(2m_r + 1)$ or $MB_{m_r}(p)$. Consider those dimensions in which W_r has nonzero cohomology and those dimensions in which possible obstructions can occur; these dimensions are calculable from knowledge of the direct sum decomposition of M_Y and determination of certain modules in convergent resolutions of the summands $\mathrm{Tr}(2m_j + 1)$ and $MB_{m_i}(p)$. If these two sets of numbers can be shown to be disjoint then the obstruction theory implies that a map exists realzing each projection. With this in mind we provide the following table which lists the dimensions in which an obstruction might occur when M_Y has the appropriate summand. To obtain the table one would compute the first few modules (X_0, X_1, X_2, and X_3) in a convergent resolution of each possible summand. The calculations only involve a routine application of the Adem relations and the unstable axioms and so are left to the reader.

TABLE 1

	Tr(3)-summand	Tr(2m+1)-summand	$MB_1(p)$-summand	$MB_m(p)$-summand
$\mathcal{O}_1$	$4p-1, 4p-2$	$2m+4p-3$	$4p-1$	$2m+4p-3$
$\mathcal{O}_2$	$6p-4$	$2m+6p-5$	$6p-3$	$2m+6p-5$

Proof of Theorem A. Recall that the dimension of $\mathscr{P}^1 x_r$ is $r + 2(p-1)$. If

$$r = 2n_i + 1 \text{ then } r+2(p-1) = 2n_i+1+2(p-1) \geqq 2n_i+1+2(n_l-n_1+1)$$
$$= 2n_l + 3 + 2n_i - 2n_1$$
$$> 2n_l + 1$$

since $n_1 \leqq n_i$ for all i. The image of a primitive under the action of $\mathscr{A}(p)$ is also primitive and since all of the primitives lie in dimensions less than or equal to $2n_l + 1$, then we can see that $\mathscr{P}^1$ acts trivially on $H^*(Y; Z_p)$. Thus $H^*(Y; Z_p) = U(M_Y)$ where $M_Y = \mathrm{Tr}(2n_1 + 1) \oplus \cdots \oplus \mathrm{Tr}(2n_l + 1)$.

Suppose we wish to realize a projection $M_Y \to \mathrm{Tr}(2n_i + 1)$ by a map $S^{2n_i+1}_{(p)} \to Y_{(p)}$. From table 1 we see that the lowest dimension in which an obstruction may occur is $2n_1 + 4p - 3$. The inequality $p \geqq n_l - n_1 + 2$ implies $2n_1 + 4p - 3 > 2n_l + 1$ and so any obstruction must vanish since the $(2n_i + 1)$-sphere has cohomology only in dimension $2n_i + 1$. Hence there is a map $S^{2n_i+1}_{(p)} \to Y_{(p)}$ realizing each projection $M_Y \to \mathrm{Tr}(2n_i + 1)$. By the discussion in the beginning of the section, this proves the theorem.

Before proving Theorem C, we first observe the following

LEMMA 4.1. *If Y and Y' are* mod p *H-spaces whose cohomology is primitively generated and if Y and Y' are very nice spaces and $g\colon M_Y \to M_{Y'}$ a morphism in $\mathscr{U}\mathscr{M}$, then the class $[f_2' \circ \gamma \circ p_1] \in \mathscr{O}_1(g)$ is primitive.*

Proof. By Corollary 2.3, E_1 and E_1' are mod p H-spaces and f_1^2 is an H-map. From 2.1E) we see that $p_1\colon Y \to E_1$ is an H-map. It suffices to note that γ is an H-map. However this is clear since γ lifts the commutative square

$$\begin{array}{ccc} E_0' & \xrightarrow{\phi_1} & E_0 \\ {\scriptstyle f_1'}\downarrow & & \downarrow{\scriptstyle f_1} \\ F_1' & \xrightarrow[\psi_1]{} & F_1 \end{array}$$

and the assumption that Y and Y' are primitively generated gives that ϕ_0, f_1', f_1 and ψ_1 are all H-maps.

Proof of Theorem B. The spaces $B_{n_i}(p)$ have nonzero cohomology in dimensions $2n_i + 1$, $2n_i + 1 + 2(p-1)$ and $2(2n_i + 1) + 2(p-1)$. When $p \geqq 5$ the spaces $B_{n_i}(p)$ are mod p H-spaces [12] and so we need only consider primitives as $\mathscr{O}_1$ obstructions. The inequality

$2p > n_l - n_1 + 2$ implies that the first obstructions to realizing maps $M_Y \to MB_{n_i}(p)$ or $M_Y \to \mathrm{Tr}(2n_j + 1)$ lie in dimensions larger than $2n_l + 1$ and hence vanish for dimension reasons.

Now observe that the inequality $2p > n_l - n_1 + 2$ guarantees that the highest dimension in which a product class $x_r \cup \mathscr{P}^1 x_r$ can occur is less than $6p - 6$. Thus the $\mathscr{O}_2$ obstructions all vanish for dimension reasons. Since any higher obstructions lie in still higher dimensions, we have that any projection $M_Y \to MB_{n_i}(p)$ can be realized. Similarly any projection $M_Y \to \mathrm{Tr}(2n_j + 1)$ can be realized. This completes the proof of Theorem B.

We add that more can be said when the mod p cohomology data for Y is known. In [11] the author obtains results of Mimura and Toda [14] on the quasi-regularity of primes for compact Lie groups without the need of the restriction $p \geqq 5$.

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