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GROUPS OF INTEGRAL REPRESENTATION TYPE

HYMAN BASS

Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

Introduction. Let Γ be a group, $\rho: \Gamma \to GL_n(K)$ a matrix representation over some field K, and χ_ρ its character: $\chi_\rho(s) = \mathrm{Tr}(\rho(s))$. The theme of this paper is, generally spaking, to draw conclusions about Γ or $\rho(\Gamma)$ from finiteness assumptions on $\chi_\rho(\Gamma)$. The prototype of such results is Burnside's theorem saying, when ρ is absolutely irreducible, that if $\chi_\rho(\Gamma)$ is finite then $\rho(\Gamma)$ is finite. This yielded his affirmative solution of the "Burnside problem" for linear groups. The same argument shows, when K is a locally compact field (like $\boldsymbol{R}$ or $\boldsymbol{C}$) that we may replace "finite" by "bounded", and conclude from boundedness of $\chi_\rho(\Gamma)$ that the closure of $\rho(\Gamma)$ is compact. This yields an affirmative answer to a question posed to us by Ken Millett. Independently, Kaplansky asked us whether a subgroup of $GL_n(\boldsymbol{C})$, each element of which is conjugate to a unitary matrix, is itself conjugate to a subgroup of $U_n(\boldsymbol{C})$. We give a counterexample. These results occupy § 1.

The rest of the paper is devoted to the introduction and study of the following notion. Let A be a commutative ring and n an integer $\geqq 1$. A group Γ is said to have *integral n-representation type* over A if, for any field K which is an A-algebra, and any representation $\rho: \Gamma \to GL_n(K)$, the elements $\chi_\rho(s) \in K$ for $s \in \Gamma$ are all integral over A, i.e., roots of a monic polynomial with coefficients in A (Def. 5.1). We conclude from this, when A is noetherian, that, for any finite subset X of Γ, the sub A-algebra of $M_n(K)$ generated by $\rho(X)$ is a finitely generated A-module, (Prop. 5.2). Further, Γ has only finitely many conjugacy classes of irreducible representations of dimension $\leqq n$ over any field K as above (Prop. 5.3). These and other strong finiteness properties are deduced from the theory of rings with polynomial identities, as developed in Procesi's book [12]. In §§ 2-4 we give a rendering of this source material adapted to the present applications.

The case of main interest is when $A = Z$, which we now assume. A group Γ has integral 1-representation type if and only if Γ^{ab} is a torsion group (Prop. 5.5). Serre [15] has furnished a class of finitely generated groups Γ of integral 2-representation type, namely those with the fixed point property for actions on trees (Th. 6.4). This is equivalent to Γ^{ab} being finite and Γ not being a nontrivial amalgamated free product (Th. 6.2). We derive a useful refinement (Th. 6.5) of Serre's theorem in order to prove (Cor. 6.7) a conjecture of P. Shalen asserting that certain subgroups of $GL_2(\boldsymbol{C})$ are non-trivial amalgamated free products. (Shalen's persistent solicitation of this proof is responsible for the present paper.)

We say that Γ has integral representation type if it has integral n-representation type for all $n \geqq 1$. Examples include all torsion groups and many arithmetic groups, e.g., $SL_n(\boldsymbol{Z})$ for $n \geqq 3$ (but not $n = 2$). In fact variations on these examples account for essentially all known finitely generated linear groups of integral representation type (cf. § 10). Groups of integral representation type are stable under passage to quotients and to subgroups of finite index (Cor. 5.8), and under formation of direct products (Prop. 5.11) (but not free products (Cor. 8.4), and even of arbitrary group extensions (Cor. 9.9)). The proof of the latter result is slightly intricate. The only finitely generated solvable groups of integral representation type are the finite ones (Cor. 5.9).

1. Groups of bounded character.

NOTATION 1.1. We fix an algebraically closed field K, a multiplicative monoid Γ in the K-algebra $M_n(K)$ of n by n matrices, and we write $\operatorname{Tr}(\Gamma) = \{\operatorname{Tr}(s) \mid s \in \Gamma\}$.

We answer below the following questions of Ken Millett and I. Kaplansky. Suppose that $K = \boldsymbol{C}$ and that Γ is a group whose elements all have eigenvalues all of absolute value 1. Then Γ is conjugate to a subgroup of

$$\begin{pmatrix} U_{n_1} & & * \\ & \ddots & \\ 0 & & U_{n_r} \end{pmatrix}$$

where $U_{n_i} = U_{n_i}(\boldsymbol{C})$ denotes the unitary group (Cor. 1.8). This affirmatively responds to a question posed to us by Millett. Kaplansky independently asked us whether, under the additional assumption that each element of Γ is semi-simple (i.e., diagonalizable), one can take $* = 0$ above. Equivalently, if each element of Γ is conjugate to an element of $U_n(\boldsymbol{C})$, is Γ conjugate to a subgroup of $U_n(\boldsymbol{C})$? We furnish a counterexample in 1.10 below. Our results are based on a classical argument of Burnside which we now recall.

THE BURNSIDE LEMMA 1.2. *Suppose that* Γ *acts irreducibly on* K^n.

(a) Γ *contains a basis* $s_1, \cdots, s_{n^2}$ *of* $M_n(K)$. *Let* $t_1, \cdots, t_{n^2}$ *be the dual basis relative to the trace form*: $\operatorname{Tr}(t_i s_j) = \delta_{ij} (1 \leqq i, j \leqq n^2)$.

(b) *For any* $s \in M_n(K)$ *we have* $s = \sum_i \operatorname{Tr}(s s_i) t_i$. *Hence*

$$\Gamma \subset \sum_i \mathrm{Tr}\,(\Gamma) t_i \,. \tag{1}$$

This is classical. Briefly, the K-linear span $K\Gamma$ of Γ is a K-algebra with faithful simple module $V = K^n$. Schur's lemma says that the division algebra $\mathrm{End}_{K\Gamma}(V)$ is K (since K is algebraically closed), and Wedderburn theory then gives $K\Gamma = \mathrm{End}_K(V) = M_n(K)$. The trace form $(x, y) \mapsto \mathrm{Tr}\,(xy)$ on $M_n(K)$ is nondegenerate, whence the existence of the dual basis (t_i). If $s = \sum_i a_i t_i \in M_n(K)$ with $a_i \in K$ then $\mathrm{Tr}\,(ss_j) = \sum_i a_i \,\mathrm{Tr}\,(t_i s_j) = a_j$. If $s \in \Gamma$ then $ss_j \in \Gamma$ for all j, so $a_j \in \mathrm{Tr}\,(\Gamma)$. Whence the lemma.

COROLLARY 1.3. *Suppose that Γ acts irreducibly on K^n.*

(a) (Burnside) *If $\mathrm{Tr}\,(\Gamma)$ is finite then Γ is finite. In fact* $\mathrm{Card}\,(\Gamma) \leqq (\mathrm{Card}\,(\mathrm{Tr}\,(\Gamma)))^{n^2}$.

(b) *Suppose that K admits an absolute value relative to which* $\mathrm{Tr}\,(\Gamma)$ *is bounded. Then Γ is bounded in $M_n(K)$.*

(c) *Suppose that K is a nondiscrete locally compact field and that* $\mathrm{Tr}\,(\Gamma)$ *is bounded. Then the closure $\bar{\Gamma}$ of Γ in $M_n(K)$ is compact.*

Both (a) and (b) are immediate from (1) above, and (c) follows from (b).

REMARK 1.4. The only algebraically closed locally compact nondiscrete field is $\boldsymbol{C}$. However other locally compact fields may be admitted in (c) provided that we assume the action of Γ on K^n is *absolutely* completely reducible.

1.5. When Γ acts not necessarily irreducibly on K^n we can choose a Jordan-Holder series

$$0 = V_0 \subset V_1 \subset \cdots \subset V_r = K^n \tag{2}$$

for the Γ-module K^n. Relative to a basis of K^n adapted to (2) the elements $s \in \Gamma$ take the matrix form

$$s = \begin{pmatrix} s_1 & & & * \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_r \end{pmatrix} \tag{3}$$

where $s_i \in M_{n_i}(K)$ is the matrix of the action s induces on the n_i-dimensional space $V_i/V_{i-1} (i = 1, \cdots, r)$. The map $s \mapsto s_i$ is a homomorphism of Γ onto a monoid $\Gamma_i \subset M_{n_i}(K)$ which acts irreducibly on K^{n_i}, so the preceeding results apply to each Γ_i. If Γ acts completely reducibly on K^n then we may choose the basis above so that

$* = 0$ in (3), i.e., s is in block diagonal form.

1.6. Suppose now that Γ is a *group*. The map

$$s = \begin{pmatrix} s_1 & & * \\ & \ddots & \\ 0 & & s_r \end{pmatrix} \longrightarrow s' = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_r \end{pmatrix} \tag{4}$$

is a homomorphism from Γ onto a subgroup Γ' of $GL_n(K)$, isomorphic to a subgroup of $\Gamma_1 \times \cdots \times \Gamma_r$. The kernel Γ_u of (4) consists of unipotent matrices $\left(\text{of the form } \begin{pmatrix} 1 \cdot & * \\ 0 & \cdot 1 \end{pmatrix}\right)$. If $\operatorname{char}(K) = 0$ then Γ_u is a torsion free group. If $\operatorname{char}(K) = p > 0$ then Γ_u has exponent p^n: $s^{p^n} = 1$ for all $s \in \Gamma_u$.

COROLLARY 1.7 (*Burnside*). *Suppose that Γ is a group of exponent e: $s^e = 1$ for all $s \in \Gamma$. Then there is a constant $c = c(n, e)$ such that the unipotent group Γ_u has index $\leqq c$ in Γ. If* $\operatorname{char}(K)$ *does not divide e then $\Gamma_u = \{1\}$ so* $\operatorname{Card}(\Gamma) \leqq c$.

In view of 1.6 it suffices to bound each $\operatorname{Card}(\Gamma_i)$ by a constant depending on n and e alone. From 1.3 (a) we have $\operatorname{Card}(\Gamma_i) \leqq \operatorname{Card}(\operatorname{Tr}(\Gamma_i))^{n_i^2}$. Since $n_i \leqq n$ it suffices to bound $\operatorname{Tr}(\Gamma_i)$. If $s \in \Gamma_i$ then $\operatorname{Tr}(s)$ is a sum of n_i eth roots of unity, and there are at most e^{n_i} such sums. This proves the corollary. One can, for example, take $c = ((e^n)^{n^2})^n = e^{n^4}$.

COROLLARY 1.8. *Let Γ be a subgroup of $GL_n(\boldsymbol{C})$ such that the set of eigenvalues of elements of Γ is bounded.*

(a) *Γ is conjugate to a subgroup of*

$$\begin{pmatrix} U_{n_1} & & & * \\ & U_{n_2} & & \\ & & \ddots & \\ 0 & & & U_{n_r} \end{pmatrix} \tag{5}$$

where U_{n_i} denotes the unitary group $U_{n_i}(\boldsymbol{C})$.

(b) *The set Γ_u of unipotent elements of Γ is a normal subgroup of Γ, and Γ/Γ_u is isomorphic to a subgroup of $U_n(\boldsymbol{C})$.*

(c) *If Γ acts completely reducibly on $\boldsymbol{C}^n$ then Γ is conjugate to a subgroup of $U_n(\boldsymbol{C})$ (and conversely).*

With the notation of 1.6 above, the hypothesis on eigenvalues implies that each $\Gamma_i \subset GL_{n_i}(\boldsymbol{C})$ has bounded trace, so 1.3 (c) implies that the closure $\bar{\Gamma}_i$ is compact, hence conjugate to a subgroup of

$U_{n_i}(C)$; whence (a). The homomorphism (4) then maps Γ onto a subgroup Γ' of $\begin{pmatrix} U_{n_1} & & 0 \\ & \ddots & \\ 0 & & U_{n_r} \end{pmatrix} \subset U_n(C)$, which has no unipotent elements $\neq 1$. It follows that the kernel Γ_u of (4) consists of all unipotent elements of Γ, thus proving (b). If Γ acts completely reducibly on C^n, then, as remarked in 1.5, we can take $* = 0$ in (5), whence (c).

Kaplansky's Problem 1.9. Let Γ be as in 1.8, and assume that each element of Γ is semi-simple (i.e., diagonalizable). Kaplansky asked whether it then follows that Γ is conjugate to a subgroup of $U_n(C)$. Note that $\Gamma_u = \{1\}$, so it follows from 1.8 (b) that Γ is isomorphic to a subgroup of $U_n(C)$. We shall show, nonetheless, that the answer to Kaplansky's problem is negative.

Counterexample 1.10. If $w \in SU_2(C)$ and $w \neq 1$ then neither eigenvalue of w can equal 1 (because their product is 1, and $SU_2(C)$ contains no unipotent elements $\neq 1$); hence $w - 1$ is invertible. It follows that for any $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \in C^2$ the element

$$\begin{pmatrix} w & c \\ 0 & 1 \end{pmatrix} \in SL_3(C) \tag{6}$$

is semi-simple. Indeed, if $b \in C^2$ then

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} w & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} w & c - (w-1)b \\ 0 & 1 \end{pmatrix} . \tag{7}$$

Taking $b = (w - 1)^{-1}c$ we obtain the semi-simple matrix $\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix}$.

Now let Γ' be a *free* subgroup of $SU_2(C)$ with free basis u, v. Such free groups are well known to exist, for example by Tits' theorem [T] (since $SU_2(C)$ is connected and nonsolvable). Since $u \neq v$ we have $(u - 1)^{-1} \neq (v - 1)^{-1}$, so there is an $a \in C^2$ such that

$$(u - 1)^{-1}a \neq (v - 1)^{-1}a . \tag{8}$$

Now let Γ denote the group

$$\Gamma = \left\langle \begin{pmatrix} u & a \\ 0 & 1 \end{pmatrix} , \quad \begin{pmatrix} v & a \\ 0 & 1 \end{pmatrix} \right\rangle \subset SL_3(C) .$$

The obvious projection $\Gamma \to \Gamma'$ is an isomorphism since Γ' is free with basis u, v. It follows that each nontrivial element of Γ is of the form (6) with $w \neq 1$, and hence is semi-simple. Moreover the elements of Γ evidently have eigenvalues on the unit circle. Thus Γ provides the promised counterexample to Kaplansky's problem once we show that Γ is not conjugate to a subgroup of $U_3(\boldsymbol{C})$. For this it suffices to show that Γ does not act completely reducibly on $\boldsymbol{C}^3$. Let e_1, e_2, e_3 be the standard basis of $\boldsymbol{C}^3$. Then Γ leaves $\boldsymbol{C}^2 = \boldsymbol{C}_{e_1} + \boldsymbol{C}e_2$ invariant. If the action were completely reducible then we could find a vector $f = b_1e_1 + b_2e_2 + e_3$ such that Γ leaves $\boldsymbol{C}f$ invariant. Putting $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ it would then follow that $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ conjugates Γ into (block) diagonal form. According to (7) we must then have $b = (w - 1)^{-1}c$ for each nontrivial element (6) of Γ. But this contradicts (8), whence our claim.

REMARK 1.11. The closure G of Γ is the full group

$$\begin{bmatrix} SU_2(\boldsymbol{C}) & \boldsymbol{C}^2 \\ 0 & 1 \end{bmatrix}.$$

I owe the following proof of this to Serre. Since each element $\neq 1$ in Γ is conjugate to an element of infinite order in $SU_2(\boldsymbol{C})$, it generates a dense subgroup of a circle group in G; it follows that G is connected. It's projection in $SU_2(\boldsymbol{C})$ is connected and nonsolvable, since it contains the free group Γ'. The proper connected subgroups of $SU_2(\boldsymbol{C})$ are abelian, so G projects onto $SU_2(\boldsymbol{C})$. Let g be the Lie algebra of G and n the kernel of the Lie algebra projection $p \colon g \to su_2(\boldsymbol{C})$. If $n = 0$ then g is conjugate to $su_2(\boldsymbol{C})$ so G is conjugate to $SU_2(\boldsymbol{C})$, contrary to what we proved above. Therefore n is a (real) vector space $\neq 0$ in $\boldsymbol{C}^2$. Since p above is surjective, n is stable under $su_2(\boldsymbol{C})$, which acts irreducibly on $\boldsymbol{C}^2$. It follows that $n = \boldsymbol{C}^2$, so

$$g = \begin{bmatrix} su_2(\boldsymbol{C}) & \boldsymbol{C}^2 \\ 0 & 0 \end{bmatrix},$$

whence the result.

2. Absolutely irreducible monoids of integral character.

NOTATION 2.1. As in 1.1, K is an algebraically closed field and Γ is a multiplicative submonoid of $M_n(K)$. We further assume that Γ acts *irreducibly* on K^n, so that the Burnside Lemma 1.2 furnishes a K-basis $s_1, \cdots, s_{n^2}$ in Γ of $M_n(K)$, and the dual basis $t_1, \cdots, t_{n^2}$: $\mathrm{Tr}\,(t_is_j) = \delta_{ij}(i, j = 1, \cdots, n^2)$. For any subring A of K we write $A\Gamma$ for the sub A-module of $M_n(K)$ generated by Γ: $A\Gamma$ is an A-algebra.

The next result records some more or less standard facts. We shall draw some simple consequences to be applied later.

PROPOSITION 2.2. *Let A be a subring of K with field of fractions F. Assume that*

$$\text{(1)} \qquad \operatorname{Tr}(\Gamma) \subset A .$$

(a) *We have*

$$\sum_i As_i \subset A\Gamma \subset \sum_i At_i$$

where the right and left hand terms are free A-modules of rank n^2.

(b) *The natural homomorphism $K \otimes_A A\Gamma \to K\Gamma = M_n(K)$ is bijective* (i.e., *$A\Gamma$ is an "A-structure" on $M_n(K)$*).

(c) *$F\Gamma$ is a central simple F-algebra of dimension n^2, hence isomorphic to $M_r(D)$ where D is a division algebra of dimension s^2 over the center F, with $rs = n$. (The integer s is called the Schur index of the representation of Γ on K^n.)*

(d) *There is an extension E of degree s of F in K, isomorphic to a maximal subfield of D, such that $E\Gamma(\cong E \otimes_F F\Gamma)$ is isomorphic to the E-algebra $M_n(E)$.*

(e) *$E\Gamma$ is conjugate in $M_n(K)$ to $M_n(E)$.*

Assertion (a) is immediate from 1.2 (b), and it clearly implies (b). Assertion (b) applied to $A = F$ implies that $F\Gamma$ is a central simple F-algebra of dimension n^2. The rest of (c) follows from Wedderburn structure theory. A maximal subfield E of D can be embedded in K as an extension of F. Such an E has degree s over F and splits the F-algebra $F\Gamma \cong M_r(D)$ (cf. [9], Th. 68.6) whence (d). We now have an E-algebra isomorphism $f\colon M_n(E) \to E\Gamma \subset M_n(K)$, as well as the inclusion $g\colon M_n(E) \to M_n(K)$. Tensoring with K produces two K-algebra isomorphisms f' and g' from $M_n(K)$ to $M_n(K)$. They define $M_n(K)$-module structures on K^n which must clearly be isomorphic, i.e., f' is conjugate to g', whence (e).

COROLLARY 2.3. *Let A be a neotherian subring of K with field of fractions F. Assume that the integral closure of A in any finite extension of F is a finitely generated A-module. Assume further that*

$$\text{(2)} \qquad \operatorname{Tr}(\Gamma) \text{ is integral over } A$$

(i.e., *that each element of* $\operatorname{Tr}(\Gamma)$ *is so) and that*

(3) $\operatorname{Tr}(\Gamma) \subset L$ *for some finitely generated field extension L of F.*

Then $A\Gamma$ is a finitely generated A-module.

The algebraic closure E of F in L is a finite extension of F (cf. Lemma 3.7 below) so the integral closure B of A in E (or L) is, by hypothesis, a finitely generated A-module. Conditions (2) and (3) imply that $\mathrm{Tr}(\Gamma) \subset B$, so 2.2 (a) implies that $B\Gamma$ is contained in a finitely generated B-module. Since A is noetherian it follows that $A\Gamma \subset B\Gamma$ is a finitely generated A-module.

REMARK 2.4. In practice condition (3) is typically assured by having $\Gamma \subset M_n(L)$. This is the case (for suitable L) whenever Γ is contained in a finitely generated sub A-algebra of $M_n(K)$. The corollary applies notably when A is a finite field, and then implies that Γ is finite.

COROLLARY 2.5. *In Proposition* 2.2 *suppose that* F *is a finite extension of the prime field of* K. *Then there is a finite extension* L *of* F *in* K *such that, if* B *is the integral closure of* A *in* L, *then* $s\Gamma s^{-1} \subset M_n(B)$ *for some* $s \in GL_n(K)$.

If $\mathrm{char}(K) > 0$ then $A = F$, a finite field, and the corollary is just part (e) of 2.2. Assume therefore that F is a finite extension of $\boldsymbol{Q}$. By 2.2 (e) we may, after a conjugation, assume that $\Gamma \subset M_n(E)$ for some finite extension E of F. Let C denote the integral closure of A in E. Then C is the ring of S-integers of E, where S is a (possibly infinite) set of primes of E containing all archimedean primes. In particular C is a ring of fractions of the ring of algebraic integers of E so it is a Dedekind ring and, by 2.2 (a), $C\Gamma$ is a C-order in the E-algebra $M_n(E)$. Therefore Γ leaves invariant a finitely generated C-module $P \subset E^n$ which contains an E-basis of E^n. There is a finite extension L of E (for example the Hilbert class field of E) such that all ideals of C become principal in the integral closure B of C in L. Then the B-module $B \otimes_C P \subset L^n$ is free. Choosing a B-basis of $B \otimes_C P$ produces the desired conjugation of Γ into $M_n(B)$.

Elementary matrices 2.6. Let e_{ij} denote the matrix with 1 in the (i, j)-coordinate and zero elsewhere. If $i \neq j$ then $e_{ij}^2 = 0$ so we have the group homomorphism

$$a \longmapsto e_{ij}^a = I + ae_{ij}$$

from K to $SL_n(K)$.

COROLLARY 2.7. *Let* N *be an additive subgroup of* K *such that* $e_{12}^N \subset \Gamma$. *Let* A *be a subring of* K *containing* $\mathrm{Tr}(\Gamma)$. *Then there is an element* $c \neq 0$ *in* K *such that* $cN \subset A$. *If* A *has transcendence degree* d *over a subfield* K_0 *of* K *then the field* $K_0(N)$ *has trans-*

cendence degree $\leqq d+1$ *over* K_0.

The last assertion follows from the first one in view of the field inclusion $K_0(N) \subset K_0(A, c)$. If $e_{12}^N \subset \Gamma$ then $Ne_{12} \subset A\Gamma$. There is nothing to prove if $N = \{0\}$, so choose $a \neq 0$ in N. Then $f = ae_{12}$ belongs to $A\Gamma$, and $Na^{-1}f \subset A\Gamma$. Write $f = \sum_i a_i t_i$ as in 2.2 (a), with $a_i \in A(i = 1, \cdots, n^2)$. Then we have $Na^{-1}a_i \subset A$ for each i. Choosing i so that $a_i \neq 0$ we can take $c = a^{-1}a_i$ in the corollary.

PROPOSITION 2.8. *Let A be a subring of K whose field of fractions F is a finite extension of the prime field of K. Assume that*

(2) $\qquad\qquad\qquad \mathrm{Tr}\,(\Gamma)$ *is integral over* A.

Let G be the normalizer of Γ in $SL_n(K)$. Then G is integral over A.

Let $s \in G$, i.e., $s\Gamma s^{-1} = \Gamma$ and $\det(s) = 1$. We must show that s is integral over A. Choose a finitely generated submonoid Γ_1 of Γ containing a K-basis of $M_n(K)$, and let Γ' denote the submonoid of Γ generated by $\bigcup_{m \in Z} s^m \Gamma_1 s^{-m}$. Then Γ' acts irreducibly on K^n, $s\Gamma' s^{-1} = \Gamma'$ and $\Gamma' \subset M_n(L)$ for some finitely generated field extension L of F. Replacing Γ by Γ' therefore, we are allowed to add condition (3) of 2.3 to our assumptions. We may enlarge A to its integral closure, so that A is a ring of fractions of the integral closure of Z in F. It follows that A is an "excellent ring" (see [11], § 34) and, in particular, satisfies the hypothesis of 2.3. We are now entitled to conclude from Corollary 2.3 that $A\Gamma$ is a finitely generated A-module, hence so also is $\mathrm{Tr}\,(A\Gamma)$. Now enlarging A to the integral closure of $A[\mathrm{Tr}\,(\Gamma)]$, we may assume further that $\mathrm{Tr}\,(\Gamma) \subset A$ (condition (1) of 2.2). Then, in view of Corollary 2.5, we may enlarge A again to its integral closure in a finite extension of F, and conjugate Γ (and s), so as to arrange that $\Gamma \subset M_n(A)$. Then $A\Gamma$ is an A-order in $M_n(F)$. If $\mathrm{char}\,(K) = p > 0$ then $A = F$ is a finite field so $A\Gamma$ is finite, and its centralizer has finite index in its normalizer. The centralizer of $A\Gamma$ consists of scalars, and the scalars in $SL_n(K)$ form a finite group, so G is finite in this case, hence integral over Z. Suppose therefore that $\mathrm{char}\,(K) = 0$, so that F is a number field and A is a ring of fractions of the ring of algebraic integers in F. It follows that A-orders in semi-simple F-algebras satisfy the Jordan-Zassenhaus theorem (see [1], Ch. X, Th. 2.4). Therefore, as in the proof of Th. 2.9 of [1], Ch. X, one concludes that the group In Aut$\,(A\Gamma)$ of inner automorphisms of $A\Gamma$ has finite index in Aut $(A\Gamma)$. The normalizer G of Γ in $SL_n(K)$ maps naturally to Aut $(A\Gamma)$, and the inverse image G_1 of In Aut (Γ) has finite index in G. Hence some power $t = s^q$ of s yields an inner automorphism of $A\Gamma$, in other words there is a unit u of $A\Gamma$ such that $uxu^{-1} = t \times t^{-1}$ for all $x \in A\Gamma$. Since the centralizer of $A\Gamma$ is K we have $t = wu$ for some $w \in K$. But $1 = \det(t) = w^n \det(u)$. Since $A\Gamma \subset M_n(A)$ we have $w^n = \det(u)^{-1} \in A$ so w is integral over A. Hence $t = wu$ is integral over A, and so also is s (because $s^q = t$); whence the proposition.

COROLLARY 2.9. *Let A and Γ be as in 2.8. Let G be a subgroup of $GL_n(K)$ that normalizes Γ and such that G^{ab} is a torsion group. Then G is integral over A.*

Let $G_1 = G \cap SL_n(K)$. Then G_1 is integral over A by Proposition 2.8. Since G/G_1 is an abelian group the hypothesis implies that every $s \in G$ has some positive power in G_1, so s is integral over A.

3. A finiteness theorem.

NOTATION 3.1. Let B be a commutative ring and let Γ be a multiplicative submonoid of $M_n(B)$ which contains a set $s_1, \cdots, s_m$ which generates $M_n(B)$ as a B-module. Let A be a subring of B, and let $A\Gamma$ denote the sub A-module of $M_n(B)$ generated by Γ; it is an A-algebra.

Our aim is to show, under suitable finiteness assumptions, that $A\Gamma$ is a finitely generated A-module. The proof is a slight refinement of arguments in Procesi [12], Ch. VI, but the formulation below is more convenient for our applications.

We begin with an integral form of the Burnside lemma.

LEMMA 3.2. (a) *There exist elements $t_1, \cdots, t_m \in M_n(B)$ such that, for all $s \in M_n(B)$ we have*

$$(1) \qquad s = \sum_i \operatorname{Tr}(ss_i)t_i\,.$$

(b) *If $\operatorname{Tr}(\Gamma) \subset A$ then*

$$\sum As_i \subset A\Gamma \subset \sum_i At_i\,.$$

(c) *If $\operatorname{Tr}(\Gamma) \subset A$ and A is noetherian then $A\Gamma$ is a finitely generated A-module.*

The trace form on the finitely generated free B-module $M_n(B)$ induces an isomorphism from $M_n(B)$ to $\operatorname{Hom}_B(M_n(B), B)$, so (a) follows from [7], p. II. 46, Prop. 12.

The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are immediate, whence the lemma.

THEOREM 3.3. *Suppose A is a noetherian subring of B and that (each element of) $\mathrm{Tr}(\Gamma)$ is integral over A. Suppose further that Γ is a finitely generated monoid. Then $A\Gamma$ is a finitely generated A-module.*

The proof proceeds by several successive reductions.

1. We may assume that *B is a finitely generated A-algebra.* Indeed we may replace B by the sub A-algebra B' of B generated by the matrix entries of the elements of a finite set of generators of Γ, including $s_1, \cdots, s_m$, plus the coefficients in B used to express the basic matrices e_{ij} as linear combinations of $s_1, \cdots, s_m$. Then $\Gamma \subset M_n(B')$ still satisfies our hypotheses, and B' is a finitely generated A-algebra.

2. We may assume that *A is a local ring.* In fact it follows from [12], VI, Lemma 2.4 that $A\Gamma$ is a finitely generated A-module provided that $A_{\mathfrak{p}}\Gamma$ is a finitely generated $A_{\mathfrak{p}}$-module for all primes $\mathfrak{p}$ of A.

3. We may assume that *the local ring A is complete.* In fact the completion $\hat{A}$ of A is a faithfully flat A-module so that $A\Gamma$ is a finitely generated A-module provided that $\hat{A} \otimes_A A\Gamma$ is a finitely generated $\hat{A}$-module ([8], Ch. I, § 3, no. 6, Prop. 11). But we have $\hat{A} \otimes_A A\Gamma \subset \hat{A} \otimes_A M_n(B) = M_n(\hat{B})$ where $\hat{B} = \hat{A} \otimes_A B$, a finitely generated $\hat{A}$-algebra. Further $\hat{A} \otimes_A A\Gamma = \hat{A}\hat{\Gamma}$ where $\hat{\Gamma}$ denotes the image of Γ in $M_n(\hat{B})$ and $\mathrm{Tr}(\hat{\Gamma})$ is integral over $\hat{A}$ since Tr commutes with base change.

4. We may assume that *B is reduced*, i.e., that the nil radical N of B is zero. In fact let A' denote the image of A in $B' = B/N$, and Γ' the image of Γ in $M_n(B')$. We have a commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_n(N) & \longrightarrow & M_n(B) & \longrightarrow & M_n(B') & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & J & \longrightarrow & A\Gamma & \longrightarrow & A'\Gamma' & \longrightarrow & 0 \end{array}$$

where $J = M_n(N) \cap A\Gamma$ is a nilpotent ideal (because B is noetherian). If we know the theorem for: B', which is a reduced finitely generated A'-algebra; A', which is complete local; and Γ', which is finitely generated with $\mathrm{Tr}(\Gamma')$ integral over A', then it follows that $A'\Gamma'$ is a finitely generated A'-module. Let P be any prime ideal of $A\Gamma$. Then p contains the nilpotent ideal J so $A\Gamma/p$, being a quotient of

$A'\Gamma'$, is a finitely generated A-module. It now follows from [12], VI, Lemma 2.6 that $A\Gamma$ is a finitely generated A-module.

5. *The integral closure A_1 of A in B is a finitely generated A-module.* This follows from Lemma 3.4 below.

Now to prove the theorem note that $\mathrm{Tr}(\Gamma) \subset A_1$, and A_1 is noetherian, by (5), so it follows from 3.2 (c) that $A_1\Gamma$ is a finitely generated A_1-module. By (5) again it follows that $A\Gamma \subset A_1\Gamma$ is a finitely generated A-module.

LEMMA 3.4. *Let A be a complete noetherian local ring, let B be a finitely generated commutative A-algebra which is reduced, and let A' be the integral closure of A in B. Then A' is a finitely generated A-module.*

Since B is noetherian and reduced we have $0 = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ where the $\mathfrak{p}_i$ are the minimal primes of B. Let $B_i = B/\mathfrak{p}_i$ and let A_i denote the integral closure of A in B_i. Then clearly $A' \subset \prod_i A_i$ so if each A_i is a finitely generated A-module so also is A'. We may therefore assume that B is an integral domain. Replacing A by its image we may further assume that $A \subset B$. Let $F \subset L$ be the corresponding fields of fractions and let E denote the algebraic closure of F in L. Since L is a finitely generated field extension of F the same is true of the intermediate extensions (Lemma 3.7 below), so E is a finite extension of F. Clearly A' is the integral closure of A in E, so it is a finitely generated A-module by Nagata's theorem ([11], Cor. 2 of Th. 31. C).

COROLLARY 3.5. *Suppose that A is a noetherian subring of B and that $\mathrm{Tr}(\Gamma)$ is integral over A. Then every finitely generated sub A-algebra of $A\Gamma$ is a finitely generated A-module. In particular $A\Gamma$ is integral over A.*

Let X be a finite subset of $A\Gamma$. Let Y be a finite subset of Γ, and Γ' the submonoid of Γ generated by Y. We can choose Y large enough to contain $s_1, \cdots, s_m$ (see 3.1) and so that $X \subset A\Gamma'$. Theorem 3.3 implies that $A\Gamma'$ is a finitely generated A-module. So likewise therefore is the sub A-algebra generated by X.

REMARK 3.6. The results 3.2, 3.3, and 3.5 remain valid if $M_n(B)$ is replaced by any Azumaya B-algebra S of rank n^2, and Tr by the reduced trace. In fact there is a faithfully flat commutative B-algebra B' such that $B' \otimes_B S$ is isomorphic to $M_n(B')$, and this reduces these questions to the case treated above.

We close this section with a lemma used in the proof of 3.4

above, for which we could not locate a convenient reference.

LEMMA 3.7. *Let $F \subset E \subset L$ be fields. If L is a finitely generated extension of F then so also is E. In particular the algebraic closure of F in L is a finite extension of F.*

Let T be a transcendence base of L over F such that $T_0 = T \cap E$ is one of E over F. Then E is algebraic over $F(T_0)$, and of degree at most that of L over $F(T)$, which is finite. Whence the lemma.

4. Integral PI algebras and integral n-representation type. We first recall some terminology from Procesi [12].

Polynomial identities 4.1. Let A be a commutative ring, and let R be an A-algebra. A polynomial $f(X_1, \cdots, X_n)$ in non commuting indeterminates and coefficients in A is called a *polynomial identity* of R if $f(x_1, \cdots, x_n) = 0$ for all $(x_1, \cdots, x_n) \in R^n$. Write $c(f)$ for the ideal of A generated by the coefficients of f. One calls R a *PI A-algebra* if it satisfies a polynomial identity f such that $c(f) = A$. It then follows from a theorem of Amitsur ([12], II, Th. 4.1) that R satisfies an identity of the form s_n^m for some integers $n, m \geqq 1$, where

$$s_n(X_1, \cdots, X_n) = \sum_\sigma \operatorname{sgn}(\sigma) X_{\sigma(1)} \cdots X_{\sigma(n)} , \tag{1}$$

with σ ranging over all permutations of $\{1, \cdots, n\}$. The ring $M_n(A)$ satisfies the "standard identity" s_{2n} of n by n matrices; this is the well known theorem of Amitsur-Levitzky ([12], I, Th. 5.2).

Central extensions 4.2. A ring homomorphism $\rho: R \to S$ is called an *extension* if S, as R-module via ρ, is generated by the centralizer $Z_R(S)$ of R in S: $Z_R(S) = \{s \in S \mid s\rho(r) = \rho(r)s$ for all $r \in R\}$. If S, as R-module, is even generated by the center $Z(S) = Z_S(S)$ of S then ρ is called a *central extension.*

Absolutely irreducible representations 4.3. Let R be a ring. A (matrix) *representation* (of dimension n over a commutative ring K) is a ring homomorphism $\rho: R \to M_n(K)$. The *character* of ρ is $\chi_\rho = \operatorname{Tr} \circ \rho: R \to K$. The center of $M_n(K)$ consists of the scalars K, so ρ is a central extension if and only if $\rho(R)$ generates $M_n(K)$ as a K-module. In this case ρ induces a homomorphism from the center $Z(R)$ to K. Hence if R is an A-algebra for some commutative ring A then ρ induces a homomorphism from A to K relative to which ρ and χ_ρ are A-linear. We call ρ an *absolutely irreducible* *representation* if ρ is a central extension and K is a field. Then $\rho(R)$ contains a K-basis of $M_n(K)$, and $\rho(R)$ acts irreducibly on L^n for all field extensions L of K. Indeed these conditions are each equivalent to the absolute irreducibility of ρ, by the Burnside lemma 1.2.

The following theorem slightly elaborates some results of Procesi ([12], Ch. VI).

THEOREM 4.4. *Let A be a commutative noetherian ring, and let R be a PI A-algebra. The following conditions are equivalent.*

(a) *For every absolutely irreducible representation $\rho: R \to M_n(K)$ of R, $\chi_\rho(R)$ (in K) is integral over A.*

(b) *Every finitely generated sub A-algebra of R is a finitely generated A-module.*

(c) *(Each element of) R is integral over A.*

(d) *R is generated as an A-module by elements which are integral over A.*

The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a) are evident. Consider the supplementary condition:

(f) R is a finitely generated A-algebra.

We shall prove simultaneously that (a) $\Rightarrow$ (c) and that (a)+(f) implies

(b′) R is a finitely generated A-module.

To prove (c) it suffices, by [12], VI, Lemma 2.3, to do so for each R/P with P a prime ideal of R. Similarly, assuming (f), then (b′) follows once it is known for each R/P, by [12], VI, Lemma 2.6. Therefore we may assume for both implications that R is a prime ring. By [12], II, Th. 3.2 there is then an embedding $R \subset M_n(K)$ which is an absolutely irreducible representation. Replacing A by its image we may assume that A is a subring of K. Let X be any finite subset of R containing a K-basis of $M_n(K)$, and let Γ denote the multiplicative monoid generated by X. Then (a) implies that $\mathrm{Tr}(\Gamma)$ is integral over A, so Theorem 3.3 implies that $A\Gamma$ is a finitely generated A-module. It is clear that both implications (a) $\Rightarrow$ (c) and (a) + (f) $\Rightarrow$ (b′) follow from this. Finally, to prove (a) $\Rightarrow$ (b) let R' be a finitely generated sub A-algebra of R. The implication (a) $\Rightarrow$ (c) shows, assuming (a), that R' is integral over A. Therefore the implication (a) + (f) $\Rightarrow$ (b′) shows that R' is a finitely generated A-module, whence (b).

DEFINITION 4.5. Let A be a commutative ring, let R be an A-algebra, and let n be an integer $\geqq 1$. We say that R has *integral n-representation type* (over A) if, for all absolutely irreducible representations $\rho: R \to M_m(K)$ of dimension $m \leqq n$, $\chi_\rho(R)$ is integral over A.

COROLLARY 4.6. *Let A be a commutative noetherian ring and let R be an A-algebra of integral n-representation type. Let h be an integer $\geqq 1$, and let J denote the ideal of R generated by all elements $s_{2n}(x_1, \cdots, x_{2n})^h$ with $(x_1, \cdots, x_{2n}) \in R^{2n}$. Then every finitely generated sub A-algebra of R/J is a finitely generated A-module.*

The absolutely irreducible representations of R/J correspond bijectively with those of R of dimension $\geqq n$ (see [12], II, Prop. 7.6 and III, Prop. 2.2). Hence R/J is a *PI* A-algebra satisfying condition (a) of 4.4, therefore also condition (b). Whence the corollary.

The next proposition gives a convenient "geometric" picture of the absolutely irreducible representations of an A-algebra finitely generated as an A-module.

PROPOSITION 4.7. *Let A be a commutative noetherian ring and let R be an A-algebra finitely generated as an A-module. There exists a finite family of central extensions (see 4.2) $\rho_i: R \to M_{n_i}(B_i)$ $(i \in I)$ with the following properties.*

(1) For each $i \in I$, B_i is an integral domain which is a finitely generated A_{s_i}-module for some $s_i \in A$ such that $\rho_i(s_i) \neq 0$.

(2) For any absolutely irreducible representation $\rho: R \to M_n(K)$ with K an algebraically closed field, there is an $i \in I$ such that $n_i = n$ and an A-algebra homomorphism $\sigma: B_i \to K$ such that ρ is conjugate to the composite $R \xrightarrow{\rho i} M_n(B_i) \xrightarrow{\sigma} M_n(K)$.

If $A = \boldsymbol{Z}$ then we may arrange that, for each $i \in I$, $\rho_i(R) \subset M_{n_i}(A_i)$, where A_i is the integral closure of A in B_i, and is a finitely generated A-module.

Let $P \in \operatorname{spec}(R)$. By [12], II, Th. 3.2, P is the kernel of an absolutely irreducible representation $\rho_P: R \to M_{n_P}(K_P)$. By Proposition 2.2 we may take K_P to be a *finite* extension of the field of fractions of $\rho_P(A)$. Then we can choose a fintely generated sub A-albebra B' of K_P such that $\rho_P(R) \subset M_{n_P}(B')$ and $\rho_P(R)$ generates $M_{n_P}(B')$ as a B'-module. For some $s_P \in A$ such that $\rho_P(s_P) \neq 0$ the A_{s_P}-algebra $B_P = B'_{s_P}$ will be a finitely generated A_{s_P}-module. In case $A = \boldsymbol{Z}$ then it follows from Corollary 2.5 that, after extending K_P if necessary, we may further arrange that $\rho_P(R) \subset M_{n_P}(A_P)$, where A_P is the integral closure of A in B_P, and is a finitely generated A-module.

The finite central extension $\rho(R_{s_P}) \subset M_{n_P}(B_P)$ induces a map spec

$(M_{n_P}(B_P)) \to \operatorname{spec}(R_{s_P})$ which, by Lemma 4.9 below, is surjective. Hence the image of ρ_P^*: $\operatorname{spec}(M_{n_P}(B_P)) \to \operatorname{spec}(R)$ is the locally closed set

$$U_P = V(P) - V(s_P R) ,$$

which is a neighborhood of P in $V(P)$. (For $X \subset R$ we write $V(X)$ for the set of primes Q in $\operatorname{spec}(R)$ which contains X; these are the closed sets in $\operatorname{spec}(R)$.) The proposition now clearly follows from the next two claims.

1. $\operatorname{spec}(R)$ is covered by a finite number of the sets U_P.
2. Let $\rho: R \to M_n(K)$ be an absolutely irreducible representation with K algebraically closed and with kernel $Q \in U_P$. Then $n_P = n$ and there is an A-algebra homomorphism $\sigma: B_P \to K$ such that ρ is conjugate to the composition $R \xrightarrow{\rho_P} M_n(B_P) \xrightarrow{\sigma} M_n(K)$.

We prove (1) by noetherian induction. Specifically, let $P_i (i \in I_0)$ be the minimal primes of R (i.e., the generic points of the irreducible components of $\operatorname{spec}(R)$). Then $\bigcup_{i \in I_0} U_{P_i}$ contains an open dense set $\operatorname{spec}(R) - V(J_1)$ of $\operatorname{spec}(R)$, where J_1 is some ideal of R. Next let $P_i (i \in I_1)$ be the generic points of the irreducible components of of $V(J_1)$. Then $\bigcup_{i \in I_0 \cup I_1} U_{P_i}$ contains $\operatorname{spec}(R) - V(J_2)$, where $V(J_2) \subset V(J_1)$ and $V(J_2)$ contains no generic point of $V(J_1)$. Continuing in this way, one exhausts $\operatorname{spec}(R)$ in a finite number of steps because $\operatorname{spec}(R)$ is noetherian.

To prove (2) we first write $Q = \operatorname{Ker}(\rho) = \rho_P^{-1}(P')$ for some prime $P' \in \operatorname{spec}(M_{n_P}(B_P))$. All primes of $M_{n_P}(B_P)$ come from B_P, so P' is the kernel of $M_{n_P}(B_P) \to M_{n_P}(B')$ for some quotient B' of B_P. Then the composite $\rho': R \to M_{n_P}(B_P) \to M_{n_P}(B')$ is a central extension with the same kernel Q as ρ. Passing to the field of fractions of B' to make ρ' an absolutely irreducible representation, and then applying [12], II, Th. 7.4, we see that $n_P = n$ and we obtain an A-algebra embedding σ' of B' into some field extension L of K such that the composites $R \xrightarrow{\rho'} M_n(B') \xrightarrow{\sigma'} M_n(L)$ and $R \xrightarrow{\rho} M_n(K) \hookrightarrow M_n(L)$ are conjugate. Since $\rho'(s_P) \neq 0$, B' is algebraic over the field of fractions of $\rho'(A)$, so we must have $\sigma'(B') \subset K$ (since K is algebraically closed). Now the composite $\sigma: B_P \to B' \xrightarrow{\sigma'} K$ satisfies claim 2.

REMARK 4.8. The refinement of Proposition 4.7 made when $A = \boldsymbol{Z}$ applies, more generally, when A is a ring of S-integers in a global field.

LEMMA 4.9. *Let A be a commutative ring, let R be a PI A-algebra, and let $R \subset S$ be a central extension (see 4.2). If $Z(S)$ is integral over R then S is integral over R, and* $\operatorname{spec}(S) \to \operatorname{spec}(R)$

is surjective.

[We say that $s \in S$ is integral over R if there exist elements $r_0, \cdots, r_{n-1}$ in R such that $r_0 + \cdots + r_{n-1}s^{n-1} + s^n = 0$.] Lemma 4.9 is proved by Shelter in [13], Cor. of Lemma 1 and Theorem 1.

PROPOSITION 4.10. *Let $R \subset S$ be a central extension of nonzero rings, and let n be an integer $\geqq 1$. Consider the conditions:*

(a) *R is an Azumaya algebra of rank n^2.*

(b) *S is an Azumaya algebra of rank n^2.*

Then (a) $\Rightarrow$ (b) *and, if $Z(S)$ is integral over R,* (b) $\Rightarrow$ (a).

According to M. Artin's theorem (see [12], Ch. VII, Th. 3.3), (a) is equivalent to (i) and (ii) below:

(i) R satisfies the identities of n by n matrices.

(ii) No nonzero quotient of R satisfies the identities of $(n-1)$ by $(n-1)$ matrices.

The implication (a) $\Rightarrow$ (b) is clear from the fact that (a) implies that $S = R \otimes_{Z(R)} Z_R(S)$, and this is a central extension only if $Z_R(S) = Z(S)$. Alternatively, one can appeal to [12], VII, Prop. 3.2. To prove (b) $\Rightarrow$ (a) when $Z(S)$ is integral over R it suffices, in view of Lemma 4.9, to show that (b) $\Rightarrow$ (a) whenever $\operatorname{spec}(S) \to \operatorname{spec}(R)$ is surjective. Clearly (i) for S implies (i) for R. To verify (ii) for R it suffices to show that, for any $P \in \operatorname{spec}(R)$, R/P does not satisfy the identities of $(n-1)$ by $(n-1)$ matrices. By assumption $P = R \cap Q$ for some $Q \in \operatorname{spec}(S)$. Then $R/P \subset S/Q$ is a central extension so, if R/P satisfies the $(n-1)$-matrix identities so also does S/Q, by [12], VII, Prop. 3.2 again. But this contradicts (ii) for S; whence the proposition.

5. Groups of integral n-reprezentation type.

DEFINITION 5.1. Let Γ be a group, n an integer $\geqq 1$, and A a commutative ring. We say that Γ has *integral n-representation type* over A if the group algebra $A[\Gamma]$ has this property, in the sense of 4.5. Equivalently, Γ satisfies:

(a) For every absolutely irreducible representation $\rho: \Gamma \to GL_m(K)$, where $m \leqq n$ and the field K is an A-algebra, $\chi_\rho(\Gamma)$ is integral over A.

By taking Jordan-Hölder series, and adding trivial representations to increase the dimension to n, if necessary, we see that (a) is equivalent to:

(b) For every representation $\rho: \Gamma \to GL_n(K)$, where K is a field which is an A-algebra, $\chi_\rho(\Gamma)$ is integral over A.

The same then clearly holds for any commutative A-algebra A' and for any quotient Γ' of Γ. Moreover, this is a property inherited by filtered inductive limits of groups. When these conditions hold for all $n \geqq 1$ we say Γ has *integral representation type* over A. When no specific ring A is mentioned it shall be understood that $A = \boldsymbol{Z}$. We shall be primarily interested in the cases when A is $\boldsymbol{Z}$ or a prime field.

PROPOSITION 5.2. *Suppose that the group Γ has integral n-representation type over a commutative noetherian ring A. Let ρ: $\Gamma \to GL_m(K)$ be a representation where $m \leqq n$ and the field K is an A-algebra. For any finite subset X of Γ, the sub A-algebra of $M_m(K)$ generated by $\rho(X)$ is a finitely generated A-module. In particular $\rho(\Gamma)$ is integral over A. If A is a finite field then $\rho(X)$ generates a finite group.*

This follows from (the implication (a) $\Rightarrow$ (b) of) Theorem 4.4.

PROPOSITION 5.3. *Let Γ be a finitely generated group of integral n-representation type (over $\boldsymbol{Z}$). There exists a finite family of representations ρ_i: $\Gamma \to GL_{n_i}(A_i) (i \in I)$ with the following properties.*

(1) *For each $i \in I$ we have $n_i \leqq n$ and A_i is an integral closed domain finitely generated as a $\boldsymbol{Z}$-module (hence either a ring of algebraic integers or a finite field). Moreover ρ_i is absolutely irreducible over the field of fractions of A_i.*

(2) *If ρ: $\Gamma \to GL_m(K)$ is an absolutely irreducible representation of Γ with $m \leqq n$ and K is an algebraically closed field then there is an $i \in I$ such that $n_i = m$ and such that there is a homomorphism σ: $A_i \to K$ such that ρ is conjugate to the composite $\Gamma \xrightarrow{\rho_i} GL_m(A_i) \xrightarrow{\sigma} GL_m(K)$.*

Let $R = \boldsymbol{Z}[\Gamma]/J$ where J is the ideal generated by all $s_{2n}(x_1, \cdots, x_{2n})$ with $(x_1, \cdots, x_{2n}) \in \boldsymbol{Z}[\Gamma]^{2n}$ (cf. 4.1). The absolutely irreducible representations of dimension $\leqq n$ of Γ are equivalent to those of the *PI* $\boldsymbol{Z}$-algebra R, and Corollary 4.6 implies that R is a finitely generated $\boldsymbol{Z}$-module. Hence the proposition follows from Proposition 4.7 applied to R.

REMARKS 5.4. 1. Proposition 6.3 expresses a strong "rigidity" property of the absolutely irreducible representations of Γ of dimension $\leqq n$. For example Card (I) bounds the number of such representations over any algebraically closed field, of any characteristic.

2. As is evident from its proof, Proposition 5.3 admits an analogue for any commutative noetherian ring A in place of $\boldsymbol{Z}$. For example when A is a prime field of characteristic p it asserts that Γ has only finitely many isomorphism classes of absolutely irreducible representations of dimension $\leqq n$ in characteristic p, and that each of them is definable over a finite extension of A. Since these classes of representations can, in general, be formed into an algebraic variety V over A (cf. [12], IV, Th. 1.8), the preceeding conclusion can be interpreted as saying that $\dim V = 0$.

PROPOSITION 5.5. *The following conditions on a group Γ are equivalent.*

(a) *Γ has integral 1-representation type over some commutative ring $A \neq 0$.*

(b) *The abelianization Γ^{ab} of Γ is a torsion group.*

Obviously (b) $\Rightarrow$ (a). Suppose, on the other hand, that some $s \in \Gamma$ has image in Γ^{ab} of infinite order. Let F be a residue class field of A, X an indeterminate, and K an algebraic closure of $F(X)$. Since $K^\times = GL_1(K)$ is a divisible group there is a linear character $\chi\colon \Gamma \to K^\times$ such that $\chi(s) = X$, an element not integral over A; whence the proposition.

Subgroups of finite index 5.6. Let Γ be a group and Γ' a subgroup of finite index r. Let A be a commutative noetherian ring, and let K be a field which is an A-algebra. If $\rho\colon \Gamma \to GL_n(K)$ is a representation and $\rho(\Gamma')$ is integral over A, then so also is $\rho(\Gamma)$, since each $s \in \Gamma$ has some positive power in Γ'. In view of Proposition 5.2, this proves assertion (i) of Proposition 5.7 below.

Let $\rho'\colon \Gamma' \to GL_n(K)$ be a representation. We then have an induced representation

$$\rho = \mathrm{Ind}_{\Gamma'}^{\Gamma}(\rho')\colon \Gamma \longrightarrow GL_{nr}(K)\ .$$

For $s \in \Gamma$ we have

$$\chi_\rho(s) = \sum_{t \in \Gamma/\Gamma'} \chi^0_{\rho'}(t^{-1}st)$$

where $\chi^0_{\rho'}$ is the extension of $\chi_{\rho'}$ to Γ vanishing on $\Gamma - \Gamma'$. In particular $\chi_\rho(s)$ is a $\boldsymbol{Z}$-linear combination of the values of $\chi_{\rho'}$. If $\rho'(\Gamma')$ is integral over A so also is $\rho(\Gamma)$. In fact Γ' contains a subgroup Γ_1 which is normal and of finite index in Γ, so it suffices to show that $\rho(\Gamma_1)$ is integral over A. If $s \in \Gamma_1$ then $t^{-1}st \in \Gamma_1 \subset \Gamma'$ for all $t \in \Gamma$, so the decomposition $K^{nr} = \bigotimes_{t \in \Gamma/\Gamma'} t \otimes K^n$ is stable under $\rho(s)$, and $\rho(s)$ acts on $t \otimes K^n$ like $\rho'(t^{-1}st)$ acts on K^n; whence the integrality of $\rho(s)$. Conversely, if $\rho(\Gamma)$ is integral over A then, since ρ' is a subrepresentation of $\rho \mid \Gamma'$, $\rho'(\Gamma')$ is likewise integral over A. Using Proposition 5.2 again this yields assertion (ii) of Proposition 5.7 which follows.

PROPOSITION 5.7. *Let Γ be a group, Γ' a subgroup of finite index r, n an integer $\geqq 1$, and A a commutative noetherian ring.*

(i) *If Γ' has integral n-representation type over A then so also does Γ.*

(ii) *If Γ has integral nr-representation type over A then Γ' has integral n-representation type over A.*

COROLLARY 5.8. *Γ has integral representation type over A if and only if Γ' has.*

COROLLARY 5.9. *Let Γ be a finitely generated solvable group. If Γ has integral representation type over some commutative ring $A \neq 0$ then Γ is finite.*

Replacing A by one of its residue class fields, we may assume that A is a field, hence noetherian. Proposition 5.5 implies that the derived group Γ' of Γ has finite index in Γ, and Corollary 5.8 then implies that Γ' has integral representation type over A. By induction on the derived length, we conclude that Γ', hence also Γ, is finite.

COROLLARY 5.10. *Let Γ be a group of integral representation type over some commutative ring $A \neq 0$. Then Γ satisfies condition:*

(T Ab) *For every subgroup Γ_1 of finite index in Γ, the group Γ_1^{ab} is torsion.*

If Γ is finitely generated it even satisfies:

(F Ab) *For every subgroup Γ_1 of finite index in Γ, the group Γ_1^{ab} is finite.*

After replacing A by one of its residue class fields, condition (T Ab) results from Corollary 5.8 and Proposition 5.5. If Γ is finitely generated then so also is each Γ_1 as above, whence (F Ab).

PROPOSITION 5.11. *Let $\Gamma = \Gamma_1 \times \Gamma_2$, a direct product of two groups, let n be an integer $\geqq 1$, and let A be a commutative ring. Then Γ has integral n-representation type over A if and only if each Γ_i has.*

In fact the absolutely irreducible representations ρ of Γ decompose as $\rho = \rho_1 \otimes \rho_2$ with ρ_i an absolutely irreducible representation of $\Gamma_i (i = 1, 2)$. Then if $s_i \in \Gamma_i$ we have $\chi_\rho(s_1, s_2) = \chi_{\rho_1}(s_1)\chi_{\rho_2}(s_2)$. The proposition results immediately from this.

EXAMPLES 5.12. 1. Groups Γ of integral 1-representation type are described in Proposition 5.5 as those for which Γ^{ab} is torsion.

2. A substantial class of groups of integral 2-representation type is furnished by Serre's Theorem 6.4 below. It contains all finitely generated groups of integral 1-representation type which are not nontrivial amalgamated free products. A construction of Shalen shows that free products are almost never of integral representation type (Corollary 8.4 below).

3. Let Γ be a group of integral n-representation type over a field K. Then there are only finitely many classes of *completely reducible* n-dimensional representations of Γ over K (Proposition 5.3). However the same need not be true of *all* n-dimensional representations of Γ over K. If $\operatorname{char}(K) = p > 0$ this can be seen already with Γ an elementary p-group of type (p, p). Examples in characteristic zero are given in § 7 below (see 7.9).

4. Groups of integral representation type clearly include all torsion groups. They are stable under passage to sub (or over) groups of finite index (Proposition 5.7), under formation of filtered inductive limits, of direct products (Proposition 5.11), and even of arbitrary group extensions (Corollary 9.9).

5. Many arithmetic groups, for example $SL_n(\boldsymbol{Z})$ with $n \geqq 3$ (but not $n = 2$), are of integral representation type. Such examples are discussed in § 10 below.

6. Groups of integral 2-representation type: Serre's theorem and Shalen's conjecture. Examples of groups of integral 2-representation type are furnished by Theorem 6.4 below of Serre. We use Serre's methods also to prove a conjecture of P. Shalen (Corollary 6.7).

The Property (FA) 6.1. A group Γ is said (by Serre [15]) to have *property* (FA) if, whenever Γ acts (without inversion of edges) on a tree X, then the tree X^Γ of fixed points of Γ is not empty. The group theoretic significance of this is expressed below in terms of "amalgams". We say, again following Serre, that Γ is an *amalgam* if Γ is a free product with amalgamation $\Gamma_0 *_A \Gamma_1$ with $A \neq \Gamma_i (i = 0, 1)$.

THEOREM 6.2 (*Serre,* [15], *Th.* 1). *A group Γ has property*

(FA) if and only if the following three conditions hold.

(a) *Γ has no infinite cyclic quotient.*

(b) *Γ is not an amalgam.*

(c) *Γ is not the union of any chain $\Gamma_1 \subset \cdots \subset \Gamma_n \subset \cdots$ of proper subgroups.*

REMARKS 6.3. 1. Consider the following conditions.

(a′) Γ^{ab} is a torsion group.

(c′) Γ is countable.

(c″) Γ is finitely generated.

It is clear that ((c) + (c′)) $\Leftrightarrow$ (c″) and that ((a′)+(c″)) $\Rightarrow$ (a). Koppelberg and Tits [10] give examples satisfying (a), (b), and (c) but not (c′).

2. It is shown in [3], Th. 3.9, that Γ satisfies (a) and (b) if and only if Γ has the following property (FA'): Whenever Γ acts (without inversion of edges) on a tree X, each element of Γ fixes some vertex of X. Therefore Serre's proof of Proposition 2 in [15] yields the following result.

THEOREM 6.4 (*Serre* [15]). *Let Γ be a group satisfying* (a) *and* (b) *of* 6.2 (i.e., *of type (FA')*). *Let $\rho\colon \Gamma \to GL_2(K)$ be a representation such that $\rho(\Gamma) \subset GL_2(F)$ for some finitely generated extension F of the prime field of K. (The latter is automatic if Γ is finitely generated.) Then for all $s \in \Gamma$ the eigenvalues of $\rho(s)$ are integral over $\mathbf{Z}$. In particular a finitely generated group of type (FA) has integral 2-representation type.*

We shall need the following refinement of this result.

THEOREM 6.5. *Let F be a field finitely generated over its prime field, and let $\bar{F}$ be an algebraic closure of F. Let Γ be a subgroup of $GL_2(F)$, and let Γ_u denote the group generated by all unipotent elements in Γ; Γ_u is a normal subgroup of Γ. Assume that the following conditions hold.*

(a″) *Γ/Γ_u has no infinite cyclic quotient.*

(c) *Γ is not an amalgam.*

Then one of the following cases occurs.

(1) *There is an $s \in GL_2(\bar{F})$ and an integer $m \geqq 1$ such that $s\Gamma s^{-1}$ consists of triangular matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a^m = d^m = 1$. If Γ is either infinite or nonabelian we may choose $s \in GL_2(F)$.*

(2) *Γ acts irreducibly on $\bar{F}^2$, and there is an $s \in GL_2(\bar{F})$ such that $s\Gamma s^{-1} \subset GL_2(A)$, where A is a subring of $\bar{F}$ finitely generated as a $\mathbf{Z}$-module.*

Before proving the theorem we draw some consequences. Note that the ring A in case (2) is either a ring of algebraic integers ($\mathrm{char}\,(F) = 0$) or a finite field ($\mathrm{char}\,(F) > 0$).

COROLLARY 6.6. *In the setting of* 6.5 *assume that* $\mathrm{char}\,(F) = p > 0$. *Assume further that either* (i) Γ *is finitely generated, or that* (ii) Γ *acts irreducibly on* F^2. *Then* Γ *is a finite group.*

In case (2) A is a finite field so the conclusion follows. Assume therefore that we are in case (1). Then condition (ii) implies that $\Gamma_u = \{1\}$, so $\mathrm{card}\,(\Gamma) \leqq m^2$. If (i) holds then also Γ is finite since a finitely generated group of triangular matrices as in (1) is clearly finite in characteristic $p > 0$.

COROLLARY 6.7 (*P. Shalen's conjecture*). *Let* Γ *be a finitely generated subgroup of* $GL_2(\boldsymbol{C})$ *with the following properties.*

(i) *There is a set* $X \subset \boldsymbol{C}$ *of transcendence degree* $\geqq 2$ *over* $\boldsymbol{C}$ *such that the elements* $e(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} (x \in X)$ *belong to* Γ *and they, together with the commutator subgroup* (Γ, Γ) *generate a subgroup of finite index in* Γ.

(ii) Γ *contains a matrix which is not upper triangular. Then* Γ *is an amalgam.*

Suppose, on the contrary, that Γ were not an amalgam, i.e., we have condition (b) of 6.5. Property (i) clearly implies condition (a″) of 6.5 as well. In applying Theorem 6.5 we take for F the field generated by the matrix coefficients of a finite set of generators of Γ. It then follows that we are in case (1) or case (2) of 6.5. We exclude case (1) by showing that Γ acts irreducibly on $\boldsymbol{C}^2$. Indeed if $x \neq 0$ belongs to X then $e(x)$ leaves a unique line L of $\boldsymbol{C}^2$ invariant, and (ii) implies that L is not invariant, under all of Γ. Thus we are in case (2). But then it follows from Corollary 2.7 that x has transcendence degree $\leqq 1$ over $\boldsymbol{Q}$, contradicting (i). This contradiction proves the corollary.

The proof of Theorem 6.5 uses the following more or less well known lemma.

LEMMA 6.8. *Let* F *be a field finitely generated over its prime field* $F_p (p = \mathrm{char}\,(F))$, *let* E *be the algebraic closure of* F_p *in* F, *and let* A *be the integral closure of* $\boldsymbol{Z}$ *in* F (*or* E).

(1) E *is a finite extension of* F_p. *If* $p > 0$ *then* $A = E$. *If* $p = 0$ *then* A *is the ring of algebraic integers of* E.

(2) *Let* V *denote the set of discrete* (*rank* 1) *valuations of* F.

For each $v \in V$ *let* $A_v = \{x \in F \mid v(x) \geqq 0\}$, *the valuation ring of* v. *Then*

$$A = \bigcap_{v \in V} A_v .$$

(3) *Given an integer* $n \geqq 1$, *there is an integer* $m = m(F, n) \geqq 1$ *such that if* w *is a root of unity of degree* $\leqq n$ *over* F *then* $w_m = 1$.

(1) follows from Lemma 3.7 and (3) is proved in [2], § 9, Prop. (A.3).

We prove (2) by induction on d, the transcendence degree of F over F_p. The case $d = 0$ is well known, so suppose that $d \geqq 1$. Let L be an intermediate extension of transcendence degree $d - 1$ over F_p and algebraically closed in F. Then F is the function field of a nonsingular projective algebraic curve x over L. If $f \in F^x$ and $v(f) \geqq 0$ for all valuations v of F over L then f is a rational function on X with no poles, hence a constant, i.e., $f \in \Gamma$. Next note that, by the extension theorem for places, each discrete valuation ring of L is contained in one of F. This remark, plus the induction hypothesis applied to Γ now establishes (2).

6.9. *Proof of* 6.5. Let V be the set of discrete valuations of F, as in 6.8 (2). Let $v \in V$, let A_v be its valuation ring, and let X_v be the tree associated to v as in [14], Ch. II, § 1. The kernel $GL_2(F)^0$ of $v \cdot \det: GL_2(F) \to \boldsymbol{Z}$ acts without inversion on X_v. Hypothesis (a″) implies that $\Gamma \subset GL_2(F)^0$. Each unipotent element of $GL_2(F)$ fixes some vertex of X_v (cf. [14], p. II-11). Hence Γ_u is contained in the subgroup Γ_0 of Γ generated by the stabilizers in Γ of vertices of X_v. The quotient group Γ/Γ_0 is isomorphic to the fundamental group of the quotient graph $\Gamma\backslash X_v$ (cf. [14], Ch. I, § 5). The latter is a free group, so (a″) again implies that it is trivial, i.e., $\Gamma\backslash X_v$ is simply connected, hence a tree. Since Γ is not an amalgam (condition (b)) one now concludes, as in Serre's proof of Theorem 6.3 (cf. also [3], Prop. 3.7), that Γ fixes some vertex of X_v. By Proposition 2 of [14], p. II-11, this implies that Γ is conjugate to a subgroup of $GL_2(A_v)$, so the coefficients of the characteristic polynomials of elements of Γ belong to A_v. This being true for every $v \in V$ it now follows from Lemma 6.8 (2) that these coefficients belong to A, the integral closure of $\boldsymbol{Z}$ in F.

Suppose Γ acts reducibly on $\bar{F}^2$. Then there is an element $s \in GL_2(\bar{F})$ such that every element $t \in s\Gamma s^{-1}$ is upper triangular: $t = \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix}$. The maps $t \mapsto t_{ii}$ are homomorphisms $\Gamma \to \bar{F}^\times$, so (a″) implies that the elements t_{ii} are roots of unity. Being eigenvalues of elements of $GL_2(F)$, they have degree $\leqq 2$ over F. It follows therefore from 6.8 (3) that there is an $m(=m(F, 2))$ such that $t_{ii}^m=1$ for all $t \in s\Gamma s^{-1}$. If Γ is either infinite or nonabelian then the projection $\begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix} \mapsto \begin{pmatrix} t_{11} & 0 \\ 0 & t_{12} \end{pmatrix}$ has nontrivial kernel, so Γ contains a unipotent element $u \neq 1$. Since u leaves a unique line L of $\bar{F}^2$ invariant, and L is defined over F, it follows that we can take s above in $GL_2(F)$. For once sus^{-1} is triangular so must $s\Gamma s^{-1}$ be also. This accounts for case (1).

Suppose finally that Γ acts irreducibly on $\bar{F}^2$. We have $\mathrm{Tr}(\Gamma) \subset A$. It follows therefore from Proposition 2.2 and Corollary 2.5 that there is a finite extension L of the field of fractions E of A, in which A has integral closure B, such that $s\Gamma s^{-1} \subset GL_2(B)$ for some $s \in GL_2(\bar{F})$. This proves Theorem 6.5.

REMARK 6.10. If, in Theorem 6.5, we drop the assumption that F is finitely generated over its prime field, the proof still shows that the coefficients of the characteristic polynomials of elements of Γ belong to $A = \bigcap_{v \in V} A_v$, where V is the set of discrete (rank 1) valuations of F. An example of such a case is $\Gamma = SL_2(\boldsymbol{Z}_p)$ in $GL_2(\boldsymbol{Q}_p)$. According to [3], Th. 5.2 any profinite group (like Γ) satisfies conditions (a) and (b) of Theorem 6.3.

Problem 6.11. *Characterize finitely generated groups of integral 2-representation type in purely group theoretic terms.*

In order that a finitely generated group Γ be of integral 2-representation type it is necessary that Γ satisfy

(a) Γ^{ab} is finite

(Proposition 5.5), and it suffices that Γ satisfy (a) and

(b) Γ is not an amalgam

(Theorem 6.4). A solution to the above problem might therefore be sought by attempting to characterize intrinsically the kinds of amalgams that arise from actions of $GL_2(F)$ on trees X_v as in the proof of Theorem 6.5.

A finitely generated group of integral 2-representation type which does not satisfy (b) can be obtained by taking a free product $\Gamma = \Gamma_1 * \Gamma_2$ of finite groups Γ_i neither of which have nontrivial linear representations of dimension 2 over any field, e.g., $\Gamma_1 = \Gamma_2 = SL_3(\boldsymbol{F}_5)$. It follows from Proposition 8.3, below, that such a Γ has faithful linear representations in any characteristic.

7. Some calculations of 2-dimensional representations.

NOTATION 7.1. Γ_p denotes a cyclic group of prime order p with generator s, and N denotes a Γ_p-module which is a finitely generated free abelian group and such that

$$(1) \qquad sx = x \Longrightarrow x = 0, \text{ for } x \in N .$$

This implies that N is a torsion free module over the ring

$$R = \boldsymbol{Z}[\Gamma_p]/(1 + s + \cdots + s^{p-1})\boldsymbol{Z}[\Gamma_p]$$

of pth cyclotomic integers; say N is of rank m over R. We propose to classify two dimensional complex representations of the semi-direct product

$$\Gamma = N \times \Gamma_p$$

7.2. Γ^{ab} is isomorphic to $H_0(\Gamma_p, N) \times \Gamma_p$ which, in view of (1), is an elementary p-group of order p^{m+1}. Hence *Γ has integral 1-representation type,* (Proposition 5.5). Since N does not have integral 1-representation type (if $N \neq 0$) it follows from Proposition 5.7 that *Γ does not have integral p-representation type.*

Diagonal representations 7.3. They are of the form

$$\delta_{\chi,\chi'}(t) = \operatorname{diag}(\chi(t), \chi'(t))$$

where $\chi, \chi'\colon \Gamma \to \boldsymbol{C}^\times$ are linear characters. In view of 7.2 there are only finitely many such. Clearly $\delta_{\chi,\chi'}$ and $\delta_{\chi_1,\chi_1'}$ are isomorphic if and only if $\{\chi, \chi'\} = \{\chi_1, \chi_1'\}$.

7.4. Let $\chi\colon \Gamma_p \to \boldsymbol{C}^\times$ be a linear character, thus making $\boldsymbol{C}$ into a Γ_p-module, which we denote $\boldsymbol{C}_\chi$. Let $\alpha\colon N \to \boldsymbol{C}_\chi$ be a Γ_p-homomorphism, i.e., $\alpha(x + y) = \alpha(x) + \alpha(y)$ and $\alpha(tx) = \chi(t)\alpha(x)$ for $x, y \in N$ and $t \in \Gamma_p$. Define $\rho_{\chi,\alpha}\colon \Gamma \to GL_2(\boldsymbol{C})$ by

$$\rho_{\chi,\alpha}(x, t) = \begin{pmatrix} \chi(t) & \alpha(x) \\ 0 & 1 \end{pmatrix}$$

for $x \in N$ and $t \in \Gamma_p$. Since $(x, t)(x', t') = (x + tx', tt')$ in Γ we see that $\rho_{\chi,\alpha}$ is indeed a representation. If $\chi = 1$ then (1) implies that $\alpha = 0$. Whenever $\alpha = 0$ we just recover

$$\delta_{\chi,1} = \rho_{\chi,0} ,$$

where χ is viewed on the left as a character of Γ.

7.5. *The case $p = 2$.* Then (1) implies that $sx = -x$ for $x \in N$. For any linear character $\chi\colon N \to \boldsymbol{C}^\times$ we can then define the "dihedral" representation σ_χ by

$$\sigma_\chi(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \sigma_\chi(x) = \begin{pmatrix} \chi(x) & 0 \\ 0 & \chi(x)^{-1} \end{pmatrix}$$

for $x \in N$. In fact σ_χ is the induced representation,

$$\sigma_\chi = \mathrm{Ind}_N^\Gamma(\chi) .$$

It is easy to see that σ_χ and $\sigma_{\chi'}$ are isomorphic if and only if $\chi' = \chi$ or $\chi' = \chi^{-1}$.

PROPOSITION 7.6. *Let $\rho: \Gamma \to GL_2(\boldsymbol{C})$ be a representation. One and only one of the following cases occurs.*

(1) *There are linear characters $\chi, \chi': \Gamma \to \boldsymbol{C}^\times$, unique up to order, such that ρ is isomorphic to $\delta_{\chi,\chi'}$ (see 7.3).*

(2) *$p = 2$ and there is a linear character $\chi: N \to \boldsymbol{C}^\times$ such that $\chi \neq \chi^{-1}$ and such that ρ is isomorphic to σ_χ (see 7.5). The set $\{\chi, \chi^{-1}\}$ is uniquely determined by ρ.*

(3) *There is a nontrivial linear character $\chi: \Gamma_p \to \boldsymbol{C}^\times$, a nonzero Γ_p-homomorphism $\alpha: N \to \boldsymbol{C}_\chi$, and a linear character $\lambda: \Gamma \to \boldsymbol{C}^x$ such that ρ is isomorphic to $\lambda \otimes \rho_{\chi,\alpha}$ (see 7.4). Two such representations $\lambda \otimes \rho_{\chi,\alpha}$ and $\lambda' \otimes \rho_{\chi',\alpha'}$ are isomorphic if and only if $\lambda = \lambda'$, $\chi = \chi'$, and $\alpha' = u\alpha$ for some $u \in \boldsymbol{C}^x$.*

Suppose first that $\rho(N)$ is diagonalizable. If $\rho(N)$ consists of scalars then $\rho(\Gamma)$ is abelian, hence finite by 7.2, and we are in case (1). Otherwise the centralizer of $\rho(N)$ (assumed now in diagonal form) is the diagonal group T, so $\rho(\Gamma)$ lies in the normalizer $W = T \times \langle w \rangle$ of T, where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Either $\rho(\Gamma) \subset T$, case (1) again, or $p = 2$ and $\rho(s) \notin T$. Conjugating by an element of T we can then make $\rho(s) = w$. Since we must then have $w\rho(x)w^{-1} = \rho(x)^{-1}$ it follows that $\rho = \sigma_\chi$ for some linear character $\chi: N \to \boldsymbol{C}^\times$. If $\chi^2 = 1$ then $\chi(N)$ consists of scalars, the case treated above. Otherwise $\rho(\Gamma)$ is nonabelian, so cases (1) and (2) are exclusive and exhaust those for which $\rho(N)$ is diagonalizable.

Assume now that $\rho(N)$ is not diagonalizable. Being abelian, it then leaves invariant a unique line L in $\boldsymbol{C}^2$. Since $N \lhd \Gamma$ the line L is Γ-invariant. Let $\lambda: \Gamma \to \boldsymbol{C}^\times$ be the character of Γ's action on $\boldsymbol{C}^2/L$; this is evidently intrinsic to ρ. Replacing ρ by $\lambda^{-1} \otimes \rho$ we reduce to the case $\lambda = 1$. Let now $\chi: \Gamma \to \boldsymbol{C}^x$ be the character of Γ's action on L. If $\chi(s) = 1$ then $\rho(s) = 1$ so $\rho(\Gamma) = \rho(N)$ is abelian, hence finite, which is impossible since $\rho(N)$ is not diagonalizable. Let L' be the $\chi(s)$ eigenspace of $\rho(s)$. A basis adapted to the decomposition $\boldsymbol{C}^2 = L \oplus L'$ then puts ρ in the form

$$\rho(t) = \begin{pmatrix} \chi(t) & \alpha(t) \\ 0 & 1 \end{pmatrix}$$

for some map $\alpha: \Gamma \to \boldsymbol{C}$ such that $\alpha(tt') = \alpha(t) + \chi(t)\alpha(t')$ for $t, t' \in \Gamma$.

Let $M = \operatorname{Ker}(\chi)$, a subgroup of index p in Γ, by 7.2. We claim that $M = N$. Otherwise $\Gamma = M \cdot N$, and $N_0 = M \cap N$ has index p in N. We have the commutator relations

$$(1 - s)N_0 = (\Gamma, N_0) = (M \cdot N, N_0) = (M, N_0) \subset (M, M) ,$$

so that M^{ab} is finite. But $\rho(M) = \begin{pmatrix} 1 & \alpha(M) \\ 0 & 1 \end{pmatrix}$ is then a finite unipotent group, so $\alpha = 0$ and $\rho = \delta_{\chi,1}$, contrary to hypothesis. Now since $\operatorname{Ker}(\chi) = N$ we can identify χ with a character of $\Gamma_p = \Gamma/N$. For $t \in \Gamma_p$, we see that $\rho(t)$ is diagonal (i.e., $\alpha(t) = 0$) and so, for $(x, t) = (x, 1) \cdot (0, t)$ in Γ, we have $\rho(x, t) = \begin{pmatrix} \chi(t) & \alpha(t) \\ 0 & 1 \end{pmatrix}$, i.e., $\rho = \sigma_{\chi,\alpha}$ as in 7.4. The basis chosen above is intrinsic to ρ up to multiplication by scalars, and such a change has the effect of multiplying α by a nonzero scalar. Whence the proposition.

7.7. Proposition 7.6 parametrizes the isomorphism classes of representations $\rho: \Gamma \to GL_2(\boldsymbol{C})$. Case (1) occurs when ρ is completely reducible, but not irreducible. There are then only finitely many possibilities.

7.8. Case (2) gives the only irreducible actions. Choosing an isomorphism $N \cong \boldsymbol{Z}^m$ the χ's in case (2) vary over elements of $\operatorname{Hom}(\boldsymbol{Z}^m, \boldsymbol{C}^\times) \cong (\boldsymbol{C}^\times)^m$ such that $\chi^2 \neq 1$. If $T = (\boldsymbol{C}^\times)^m$ and $T_2 = \{x \in T \mid x^2 = 1\}$ then the classes of ρ's in case (2) are parametrized by $T - T_2$ modulo the action $x \mapsto x^{-1}$. Thus, *when* $p = 2$ *and* $m > 0$, Γ *is not of finite* 2*-representation type over* $\boldsymbol{C}$ (cf. 7.2).

7.9. Case (3) corresponds to noncompletely reducible representations. View N as a module, say of rank m, over the ring R of pth cyclotomic integers. Then, as Γ_p-module, $\boldsymbol{C} \otimes_{\boldsymbol{Z}} N \cong (\boldsymbol{C} \otimes_{\boldsymbol{Z}} R)^m = (\bigoplus_\chi \boldsymbol{C}_\chi)^m$, where χ varies over the nontrivial linear characters of Γ_p. It follows that $\operatorname{Hom}_{\Gamma_p}(N, \boldsymbol{C}_\chi) = \operatorname{Hom}_{\boldsymbol{C}[\Gamma_p]}(\boldsymbol{C} \otimes_{\boldsymbol{Z}} N, \boldsymbol{C}_\chi) \cong \boldsymbol{C}^m$. Thus, if we fix the (discrete) parameters λ and χ in case (3), the isomorphism classes of possible ρ's are given by nonzero $\alpha \in \boldsymbol{C}^m$ modulo the action of $\boldsymbol{C}^x$, i.e., by a projective space $\boldsymbol{P}^{m-1}(\boldsymbol{C})$. This yields a *nondiscrete parametrization* as soon as $m \geqq 2$. Note that this may happen for $p \geqq 3$ *when* Γ *has integral* 2*-representation type over* $\boldsymbol{Q}$, because $\chi_\rho(\Gamma)$ is integral over $\boldsymbol{Z}$ in cases (1) and (3).

7.10. Let $N_0 = (\Gamma, \Gamma) = (1 - s)N$, the commutator subgroup of Γ. Suppose that $p \neq 2$. Then for all ρ as above, $\rho(N_0)$ is an abelian group of unipotent matrices. Let I denote the augmentation ideal of $\boldsymbol{Z}[N_0]$, and let $J = I \cdot \boldsymbol{Z}[\Gamma]$. Then for all representations $\rho: \boldsymbol{Z}[\Gamma] \to M_2(K)$ we see that $\rho(J)^2 = 0$. Moreover $\boldsymbol{Z}[\Gamma]/J^2$ is a finitely generated $\boldsymbol{Z}$-module. This illustrates the kind of phenomenon described in Corollary 4.6.

7.11. Let K be an algebraically closed field of characteristic $q > 0$. If $q \neq p$ then the arguments and conclusions above apply with no substantial change to representations $\rho: \Gamma \to GL_2(K)$.

7.12. Suppose that $q = p \neq 2$. Then arguments like those above can be used to show that every $\rho: \Gamma \to GL_2(K)$ is isomorphic to some

$$\rho_\alpha: t \longrightarrow \begin{pmatrix} 1 & \alpha(t) \\ 0 & 1 \end{pmatrix}$$

where $\alpha: \Gamma \to K$ is an additive character, determined by ρ up to multiplication by a nonzero scalar. With the notation of 7.9 we have $\Gamma^{ab} \cong (\boldsymbol{Z}/p\boldsymbol{Z})^{m+1}$, so the isomorphism classes of nontrivial ρ's are parametrized by $\boldsymbol{P}^m(K)$.

7.13. If $q = p = 2$ the same description applies to *reducible* representations. However irreducible ones may exist, and are exactly as in case (2) of 7.6.

7.14. It follows from the above that *Γ has integral 2-representation type over $\boldsymbol{Z}$ if and only if $p \neq 2$ or $p = 2$ and $m = 0$*, where m is as in 7.9. However the reducible representations $\rho: \Gamma \to GL_2(K)$ are discretely parametrized if and only if either $q \neq p$ and $m \leqq 1$ or $q = p$ and $m = 0$.

8. Linear representations of free products: Shalen's costruction. The following construction of P. Shalen [17] is used to show that free products are not of integral representation type.

PROPOSITION 8.1. *Let K be a field, n and integer $\geqq 2$, and Γ, Γ' subgroups of $GL_n(K)$ satisfying:*

(a) *$H = \Gamma \cap \Gamma'$ consists of diagonal matrices; and*

(b) *For $s \in \Gamma - H$ (resp. $s' \in \Gamma' - H$) we have $s_{n1} \neq 0$ (resp. $s'_{1n} \neq 0$). (u_{ij} denotes the (i, j) coefficient $u \in M_n(K)$.)*

Let T be an indeterminate and define

$$\rho: G = \Gamma *_H \Gamma' \longrightarrow GL_n(K[T, T^{-1}])$$

by $\rho(s) = tst^{-1}$ for $s \in \Gamma$ and $\rho(s') = s'$ for $s' \in \Gamma'$, where $t = \operatorname{diag}(T, T^2, \cdots, T^n)$. An element of G not conjugate to an element of $\Gamma \cup \Gamma'$ is conjugate to one of the form

$$u = s_1 s'_1 \cdots s_r s'_r$$

with $s_i \in \Gamma - H$ and $s'_i \in \Gamma' - H (i. = 1, \cdots, r)$. The highest degree term in the Laurent polynomial $\chi_\rho(u)$ has degree $r(n-1)$.

This is proved by Shalen in [17], Lemma 1.1 and the proof of Proposition 1.3. In applying Proposition 8.1 we use the next lemma to secure conditions (a) and (b).

LEMMA 8.2. *Let K be a field, n and integer $\geqq 2$, and $x = (x_{ij})$ an n by n matrix with indeterminate coefficients. If $s \in M_n(K)$ is not a scalar matrix, and if $u = xsx^{-1} \in M_n(K(x))$, then each coefficient u_{ij} of u is transcendental over K.*

Write $u_{ij} = u_{ij}(x)$, viewing x as a variable, which we may specialize. If some $u_{ij}(x)$ is not transcendental over K then it belongs to K, and hence is a constant c as a function of x. Specializing x to permutation matrices we conclude that $u_{\sigma(i)\sigma(j)}(x)$ is likewise the same constant c for all permutations σ of $\{1, \cdots, n\}$. We now distinguish the cases $i \neq j$ and $i = j$. If $i \neq j$ then, putting s in (upper or lower) triangular form over an algebraic closure of K, we see that $c = 0$ so all conjugates of s are diagonal, and hence s is a scalar. If $i = j$ then $s = cI + s'$ where all conjugates of s' have zero diagonal. Conjugating s' by suitable elementary matrices one concludes that $s' = 0$, i.e., s is a scalar.

PROPOSITION 8.3. *Let K be a field, n an integer $\geqq 2$, and Γ, Γ' subgroups of $GL_n(K)$. Let H denote the group of scalar matrices in $\Gamma \cap \Gamma'$ and put $G = \Gamma *_H \Gamma'$. Let L be a field extension of transcendence degree $\geqq n^2 + 1$ over K. Then there is a faithful representation $\rho: G \to GL_n(L)$ such that $\rho \mid \Gamma$ and $\rho \mid \Gamma'$ are conjugate to the inclusions, and such that $\chi_\rho(s)$ is transcendental over K for any $s \in G$ not conjugate to an element of $\Gamma \cup \Gamma'$.*

Choose $x = (x_{ij}) \in GL_n(L)$ with coefficients algebraically independent over K, and let $\Gamma_1 = x\Gamma x^{-1}$ and $\Gamma'_1 = x\Gamma' x^{-1}$ in $GL_n(K_1)$, where $K_1 = K(X)$. It follows from Lemma 8.2 that $\Gamma_1 \cap \Gamma'_1 = H$ and that, if $s \in \Gamma_1 - H$ or if $s \in \Gamma'_1 - H$, then all coefficients of s are transcendental over K, in particular $\neq 0$. Thus K_1, Γ_1, Γ'_1 satisfy the hypotheses of Proposition 8.1. Choose $T \in L$ transcendental over K_1. Then Proposition 8.1 furnishes a representation ρ with the properties claimed.

COROLLARY 8.4. *Let K be a field and n an integer $\geqq 2$. Let Γ and Γ' be groups which have nontrivial representations in $GL_n(K)$. Then $\Gamma * \Gamma'$ is not of integral n-representation type over K.*

Let $\rho: \Gamma \to GL_n(K)$ and $\rho': \Gamma'' \to GL_n(K)$ be nontrivial representations. If either of them is scalar we can replace one of its diagonal entries by 1 to make it nonscalar. Now let H denote the group of scalar matrices in $\rho(\Gamma) \cap \rho'(\Gamma')$; then $H \neq \rho(\Gamma)$ and $H \neq \rho'(\Gamma')$. If L is a suitable transcendental extension of K then Proposition 8.3 furnishes a faithful representation $\rho(\Gamma) *_H \rho'(\Gamma') \to GL_n(L)$ whose character takes values transcendental over K; whence the corollary.

Problem 8.5. *Which amalgamated free products admit faithful linear representations*? This problem is raised by Shalen in [17], where he treats some very special cases of it.

9. Integral representation type and group extensions.

NOTATION 9.1. We fix an integrally closed domain A whose field of fractions F is a finite extension of its prime field. Thus A is a ring of fractions of the integral closure of Z in F; it is a Dedekind domain. For any group G we shall write $A_n(G)$ to indicate that G has integral n-representation type over A.

We fix a group G with a normal subgroup H. Let $\rho: G \to GL_m(K)$ be an absolutely irreducible representation with K an algebraically closed field which is an A-algebra. We consider also the associated adjoint representation

$$\alpha = Ad \circ \rho = \rho \otimes \rho^*: G \longrightarrow \mathrm{Aut}_{K\text{-alg}}(M_m(K)) \subset GL_{m^2}(K) .$$

Our aim is to give criteria for the integrality of $\rho(G)$ over A.

LEMMA 9.2. *If* $\det(\rho(G))$ *and* $\alpha(G)$ *are integral over* A *then* $\rho(G)$ *is integral over* A.

Let $s \in G$ and let $w_1, \cdots, w_m$ be the eigenvalues of $\rho(s)$. Since $\alpha = \rho \otimes \rho^*$ the eigenvalues of $\alpha(s)$ are all $w_i w_j^{-1}$. By assumption the latter are integral over A. Fix an i, put $w = w_i$, and write $w_j = wv_j$, so that each v_j is integral over A. Let $d = \det(\rho(s))$. Then $d = w^m v$, where $v = \prod_j v_j$, and d and d^{-1} are integral over A. Therefore w is integral over A; whence the lemma.

PROPOSITION 9.3. *Suppose that* H *is central in* G. *Then*

$$(A_1(G) + A_{n^2}(G/H)) \Longrightarrow A_n(G) .$$

We must show that $\rho(G)$ is integral over A if $m \leqq n$. The condition $A_1(G)$ implies that $\det \rho(G)$ consists of roots of unity (cf.

Proposition 5.5). Our conclusion follows therefore from Lemma 9.2 once we show that $\alpha(G)$ is integral over A. Since H is central $\rho(H)$ consists of scalars, so α is trivial on H. Therefore α comes from a representation of G/H of dimension $m^2 \leqq n^2$, so the integrality of $\alpha(G)$ follows from $A_{n^2}(G/H)$.

REMARK 9.4. Clearly $(A_1(H) + A_1(G/H)) \Longrightarrow A_1(G)$, as the exact sequence $H^{ab} \to G^{ab} \to (G/H)^{ab} \to 1$ shows. Therefore one may substitute $A_1(H)$ for $A_1(G)$ in the hypothesis of 9.3.

LEMMA 9.5. *Assume that $\rho(H)$ is integral over A and acts irreducibly on K^m, and that $(G/H)^{ab}$ is torsion* (i.e., $A_1(G/H)$). *Then $\rho(G)$ is integral over A.*

We may first replace G and H by their images in $GL_m(K)$. Let D denote the group of scalar matrices $d \cdot I$ such that d and d^{-1} are integral over A. All of our assumptions are preserved if we enlarge G, H to $D \cdot G$, $D \cdot H$; thus we may assume further that $D \subset H$. Let $s \in G$. Since $(G/H)^{ab}$ is torsion and $\det(G)$ is abelian it follows that some power of $d = \det(s)$ belongs to $\det(H)$, and so is integral over A. Now $d^{-1}s^m \in G_1 = G \cap SL_m(K)$ because $d \cdot I \in D \subset G$. According to Corollary 2.9 G_1 is integral over A. Now the integrality of $d^{-1}s^m$ over A implies that of s, whence the lemma.

PROPOSITION 9.6. *Assume $A_{m^2}(G/H)$ and that $\rho(H)$ is integral over A. Then $\rho(G)$ is integral over A.*

Case 1. *$\rho \mid H$ is irreducible.* Then the integrality of $\rho(G)$ follows from Lemma 9.5 using only $A_1(G/H)$, not the full force of $A_{m^2}(G/H)$.

Case 2. *$\rho \mid H$ is isotypic.* Then $K \cdot \rho(H)$ is a simple K-algebra and we can choose coordinates so that

$$M_m(K) = M_p(K) \otimes M_q(K)$$
$$K \cdot \rho(H) = M_p(K) \otimes 1 .$$

Applying the Skolem-Noether theorem to $M_p(K)$ one concludes that the normalizer of $M_p(K)$ in $GL_m(K)$ is $GL_p(K) \otimes GL_q(K)$. Note that $\rho(G)$ is contained in this normalizer of $K \cdot \rho(H)$. Let $\bar{\rho}_q \colon G \to PGL_q(K)$ be the homomorphism obtained by factoring out scalars in $GL_p(K) \otimes GL_q(K)$ and projecting on the second factor. Note that $\bar{\rho}_q$ is trivial on H. Form the cartesian square (fibre product),

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\rho_q} & SL_q(K) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ G & \xrightarrow[\bar{\rho}_q]{} & PGL_q(K) . \end{array}$$

Then π is surjective, $\operatorname{Ker}(\pi)$ is a finite central subgroup of $\tilde{G}$, and π maps $\operatorname{Ker}(\rho_q)$ isomorphically to $\operatorname{Ker}(\bar{\rho}_q)$. Let $\tilde{H} \subset \operatorname{Ker}(\rho_q)$ be the inverse image by π of $H \subset \operatorname{Ker}(\bar{\rho}_q)$. Then $\tilde{H} = \pi^{-1}(H) \cap \operatorname{Ker}(\rho_q)$ is a normal subgroup of $\tilde{G}$. In the exact commutative diagram,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \tilde{H} & \longrightarrow & \tilde{G} & \longrightarrow & \tilde{G}' & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} & & \\ 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G' & \longrightarrow & 1 , \end{array}$$

π' is surjective with finite central kernel. By Proposition 9.3 and Remark 9.4 therefore we have $A_m(\tilde{G}')$. Since ρ_q is trivial on $\tilde{H}$ (and $q \leqq m$) we conclude that $\rho_q(\tilde{G})$ is integral over A. Now define ρ_p: $\tilde{G} \to GL_p(K)$ by the formula

$$\rho(\pi(s)) = \rho_p(s) \otimes \rho_q(s)$$

for $s \in \tilde{G}$. If we show that $\rho_p(\tilde{G})$ is integral over A the integrality of $\rho(G)$ over A will follow. Since $\rho_p(s) \otimes 1 = \rho(\pi(s))$ for $s \in \tilde{H}$(so $\pi(s) \in H$) it follows that $\rho_p(\tilde{H})$ is integral over A and $\rho_p \mid \tilde{H}$ is irreducible. Therefore the desired integrality of ρ_p follows from case 1.

General case. By Clifford's theorem (see [9], §§ 49–51) $\rho = \operatorname{Ind}_{G_1}^G(\rho_1)$ where G_1 is the stabilizer in G of an isotypic component of $\rho \mid H$, say of dimension m_1, and $\rho_1: G_1 \to GL_{m_1}(K)$ is the corresponding representation. Clearly $\rho_1(H)$ is integral over A. We have $m = m_1 r$ where $r = [G: G_1]$. If $\rho_1(G_1)$ is integral over A then so also is $\rho(G)$, by 5.6. Putting $G_1' = G_1/H$, the integrality of $\rho_1(G_1)$ over A will follow from case 2, once we verify the hypothesis $A_{m_1^2}(G_1')$. Since G_1' has index r in $G' = G/H$ it follows from Proposition 5.7 that $A_{m_1^2 r}(G') \Rightarrow A_{m_1^2}(G_1')$. Since $m_1^2 r \leqq m^2$ our hypothesis $A_{m^2}(G')$ therefore implies $A_{m_1^2}(G_1')$. This completes the proof of Proposition 9.6.

LEMMA 9.7. *Assume that H is virtually solvable* (i.e., *that H has a solvable subgroup of finite index*). *Assume further that for every subgroup G_1 of index deviding m in G, the group G_1^{ab} is torsion. Then there is a subgroup H_0 of finite index in H such that $\rho(H_0)$ is conjugate to a diagonal group with roots of unity on the diagonal. In particular if $s \in H$ then the eigenvalues of $\rho(s)$ are* *roots of unity.*

Replacing G by $\rho(G)$ we may assume that $G \subset GL_m(K)$. Let $\bar{H}$ denote the Zariski closure of H, $\bar{H}_0$ its identity component, and $H_0 = H \cap \bar{H}_0$. The virtual solvability of H implies the solvability of $\bar{H}_0$. The Lie-Kolchin theorem permits us to assume that $\bar{H}_0$ is upper triangular. Since G normalizes H it likewise normalizes $\bar{H}$ and $\bar{H}_0$, so $\bar{H}_0$ acts completely reducibly on K^m. Therefore we can even diagonalize $\bar{H}_0$. It remains to show that the diagonal entries in H_0 are roots of unity. Let V be an H_0 isotypic subspace of K^m, on which H_0 therefore acts by scalars, which we must show are roots of unity. Let G_1 be the stabilizer in G of V and $\rho_1: G_1 \to GL(V)$ the corresponding representation. Then G_1 has finite index dividing m in G since $\rho = \mathrm{Ind}_{G_1}^{G}(\rho_1)$ (by Clifford's theorem). By hypothesis G_1^{ab} is torsion, so $\det(\rho_1(H_0))$ is torsion. It follows that the scalars $\rho_1(H_0) = H_0 \mid V$ have finite order, as claimed.

THEOREM 9.8. *Let n be an integer $\geqq 1$ and assume $A_{n^2}(G/H)$ as well as one or the other of the following conditions*: (i) $A_n(H)$; *or* (ii) *H is virtually solvable and $A_1(G_1)$ for every subgroup G_1 of index $\leqq n$ in G. Then $A_n(G)$.*

Assuming that $m \leqq n$ we must show that $\rho(G)$ is integral over A. This will follow from Proposition 9.6 once we show that $\rho(H)$ is integral over A, a condition that follows immediately from (i), and which results from Lemma 9.7 if we assume (ii).

COROLLARY 9.9. *Assume that G/H has integral representation type over A and that either H does likewise, or else that H is virtually solvable and G_1^{ab} is torsion for all subgroups G_1 of finite index in G. Then G has integral representation type over A.*

10. Algebraic and arithmetic groups.

PROPOSITION 10.1. *Let K be an algebraically closed field of characteristic $p \geqq 0$, and let Γ be a subgroup of $GL_n(K)$.*

(a) *Suppose that $p > 0$ and that Γ has integral representation type over the prime field of K. Then every finitely generated subgroup of Γ is finite.*

(b) *Suppose that Γ satisfies condition*:

(F Ab) *If Γ_1 is a subgroup of finite index in Γ then Γ_1^{ab} is a finite group.*

This is the case for example if Γ is finitely generated and of integral representation type over some field (Corollary 5.10). *Let G be* *the Zariski closure of Γ, G_0 its identity component, and U its unipotent radical. Then we have the commutator relations $G_0=(G_0, G_0)$ and (hence) $U=(G_0, U)$, and the group G_0/U is semi-simple.*

Assertion (a) follows from Proposition 5.2. Put $\Gamma_0=\Gamma\cap G_0$ and $\Gamma_1=(\Gamma_0, \Gamma_0)$. Condition $(F\ Ab)$ implies that Γ_1 has finite index in Γ, so Γ_1 is Zariski dense in G_0, whence $G_0=(G_0, G_0)$ (cf. [5], Ch. 1, 2.4). Now G_0 is a semidirect product $U\cdot H$, with H reductive, so $G_0^{ab}=(U/(G_0, U))\times H^{ab}$, whence $U=(G_0, U)$ and $H=(H, H)$. Since the derived group of a reductive group is semi-simple this proves (b).

10.2. In view of Proposition 10.1, interesting linear groups of integral representation type are to be found only in characteristic zero. Let G be an affine algebraic group defined over a number field F, let $G=\underline{G}(F)$, and let Γ be a subgroup of G which is Zariski dense in $\underline{G}$. We seek conditions that ensure that:

(IRT) Γ has integral representation type over $\boldsymbol{Q}$.

This depends only on the commensurability class of Γ (Corollary 5.8) so, after passing to subgroups of finite index, we may assume that G is connected. Let $\underline{U}$ denote the unipotent radical of $\underline{G}$ and $U=\boldsymbol{U}(F)$. In order to have (IRT) we must (by 10.1) assume that $\underline{G}=(\underline{G}, \underline{G})$, and hence that

$$\underline{U}=(\underline{G}, \underline{U}) \text{ and } \underline{G}/\underline{U} \text{ is semi-simple.}$$

In certain cases these conditions suffice to imply condition $(F\ Ab)$ of 10.1 (b) for Γ, in which case it follows from Corollary 9.9 that (IRT) holds if and only if its analogue holds for $\Gamma/\Gamma\cap U$ in G/U. Thus we are led to consider the case when $\underline{G}$ is semi-simple. Let $\underline{H}=R_{F/\boldsymbol{Q}}\underline{G}$ be the algebraic group over $\boldsymbol{Q}$ obtained by restriction of scalars to $\boldsymbol{Q}$, so that $\underline{H}(\boldsymbol{Q})=\underline{G}(F)=G$, and let $\rho\colon \underline{H}\to GL_n$ be an algebraic representation. Then $\rho(G)$ is integral over $\boldsymbol{Q}$. Borel and Tits [6] have shown, in many cases (for example if the F-simple factors of $\underline{G}$ are all of F-rank $\geqq 1$), that every linear representation of G arises from an algebraic representation of the simply connected covering group of $\underline{H}$. In such cases therefore G has integral representation type over $\boldsymbol{Q}$. The same then follows for Γ clearly whenever Γ is "*intimately embedded*" in G, in the following sense: Given a representation $\rho\colon \Gamma\to GL_n(K)$, there is an extension K' of K and a unique representation $\rho'\colon G\to GL_n(K')$ which agrees with ρ on a subgroup of finite index in Γ. This condition is known to hold in many cases when Γ is an arithmetic or S-arithmetic subgroup of G (cf. [4], §16 and [16], 2.7). It is a consequence of the strong approximation theorem plus a qualitative form of the congruence subgroup theorem, whenever these are valid for F, S, and $\underline{G}$ (loc. cit.).

EXAMPLE 10.3. Let $\Gamma = SL_n(\boldsymbol{Z})$ with $n \geqq 3$. Γ is intimately embedded in $SL_n(\boldsymbol{Q})$, and Γ has integral representation type over $\boldsymbol{Z}$. Let Γ' be the semi-direct product of Γ with the Γ-module $\boldsymbol{Z}^n$. Then it is easily shown that Γ' satisfies condition (F Ab) of 10.1 (b), so it follows from Corollary 9.9 that Γ' likewise has integral representation type over $\boldsymbol{Z}$.

Question 10.4. Let Γ be a finitely generated group of integral representation type over $\boldsymbol{Q}$, and with a faithful linear representation over $\boldsymbol{Q}$. I know of no such groups which cannot be intimately embedded in the $\boldsymbol{Q}$-rational points of some algebraic group over $\boldsymbol{Q}$. Is this perhaps always the case? If Γ is even of integral representation type over $\boldsymbol{Z}$ must it then be an arithmetic group?

Question 10.5. Let Γ be a finitely generated group of integral representation type over $\boldsymbol{Z}$ whose linear representations separate points of Γ. Does Γ then have a faithful linear representation over $\boldsymbol{Q}$?

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COLUMBIA UNIVERSITY
NEW YORK, NY 10027