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QUASICOMPACTIFICATIONS AND SHAPE THEORY

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If $f: X \to Y$ is an embedding of a space X into a space Y such that each component of Y is a compactification of the image of a quasicomponent of X and such that f induces a homeomorphism of the space QX of quasicomponents of X onto the space of components of Y, then (f, Y) is called a *quasicompactification* of X. After some preliminary results, it is shown that a locally compact metric space X has a locally compact metric quasicompactification if and only if QX is locally compact. Two canonical quasicompactifications, F^*X and αX, of such a space are described, and it is shown that if $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$, then $\mathrm{Sh}_p F^*X = \mathrm{Sh}_p F^*Y$; the question whether also $\mathrm{Sh}_p \alpha X = \mathrm{Sh}_p \alpha Y$ is left open. Finally, some techniques of this paper are used to obtain a proper shape version of a theorem due to Y. Kodama, generalizing previous work of the author.

1. Introduction. A subset A of a topological space X is a *component* of X if it is maximal with respect to the property that no two points of A are separated in the subspace A, and is a *quasicomponent* of X if it is maximal with respect to the property that no two points of A are separated in the whole space X. Thus quasicomponents are more dependent on, and indicative of, the global structure of X than are components.

It is easily shown that both the set $\mathscr{C}_X$ of all components of X and the set $q\mathscr{C}_X$ of all quasicomponents of X are decompositions of X into disjoint closed sets. The resulting spaces $X/\mathscr{C}_X$ and $X/q\mathscr{C}_X$ (with the decomposition topologies) are not in general very nice spaces; for example, even if X is separable and metric, $X/\mathscr{C}_X$ need not be Hausdorff and $X/q\mathscr{C}_X$, while always Hausdorff, need not have a basis of open and closed sets (see [12]). Retopologizing $X/q\mathscr{C}_X$, however, avoids this latter difficulty; specfically, the *quasicomponent space of* X is the space QX whose points are the quasicomponents of X and whose topology has as a basis those sets of quasicomponents whose union is both open and closed in X. The space QX thus has a basis consisting of open and closed sets (i.e., QX is 0-dimensional) and hence is regular and totally disconnected.

Elementary, well known arguments suffice to establish the following useful facts. If p is a point of a topological space X, the component of X about p is the union of all connected subsets of X containing p, and the quasicomponent of X about p is the intersection of all open and closed subsets of X containing p. Hence any component of X contains every connected subset of X which intersects it, and every quasicomponent of X is contained in every open and closed subset of X which intersects it.

Any two components of a compact Hausdorff space X are separated in X, but this fails in the absence of compactness (even if X is locally compact and metrizable). However, any two quasicomponents of any topological space X are separated in X.

For a locally compact Hausdorff space X, any *compact* component of X is a quasicomponent of X. Combining this fact with other well known results gives the following theorem.

THEOREM 1.1. *If X is a locally compact Hausdorff space such that every component of X is compact, then $\mathscr{C}_X = q\mathscr{C}_X$, $\mathscr{C}_X$ is upper semicontinuous, $X/\mathscr{C}_X$ is locally compact and totally disconnected (and is metrizable if X is), and $QX = X/q\mathscr{C}_X = X/\mathscr{C}_X$.*

This observation suggests that it might be desirable, when possible, to construct a "nice" embedding of a given space X into a space Y which has compact components. In this paper it will be shown how this may be done for certain spaces, and some shape-theoretic properties of the resulting "quasicompactifications" will be considered.

2. Definitions and preliminary results. If $\{X_\alpha \mid \alpha \in A\}$ is a collection of disjoint, nonempty topological spaces (with $X_\alpha \neq X_\beta$ for $\alpha \neq \beta$), then $\cup X_\alpha$, with the topology in which a set is open if and only if its intersection with each X_α is open in X_α, is called the *topological sum* of the X_α's and is denoted by $\bigoplus \{X_\alpha \mid \alpha \in A\}$ or by $\bigoplus X_\alpha$. Equivalently, a space X is the topological sum of subspaces $\{X_\alpha\}$ if the X_α's are disjoint, nonempty, open and closed subsets of X and $\cup X_\alpha = X$.

LEMMA 2.1. *If $X = \bigoplus \{X_\alpha \mid \alpha \in A\}$, then $QX = \bigoplus \{QX_\alpha \mid \alpha \in A\}$; conversely, if $QX = \bigoplus \{Z_\alpha \mid \alpha \in A\}$, then $X = \bigoplus \{X_\alpha \mid \alpha \in A\}$ with $QX_\alpha = Z_\alpha$ for each α.*

Proof. The first assertion follows immediately from the fact that if K is open and closed in X, then QK is open and closed in QX. For the converse, let $p: X \to QX$ be the natural projection and let $X_\alpha = p^{-1}(Z_\alpha)$ for each α in A. It is clear that $X = \bigoplus X_\alpha$ and that $QX_\alpha = Z_\alpha$ for each α.

LEMMA 2.2. *If X is a locally compact metrizable space, then QX is a Lindelöf space (i.e., every open cover has a countable subcover) if and only if X is separable.*

Proof. If X is separable, then QX is a Lindelöf space since the Lindelöf property is preserved by continuous surjective maps.

For the converse, suppose QX has the Lindelöf property and write X in the form $X = \bigoplus\{X_\alpha | \alpha \in A\}$ with each X_α separable (see [5], p. 241, Th. 7.3). Then $QX = \bigoplus\{QX_\alpha | \alpha \in A\}$ and since QX has the Lindelöf property, it follows that A is countable. Hence X is the union of a countable number of separable spaces and hence is separable.

LEMMA 2.3. *If X is a locally compact metrizable space and QX is compact, then QX is metrizable.*

Proof. It follows from Lemma 2.2 that X is separable. Since any compact Hausdorff space which is the continuous image of a separable metric space is metrizable ([8], p. 115, Lemma 40), it follows that QX is metrizable.

LEMMA 2.4. *If X is a locally compact metric space, then QX is locally compact if and only if $X = \bigoplus\{X_\alpha | \alpha \in A\}$ with QX_α compact for each α.*

Proof. Since, as observed before, any locally compact metric space can be written as the topological sum of separable spaces, it is sufficient to prove the result for the case in which X is separable. One direction follows immediately from Lemma 2.1; for the converse, suppose QX is locally compact and X is separable. By Lemma 2.2, QX is a Lindelöf space and, since QX is 0-dimensional, it easily follows that $QX = \bigoplus\{Z_n | n = 1, 2, \cdots\}$ with Z_n compact for each n. By Lemma 2.1, $X = \bigoplus\{X_n | n = 1, 2, \cdots\}$ with $QX_n = Z_n$ for each n. The theorem follows.

It is clear that for any map (i.e., continuous function) $f: X \to Y$, the image under f of any quasicomponent of X lies in a (unique) quasicomponent of Y.

LEMMA 2.5. *If $f: X \to Y$ and for each $A \in QX$, $\phi_f(A)$ is the quasicomponent of Y containing $f(A)$, then ϕ_f is a continuous function from QX to QY.*

Proof. Let $\mathscr{U}$ be a basic open set in QY. Then $\cup\mathscr{U}$ is open and closed in Y and hence $f^{-1}(\cup\mathscr{U})$ is open and closed in X. If $\mathscr{V} = \{A \in QX | A \subset f^{-1}(\cup\mathscr{U})\}$, then $\cup\mathscr{V} = f^{-1}(\cup\mathscr{U})$, so $\mathscr{V}$ is open in QX. Using the fact that $\cup\mathscr{U}$ and $\cup\mathscr{V}$ are open and closed in Y and X, respectively, it is easily shown that $\phi_f^{-1}(\mathscr{U}) = \mathscr{V}$. Thus ϕ_f is continuous.

The map ϕ_f described in Lemma 2.2 will be called the map of QX into QY *induced by* the map $f: X \to Y$.

3. **Quasicompactifications.** Throughout the remainder of this paper, unless the contrary is specifically indicated *all spaces considered are assumed to be locally compact and metrizable.* If X is such a space, then a *quasicompactification* of X is a pair (Y, f) such that Y is a (locally compact metrizable) space with compact components and $f: X \to Y$ is an embedding satisfying (1) for each A in QX, $\mathrm{cl}_Y f(A)$ is a component of Y and (2) the map $\phi_f: QX \to QY$ induced by f is a homeomorphism. (Here, as usual, $\mathrm{cl}_Y f(A)$ denotes the closure of $f(A)$ in Y.) Following the usual practice with respect to compactifications, we will often call the space Y a quasicompactification of X, considering X as a subspace of Y with $f: X \to Y$ the inclusion map.

Note that condition (1) alone implies that ϕ_f is a continuous bijection of QX onto a subspace of QY. In general, however, ϕ_f need not be a homeomorphism, even if it is surjective.

In order that X should have a quasicompactification, it is clearly necessary that QX be locally compact since $QX \approx QY$ and QY is locally compact by Theorem 1.1. Local compactness of QX is also a sufficient condition for X to have a quasicompactification, as the following lemmas show.

Recall that the Freudenthal compactification FX of a rim-compact space X is characterized by the property that no open neighborhood of a point p of $EX(=FX - X)$ is separated by EX into two disjoint open sets each having p as a limit point. (The definition of FX and additional properties of this compactification may be found in [6], [11] and [8]; additional references and a characterization of FX in terms of nonconvergent sequences, for X locally compact and metrizable and QX compact, are given in [2].)

LEMMA 3.1. *The closure in FX of any open and closed subset of X is open and closed in FX.*

This is an easy consequence of the characterization of FX quoted above.

THEOREM 3.2. *If QX is compact, then FX is a quasicompactification of X.*

Proof. It is well known that in this case FX is metrizable. Considering X as a subspace of FX and letting $f: X \to FX$ be the inclusion map, we must show that conditions (1) and (2) of the definition are satisfied.

First suppose A is a quasicomponent of X and let C be the component of FX containing A. By Lemma 3.1, any two points of X which are separated in X are separated in FX and hence $C - A \subset EX$. Since C is connected and EX is totally disconnected, it follows that $C = \mathrm{cl}_{FX} A$. Hence condition (1) holds, and it remains only to show that the induced map $\phi(=\phi_f)$ is surjective and open.

To see that ϕ is surjective, it is sufficient to show that each component C of FX intersects X, for then C contains some $A \in QX$ and, by the argument above, $\mathrm{cl}_{FX}A = C = \phi(A)$. If C is a component of FX which does not intersect X, then $C = \{p\}$ for some point $p \in EX$. Since X is dense in FX (and FX is metrizable), there is a sequence $\alpha = (x_1, x_2, \cdots)$ of points of X converging to p in FX. If any quasicomponent A of X contains infinitely many points of α, then $p \in \mathrm{cl}_{FX} A \subset C$ and hence $A \subset C$, contrary to the assumption that $C \cap X = \emptyset$. Hence suppose that no two points of α belong to the same quasicomponent of X, and for each i, let A_i be the quasicomponent of X containing x_i. Since QX is compact, it is metrizable by Lemma 2.3 and hence some subsequence of $\{A_i\}$ converges in QX; suppose, without loss, that $\{A_i\} \rightarrow A \in QX$. Then $p \notin \mathrm{cl}_{FX} A$, so $FX = H \cup K$, with H and K closed and disjoint, $A \subset H$ and $p \in K$. Since $A_i \rightarrow A$ in QX and $H \cap X$ is an open and closed subset of X containing A, $A_i \subset H \cap X$ for almost all i; but $x_i \in K$ for almost all i since $x_i \rightarrow p \in K$, and hence $A_i \cap K \neq \emptyset$ for almost all i. This is a contradiction, and it follows that every component of FX intersects X, so ϕ is surjective.

Now suppose $\mathscr{U}$ is a basic open set in QX. Then $U = \cup \mathscr{U}$ is open and closed in X, so $\mathrm{cl}_{FX}U$ is open and closed in FX. If $p \in \mathrm{cl}_{FX}U$, then by the preceding argument, $p \in \mathrm{cl}_{FX} A$ for some $A \in QX$. Since U is open and closed in X, this implies that

$$\mathrm{cl}_{FX}U = \cup \{\mathrm{cl}_{FX} A | A \subset U\} .$$

Since $\phi(\mathscr{U}) = \{\mathrm{cl}_{FX} A | A \subset U\}$ and $\cup \{\mathrm{cl}_{FX} A | A \subset U\}$ is open and closed in FX, $\phi(\mathscr{U})$ is open (and closed) in QFX. Hence ϕ is an open map, and since it is $1-1$ and onto, $\phi: QX \rightarrow QFX$ is a homeomorphism. Thus FX is a quasicompactification of X.

COROLLARY 3.3. *If QX is locally compact, then X has a quasicompactification.*

Proof. By Lemma 2.4, $X = \bigoplus \{X_\alpha | \alpha \in A\}$ with QX_α compact for each α. By Theorem 3.2, FX_α is a quasicompactification of X_α and it readily follows that $\bigoplus \{FX_\alpha | \alpha \in A\}$ is a quasicompactification of X.

The following theorem shows that the quasicompactification of

X described in the proof of Corollary 3.3 is independent of the choice of the X_α's; this quasicompactification will be denoted by F^*X and will be called the *Freudenthal quasicompactification* of X.

THEOREM 3.4. *If $X = \bigoplus\{X_\alpha \mid \alpha \in A\}$ and $X = \bigoplus\{Y_\beta \mid \beta \in B\}$, with QX_α and QY_β compact for $\alpha \in A$ and $\beta \in B$, respectively, then $\bigoplus\{FX_\alpha \mid \alpha \in A\}$ is homeomorphic to $\bigoplus\{FY_\beta \mid \beta \in B\}$.*

Proof. We first observe that if Z is a space such that QZ is compact and $Z = \bigoplus\{Z_\tau \mid \tau \in T\}$, then $FZ = \bigoplus\{FZ_\tau \mid \tau \in T\}$. To see this, note that by Lemma 2.1, $QX = \bigoplus\{QZ_\tau \mid \tau \in T\}$, and since QZ is compact, T must be finite. Hence $\bigoplus\{FZ_\tau \mid \tau \in T\}$ is a compactification of Z. Using the characterization of the Freudenthal compactification given earlier, it is easily shown that $\bigoplus\{FZ_\tau \mid \tau \in T\} = FZ$.

Now for each $\alpha \in A$, $X_\alpha = \bigoplus\{X_\alpha \cap Y_\beta \mid \beta \in B, X_\alpha \cap Y_\beta \neq \emptyset\}$ so, by the previous remark, $FX_\alpha = \bigoplus\{F(X_\alpha \cap Y_\beta) \mid \beta \in B, X_\alpha \cap Y_\beta \neq \emptyset\}$; therefore $\bigoplus\{FX_\alpha \mid \alpha \in A\} = \bigoplus\{F(X_\alpha \cap X_\beta) \mid \alpha \in A, \beta \in B, X_\alpha \cap Y_\beta \neq \emptyset\}$. Similarly $\bigoplus\{FY_\beta \mid \beta \in B\} = \bigoplus\{F(Y_\beta \cap X_\alpha) \mid \beta \in B, \alpha \in A, Y_\beta \cap X_\alpha \neq \emptyset\}$, and hence $\bigoplus\{FX_\alpha \mid \alpha \in A\} = \bigoplus\{FY_\beta \mid \beta \in B\}$.

Simple examples show it is not the case that each component of the Freudenthal quasicompactification of X is necessarily the Freudenthal compactification of a quasicomponent of X. As the next theorem shows, however, if X has a quasicompactification at all, it has one in which each component is the Alexandroff compactification of a quasicomponent of X. (Here, the *Alexandroff compactification* of a space Z is the one-point compactification if Z is not compact, and is Z itself if Z is compact.)

THEOREM 3.5. *If QX is locally compact, there is a topologically unique quasicompactification Y of X such that each component of Y is the Alexandroff compactification of a quasicomponent of X.*

Proof. For each quasicomponent A of X, let $\bar{A}$ denote the closure of A in F^*X and let $\mathscr{G} = \{\bar{A} - A \mid A \in q\mathscr{C}_X\}$. It is easy to see that $\mathscr{G}$ is an upper semicontinuous collection of disjoint closed subsets of F^*X. If $Y = FX/\mathscr{G}$ and $p\colon F^*X \to Y$ is the projection map, it is clear that for each $A \in QX$, $p(\bar{A})$ is the Alexandroff compactification of A. Since no element of $\mathscr{G}$ intersects two components of F^*X, it easily follows that the components of Y are precisely the sets $p(A)$ for $A \in QX$.

By definition, the map $\phi\colon QX \to Q(F^*X)$ defined by $\phi(A) = \bar{A}$ for $A \in QX$ is a homeomorphism. Since the map $\psi\colon Q(F^*X) \to QY$ defined by $\psi(\bar{A}) = p(\bar{A})$ is also a homeomorphism, so is the map $\psi \circ \phi\colon QX \to QY$ and it follows that Y is a quasicompactification of X.

To see that Y is topologically unique, suppose Y' is any quasicompactification of X with each component of Y' the Alexandroff compactification of a quasicomponent of X; for simplicity, assume that $X \subset Y'$. For each noncompact quasicomponent A of X, let $p_A = (\mathrm{cl}_Y A) - A$ and $p'_A = (\mathrm{cl}_{Y'} A) - A$. If $h: Y \to Y'$ is defined by $h(x) = x$ for $x \in X$ and $h(p_A) = p'_A$ for A a noncompact quasicomponent of X, then h is a homeomorphism of Y onto Y'.

The quasicompactification Y described in the above proof will be called the *Alexandroff quasicompactification* of X, and will be denoted by αX.

4. Shape properties. For an arbitrary topological space X, we denote the shape of X in the sense of Mardešić [10] by $\mathrm{Sh}\, X$; if X is locally compact and metrizable, $\mathrm{Sh}_p X$ will denote the proper shape of X in the sense of [4] and $\mathrm{Sh}^1_p X$ the proper shape of X in the alternative sense described in [3]. The following "decomposition theorem", which was essentially proved in [4], probably has analogues for $\mathrm{Sh}^1_p X$ and for $\mathrm{Sh}\, X$, though this is by no means clear (to the present author, at least).

THEOREM 4.1. *If X and Y are locally compact metric spaces and $X = \bigoplus \{X_\alpha \mid \alpha \in A\}$, then $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ (respectively, $\mathrm{Sh}_p X \leqq \mathrm{Sh}_p Y$) if and only if $Y = \bigoplus \{Y_\alpha \mid \alpha \in A\}$ with $\mathrm{Sh}_p X_\alpha = \mathrm{Sh}_p Y_\alpha$ (resp., $\mathrm{Sh}_p X_\alpha \leqq \mathrm{Sh}_p Y_\alpha$) for each $\alpha \in A$.*

Proof. It follows from Lemma 5.5 of [4] and from the proof of Lemma 5.8 of the same paper that $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ if and only if there exist locally compact ANR's P and Q containing X and Y, respectively, as closed subsets, and proper fundamental nets $\underline{f}: X \to Y$ in (P, Q), $\underline{g}: Y \to X$ in (Q, P) such that $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_{X,P}$ and $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_{Y,Q}$; an analogous condition, requiring only that $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_{X,P}$, characterizes the relation $\mathrm{Sh}_p X \leqq \mathrm{Sh}_p Y$. It follows immediately that if $X = \bigoplus \{X_\alpha \mid \alpha \in A\}$ and $Y = \bigoplus \{Y_\alpha \mid \alpha \in A\}$ with $\mathrm{Sh}_p X_\alpha = \mathrm{Sh}_p Y_\alpha$ (respectively, $\mathrm{Sh}_p X_\alpha \leqq \mathrm{Sh}_p Y_\alpha$) for each $\alpha \in A$, then $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ (respectively, $\mathrm{Sh}_p X_\alpha \leqq \mathrm{Sh}_p Y_\alpha$).

For the converse, suppose $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ (respectively, $\mathrm{Sh}_p X \leqq \mathrm{Sh}_p Y$) and $X = \bigoplus \{X_\alpha \mid \alpha \in A\}$. As above, there exist locally compact ANR's P and and Q containing X and Y, respectively, as closed subsets, and proper fundamental nets $\underline{f}: X \to Y$ in (P, Q), $\underline{g}: Y \to X$ in (Q, P) such that $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_{X,P}$ and $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_{Y,Q}$ (or only the first of these if $\mathrm{Sh}_p X \leqq \mathrm{Sh}_p Y$). Theorem 5.2 of [4] shows that Y can be written as $Y = \bigoplus \{Y_\alpha \mid \alpha \in A\}$ with $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_{X_\alpha,P}$ and $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_{Y_\alpha,Q}$ (or only the first of these, if $\mathrm{Sh}_p X \leqq \mathrm{Sh}_p Y$) for each $\alpha \in A$. It follows that $\mathrm{Sh}_p X_\alpha = \mathrm{Sh}_p Y_\alpha$ (resp., $\mathrm{Sh}_p X_\alpha \leqq \mathrm{Sh}_p Y_\alpha$), as required.

LEMMA 4.2. *If* $\mathrm{Sh}_p X \leqq \mathrm{Sh}_p Y$ *and* QX *is compact, then* QY *is compact.*

Proof. Lemma 2.2 implies that X is separable and hence ([4], Lemma 5.3) Y is also separable. By Lemma 2.2, QY is a Lindelöf space. If QY is not compact then, since it is 0-dimensional, $QY = \bigoplus Z_n$, where $Z_1, Z_2, \cdots$ is an infinite set of disjoint (nonempty) open subsets of QY. By Lemma 2.1, $Y = \bigoplus Y_n$ with $QY_n = Z_n$ for each n, and by Theorem 4.1, $X = \bigoplus X_n$ with $\mathrm{Sh}_p X_n \leqq \mathrm{Sh}_p Y_n$ for each n. But then $QX = \bigoplus QX_n$, and this is impossible since QX is compact and therefore cannot be the union of infinitely many disjoint open sets.

THEOREM 4.3. *If* $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ *and* QX *is locally compact, then* QY *is locally compact and* $\mathrm{Sh}_p F^*X = \mathrm{Sh}_p F^*Y$.

Proof. By Lemma 2.4, $X = \bigoplus \{X_\alpha \mid \alpha \in A\}$ with each QX_α compact. By Theorem 4.1, $Y = \bigoplus \{Y_\alpha \mid \alpha \in A\}$ with $\mathrm{Sh}_p X_\alpha = \mathrm{Sh}_p Y_\alpha$ for each α; since QX_α is compact, QY_α is compact by Lemma 4.2 and since $QY = \bigoplus \{QY_\alpha \mid \alpha \in A\}$, QY is locally compact. By definition, $F^*X = \bigoplus \{FX_\alpha \mid \alpha \in A\}$ and $F^*Y = \bigoplus \{FY_\alpha \mid \alpha \in A\}$; since $\mathrm{Sh}_p X_\alpha = \mathrm{Sh}_p Y_\alpha$, Corollary 4.8 of [4] implies $\mathrm{Sh}\, FX_\alpha = \mathrm{Sh}\, FY_\alpha$, and hence by Theorem 4.1, $\mathrm{Sh}_p F^*X = \mathrm{Sh}_p F^*Y$.

NOTE. In the version of this paper presented at the 1978 Topology Conference in Warsaw, it was claimed that the hypothesis of Theorem 4.3 implies also that $\mathrm{Sh}_p \alpha X = \mathrm{Sh}_p \alpha Y$. The author's argument for this has proved to be defective, and the conjecture remains unsettled.

Finally, we show that Theorem 4.1 can be used to obtain a proper shape version of a result due to Y. Kodama ([9], Theorem 2), without the restriction that the spaces involved be finite dimensional.

LEMMA 4.4. *If each component of* X *is compact, then* X *is the topological sum of compact subspaces.*

Proof. By Theorem 1.1, $\mathscr{C}_X (= q\mathscr{C}_X)$ is upper semicontinuous. Hence the projection map $p: X \to QX (= X/\mathscr{C}_X)$ is closed and since each point-inverse under p is compact, p is a compact mapping (i.e., the inverse of any compact set is compact). Moreover, by Lemma 2.4, $X = \bigoplus \{X_\alpha \mid \alpha \in A\}$ with QX_α compact for each α. Since $X_\alpha =$

$p^{-1}(QX_\alpha)$, X_α is compact.

THEOREM 4.5. *If* $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ *and each component of* X *is compact, then each component of* Y *is compact and there is a homeomorphism* $\Lambda: QX \to QY$ *such that for each locally compact subset* F *of* QX, $\mathrm{Sh}_p p^{-1}(F) = \mathrm{Sh}_q q^{-1}(\Lambda(F))$, *where* $p: X \to QX$ *and* $q: Y \to QY$ *are the projection maps.*

Proof. By Lemma 4.4, $X = \bigoplus\{X_\alpha \mid \alpha \in A\}$ with each X_α compact and hence by Theorem 4.1, $Y = \bigoplus\{Y_\alpha \mid \alpha \in A\}$ with $\mathrm{Sh}_p X_\alpha = \mathrm{Sh}_p Y_\alpha$ for each α; since X_α is compact, Y is compact ([4], p. 172) and therefore $\mathrm{Sh}\, X_\alpha = \mathrm{Sh}\, Y_\alpha$ by Theorem 3.15 of [4]. Hence by Theorem 2.2 of [1], for each $\alpha \in A$ there is a homeomorphism $\Lambda_\alpha: QX_\alpha \to QY_\alpha$ such that for every compact subset K_α of QX_α, $\mathrm{Sh}\, p^{-1}(K_\alpha) = \mathrm{Sh}\, q^{-1}(\Lambda(K_\alpha))$. Let $\Lambda: QX \to QY$ be the combination of the Λ_α's(i.e., $\Lambda(a) = \Lambda_\alpha(a)$ if $a \in X_\alpha$). If F is a locally compact subset of QX and for each α, $F_\alpha = F \cap QX_\alpha$, then F_α is a locally compact subset of QX. The argument for Lemma 2.3 of [1], using Theorem 4.1 in in place of Theorem 4.2 of [7], shows that $\mathrm{Sh}_p\, p^{-1}(F_\alpha) = \mathrm{Sh}_p q^{-1}(\Lambda_\alpha(F_\alpha))$. Since $p^{-1}(F) = \bigoplus\{p^{-1}(F_\alpha) \mid \alpha \in A\}$ and $q^{-1}(\Lambda(F)) = \bigoplus\{q^{-1}(\Lambda_\alpha(F_\alpha)) \mid \alpha \in A\}$, it follows from Theorem 4.1 that $\mathrm{Sh}_p\, p^{-1}(F) = \mathrm{Sh}_p\, q^{-1}(\Lambda(F))$.

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