

COUNTABLE SPACES WITHOUT POINTS OF FIRST COUNTABILITY

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In this paper we show that there are 2^c non-homeomorphic countable regular spaces, each of which has no point of first countability. Several specific countable regular spaces are shown not to be homeomorphic.

1. Preliminaries. A countable space need not be first countable. One example of such a space is $N \cup \{p\}$ where $p \in \beta N - N$ and the topology is the relative topology of βN . This space, however, has many points of first countability—indeed all of the points of N are isolated. Several examples of countable spaces without points of first countability are known to exist.

N denotes the space of natural numbers including 0, Q denotes the space of rational numbers, and \mathbf{R} denotes the space of reals. The cardinal of \mathbf{R} is denoted c . If X is a completely regular Hausdorff space, βX is the Stone–Cech compactification of X . If X and Y are spaces and $f: X \rightarrow Y$ is a continuous surjection, f is *irreducible* if there is no proper closed subset K of X such that $f(K) = Y$. It is well-known (see for example [11], 10.48) that if X and Y are compact Hausdorff spaces and $f: X \rightarrow Y$ is a continuous surjection, there is a closed subset K of X such that $f(K) = Y$ and the restriction of f to K is irreducible. A space X is *resolvable* if X contains disjoint dense subsets. A space X is *homogeneous* if for any pair of points $p, q \in X$, there is a homeomorphism $f: X \rightarrow X$ such that $f(p) = q$. A *rigid* space is a space whose only auto-homeomorphism is the identity.

For $n \in N$, let R_n be $N^{\{1, \dots, n\}}$. Since for $n = 0$, $\{1, \dots, n\} = \emptyset$, $R_0 = \{\emptyset\}$. The empty set, when viewed as the element of R_0 , is denoted p_0 . Let $S = \bigcup_{n \in N} R_n$ and define an order \leq on S by $p \leq q$ if and only if $p \in R_m$, $q \in R_n$ with $m \leq n$ and $q \upharpoonright \{1, \dots, m\} = p$. (S, \leq) is a tree (see [6]) and is clearly countably infinite. For $x \in S$, A_x is the set $\{p \in S: x \leq p, x \neq p, \text{ and } x \leq y \leq p \text{ implies } x = y \text{ or } y = p\}$; thus, A_x is the set of immediate successors of x . For $x \in S$, $U \subseteq N$, let $K_x^U = \{x\} \cup \{p \in S: \text{There is a } q \in A_x \text{ such that the last entry of } q \text{ is an element of } U \text{ and } q \leq p\}$.

If $p \in \beta N - N$, that is, p is a free ultrafilter on N , then Σ_p denotes the subspace $N \cup \{p\}$ of βN . Two points p and q of $\beta N - N$ are the *same βN -type*, or simply the *same type*, if Σ_p is homeomorphic to Σ_q , or,

equivalently, if there is a permutation φ of N such that the elements of q are precisely the images under φ of the elements of p .

All given spaces are assumed to be completely regular and Hausdorff.

For general background see [3] or [11].

2. Construction of 2^c countable spaces. Suppose $\mathcal{U} \in \beta N - N$. A topology on S is defined by taking as a subbase the collection $\{K_x^U : x \in S, U \in \mathcal{U}\}$. (It is not hard to show that in fact $\{K_x^U : x \in S, U \in \mathcal{U}\}$ is a base for a topology, although we do not need this fact.) The topological space which arises in this way is denoted $S_{\mathcal{U}}$. For each \mathcal{U} , $S_{\mathcal{U}}$ is Hausdorff. To see this, notice first that if $n \in N$, $N - \{n\} \in \mathcal{U}$. Suppose $p = (p_1, \dots, p_m)$ and $q = (q_1, \dots, q_n)$ are distinct elements of S , with $m \leq n$. If p and q are not comparable under \leq , the sets K_p^N and K_q^N are disjoint neighborhoods of p and q . If $p \leq q$, $K_p^{N-(p_{m+1})}$ and K_q^N are disjoint neighborhoods of p and q . Each $S_{\mathcal{U}}$ is also zero-dimensional. To check this, we show that each set of the form K_p^U is closed, whether or not $U \in \mathcal{U}$. Suppose $q \notin K_p^U$. If q and p are not comparable, K_q^N is a neighborhood of q which does not intersect K_p^U . If $q \leq p$, let $r = (r_1, \dots, r_k)$ be the element of A_q such that $r \leq p$. Then $K_q^{N-(r_k)}$ is a neighborhood of q which does not intersect K_p^U . If $p \leq q$, then since $q \notin K_p^U$, it follows that $x \notin K_p^U$ for any x such that $q \leq x$, that is, $K_q^N \cap K_p^U \neq \emptyset$. This proves that $S_{\mathcal{U}}$ is zero-dimensional. Furthermore, since a zero-dimensional Hausdorff space is regular, and since a regular Lindelof space is normal, $S_{\mathcal{U}}$ is normal for each \mathcal{U} .

We now claim that for each \mathcal{U} , $S_{\mathcal{U}}$ is homogeneous. It suffices to show that if $q \in S$, there is a homeomorphism $f_q : S_{\mathcal{U}} \rightarrow S_{\mathcal{U}}$ such that $f_q(q) = p_0$, because then $f_p^{-1} \circ f_q$ is a homeomorphism of $S_{\mathcal{U}}$ mapping q to p . We first show that there is an order-preserving homeomorphism $g : S_{\mathcal{U}} \rightarrow K_q^N$. Suppose $q = (n_1, \dots, n_k)$. Define g by $g(p_0) = q$, and $g(m_1, \dots, m_l) = (n_1, \dots, n_k, m_1, \dots, m_l)$. The function g is clearly one-to-one and onto. $g(K_{(m_1, \dots, m_l)}^U) = K_{(n_1, \dots, n_k, m_1, \dots, m_l)}^U$ so g is an open map. $g^{-1}(K_{(n_1, \dots, n_k, m_1, \dots, m_l)}^U) = K_{(m_1, \dots, m_l)}^U$ so g is continuous. Now define $f : S_{\mathcal{U}} \rightarrow S_{\mathcal{U}}$ by $f|K_{g(q)}^N$ is the identity map, $f(S_{\mathcal{U}} - K_q^N) = g(S_{\mathcal{U}} - K_q^N)$, and $f|(K_q^N - K_{g(q)}^N) = g^{-1}(K_q^N - K_{g(q)}^N)$. Then $f \circ f$ is the identity map on $S_{\mathcal{U}}$, so f is one-to-one and onto. $f(p_0) = q$, so $f(q) = f(f(p_0)) = p_0$. Furthermore, the restriction of f to each of the clopen sets $(S_{\mathcal{U}} - K_q^N)$, $(K_q^N - K_{g(q)}^N)$, and $K_{g(q)}^N$ is a homeomorphism, so f is a homeomorphism.

$S_{\mathcal{U}}$ is not first countable. To see this, let the map $h : \Sigma_{\mathcal{U}} \rightarrow \{p_0\} \cup A_{p_0}$ be defined by

$$h(x) = \begin{cases} p_0 & \text{if } x = \mathcal{U} \\ n & \text{if } x = n. \end{cases}$$

Then h is a homeomorphism, so p_0 is not a point of first countability. Furthermore, since $S_{\mathcal{U}}$ is homogeneous, $S_{\mathcal{U}}$ has no point of first countability.

Finally, we observe that there are 2^c nonhomeomorphic spaces $S_{\mathcal{U}}$. For there are 2^c types of ultrafilters on N . In a given space $S_{\mathcal{U}}$, at most c spaces Σ_p can appear as subspaces. Thus, if there were fewer than 2^c homeomorphism classes of $S_{\mathcal{U}}$'s, there would be fewer than 2^c types of ultrafilters, a contradiction.

We summarize in the following proposition.

PROPOSITION 2.1. *There are 2^c nonhomeomorphic countable regular homogeneous spaces each of which fails to have any point of first countability.*

REMARK. The construction of $S_{\mathcal{U}}$ can be modified to give rigid countable regular spaces without point of first countability. To do this, enumerate the elements of S as $\{p_k : k \in N\}$ and let $\{\mathcal{U}_k : k \in N\}$ be a collection of free ultrafilters on N such that if $k \neq n$, $\Sigma_{\mathcal{U}_k}$ is not homeomorphic to any subspace of $\Sigma_{\mathcal{U}_n}$. Now take as a neighborhood base at p_k the set $\{K_{p_k}^U : U \in \mathcal{U}_k\}$. Then S with this topology can be shown to have the required properties. An argument similar to the counting argument used in proving 2.1 shows that there are 2^c such spaces no two of which are homeomorphic.

3. Other countable spaces. In this section we examine some other countable spaces without points of first countability. We will list for the sake of later reference some well-known and some easy facts.

LEMMA 1. ([3], 9H). *Every countable subset of an extremally disconnected space is C^* -embedded.*

LEMMA 2. ([8]). *A space is extremally disconnected if and only if every dense subset is C^* -embedded.*

LEMMA 3. *A countable subspace of an extremally disconnected space is extremally disconnected.*

Proof. If A is a countable subspace of the extremally disconnected space X , every subset A is C^* -embedded in X (Lemma 1) and hence in A . Therefore, by Lemma 2, A is extremally disconnected.

LEMMA 4. *An extremally disconnected space without isolated points has no convergent sequence of distinct points and hence no point of first countability.*

Proof. If $a_n \rightarrow b$ where $a_n \neq a_k$, if $n \neq k$, the function $f: \{a_n\} \rightarrow \mathbf{R}$ defined by $f(a_n) = (-1)^n$ does not extend continuously to b , so the countable set $\{a_n\}$ is not C^* -embedded. Hence by Lemma 1, no space which contains a nontrivial convergent sequence is extremally disconnected.

The following example was described in a letter by Professor W. Comfort. Let $\varphi: \beta N - N \rightarrow [0, 1]$ be a continuous surjection. Let K be a closed subset of $\beta N - N$ such that $\varphi(K) = [0, 1]$ and $\varphi|_K$ is irreducible. Let A be a countable dense subset of $[0, 1]$ and for each $a \in A$ choose $c_a \in (\varphi|_K)^{-1}(a)$. Then $\{c_a: a \in A\}$ is a countable subset of $\beta N - N$ whose closure maps onto $[0, 1]$ by an irreducible map. Let \mathcal{C} be the set of all countable subsets of $\beta N - N$ whose closures map onto $[0, 1]$ by a continuous irreducible map. We have seen that $\mathcal{C} \neq \emptyset$. Now suppose $C \in \mathcal{C}$ and $f: C1_{\beta N}C \rightarrow [0, 1]$ is an irreducible map.

C is resolvable. To see this, let A_1 and A_2 be disjoint dense subsets of $f(C)$. Then $f^{-1}(A_1) \cap C$ and $f^{-1}(A_2) \cap C$ are disjoint subsets of C . We claim that each is dense in C . $f(C1_{\beta N}(f^{-1}(A_1) \cap C))$ is a compact subset of $[0, 1]$ containing the dense subset A_1 of $[0, 1]$. Therefore, $f(C1_{\beta N}(f^{-1}(A_1) \cap C)) = [0, 1]$. Since f is irreducible, $C1_{\beta N}(f^{-1}(A_1) \cap C) = C1_{\beta N}C$, so $f^{-1}(A_1) \cap C$ is dense in C . Similarly, $f^{-1}(A_2) \cap C$ is dense in C .

Since a resolvable space can have no isolated points, C is dense-in-itself. By Lemma 3, the space C is extremally disconnected. Lemma 4 implies that C has no points of first countability, and, in fact, no nontrivial convergent sequences.

Every dense subset of $C1_{\beta N}C$ is separable. The proof of this fact consists of observing that in the above argument that $f^{-1}(A_1) \cap C$ is dense in C , the crucial facts are that C is dense in $C1_{\beta N}C$ and that $f^{-1}(A_1) \cap C$ maps via f onto a dense subset of $[0, 1]$. Thus if X is any dense subset of $C1_{\beta N}C$ and Y is any countable subset of X such that $f(Y)$ is dense in $f(X)$, then Y is dense in X . This shows that every dense subset of $C1_{\beta N}C$ contains a dense subset which is in \mathcal{C} .

We summarize the above properties of elements of \mathcal{C} in the following proposition.

PROPOSITION 3.1. *If \mathcal{C} is the set of all countable subsets of $\beta N - N$ whose closures in βN map onto $[0, 1]$ by an irreducible map, then if $C \in \mathcal{C}$, C is extremally disconnected, resolvable, and has no point of first countability. Furthermore, every dense subset of $C1_{\beta N}C$ contains a dense subset which is itself an element of \mathcal{C} .*

E. van Dowen, in [1], using a technique of Hewitt [5], shows that there are nonresolvable countable regular spaces without isolated points and with the property that any subset which has no isolated point and

whose complement has no isolated point is open. Let \mathcal{D} denote the collection of all such spaces. Then if $D \in \mathcal{D}$ and U is an open subset of D , neither $C1_D U$ nor $D - C1_D U$ has isolated points. Hence, the closure of any open subset of D is open, that is, D is extremally disconnected. Therefore, by Lemma 4, the space D has no point of first countability.

PROPOSITION 3.2. *There are 2^c nonhomeomorphic extremally disconnected resolvable countable spaces without points of first countability and 2^c nonhomeomorphic extremally disconnected nonresolvable countable spaces without points of first countability.*

Proof. Let X be a countable extremally disconnected space without points of first countability (for example, $X \in \mathcal{C}$ or $X \in \mathcal{D}$). X is not countably compact—a countably compact countable space is compact and a compact extremally disconnected space has cardinal at least 2^c . Therefore X contains a closed copy of N , that is, N may be viewed as a closed subspace of X . By Lemma 1, the subspace N is C^* -embedded in X , so $\beta N = C1_{\beta X} N$ and $\beta N - N \subseteq \beta X - X$. Let $X_0 = X$ and suppose a countable space X_α is defined for each $\alpha < \gamma$ where γ is an ordinal less than 2^c . Since a countable set has only c subsets, at most $c \mid \gamma \mid = \max \{c, \mid \gamma \mid\}$ spaces Σ_p are homeomorphic to a subset of some X_α , $\alpha < \gamma$. Let $p_\gamma \in \beta N - N$ be an ultrafilter such that Σ_{p_γ} is not homeomorphic to any subset of any X_α , $\alpha < \gamma$. Let $X_\gamma = X \cup \{p_\gamma\}$. Then $\{X_\gamma : 1 \leq \gamma < 2^c\}$ has cardinal 2^c . If $\alpha < \gamma$, X_γ contains the space Σ_{p_γ} , whereas X_α contains no subset homeomorphic to Σ_{p_γ} . Therefore, no two elements of $\{X_\gamma : 1 \leq \gamma < 2^c\}$ are homeomorphic. For each γ , $X \subseteq X_\gamma \subseteq \beta X$, so each X_α is extremally disconnected and hence by Lemma 4 has no point of first countability. Since $X_\alpha - X$ consists of exactly one point for $\alpha \geq 1$, each X_α is countable. Furthermore, since for a given α , X is dense and open in X_α , every space X_α is resolvable if and only if X is. Thus, if $X \in \mathcal{C}$, each X_α is resolvable and if $X \in \mathcal{D}$, each X_α is nonresolvable.

In [7] it is shown that every dense subset of $\beta Q - Q$ is separable. This fact is established by showing that there is a continuous surjection $F: \beta Q \rightarrow [0, 1]$ such that A is dense in βQ if and only if $F(A)$ is dense in $[0, 1]$. Let \mathcal{B} be the set of countable dense subsets of $\beta Q - Q$. If $B \in \mathcal{B}$, then since $\beta Q - Q$ is dense in βQ ([3], 7E), B is dense in βQ and therefore dense-in-itself. If A_1 and A_2 are disjoint dense subsets of $F(B)$, then by the above property of F , $F^{-1}(A_1) \cap B$ and $F^{-1}(A_2) \cap B$ are disjoint dense subsets of B . Therefore, B is resolvable. Finally, since B is dense in βQ , any point of first countability of B would be a G_δ of βQ ([3], 9.7), and no point of $\beta Q - Q$ is a G_δ of βQ ([3], 9.6). Thus, B has no point of first countability.

Mrowka proves in [9] that any countable dense subset of $\mathbf{R}^{\mathbf{R}}$ with the product topology fails to have any point of first countability. Let \mathcal{M} be the set of countable dense subsets of $\mathbf{R}^{\mathbf{R}}$. Suppose $M \in \mathcal{M}$. Then M is resolvable, since if A_1 and A_2 are disjoint dense subsets of $\{f(0) : f \in M\}$, then $\{f \in M : f(0) \in A_1\}$ and $\{f \in M : f(0) \in A_2\}$ are disjoint dense subsets of M . It is proved in [7] that M has no compactification each of whose dense subsets is separable.

Let \mathcal{U} be a free ultrafilter on N . For each $p_k \in A_{p_k}$ choose $x_k \in A_{p_k}$. Let $W = \bigcup_{k=1}^{\infty} K_{x_k}^N$. The subset W is open in $S_{\mathcal{U}}$ and $C1_{S_{\mathcal{U}}}W = W \cup \{p_0\}$ which is not open in $S_{\mathcal{U}}$. Thus $S_{\mathcal{U}}$ is not extremally disconnected. $S_{\mathcal{U}}$ can be written as the union of disjoint dense subsets by $S_{\mathcal{U}} = \bigcup_{k=0}^{\infty} R_{2k} \cup \bigcup_{k=0}^{\infty} R_{2k+1}$. Therefore, $S_{\mathcal{U}}$ is resolvable.

We now combine the above comments.

PROPOSITION 3.3. *Let \mathcal{C} denote the set of all countable subsets of $\beta N - N$ whose closures map irreducibly onto $[0, 1]$. Let \mathcal{D} denote the class of countable, nonresolvable spaces without isolated points such that each set without isolated points and whose complement has no isolated points is open. Let \mathcal{B} be the set of countable dense subsets of $\beta Q - Q$. Let \mathcal{M} denote the set of countable dense subsets of $\mathbf{R}^{\mathbf{R}}$. Suppose $C \in \mathcal{C}$, $D \in \mathcal{D}$, $B \in \mathcal{B}$, $M \in \mathcal{M}$, and $\mathcal{U} \in \beta N - N$.*

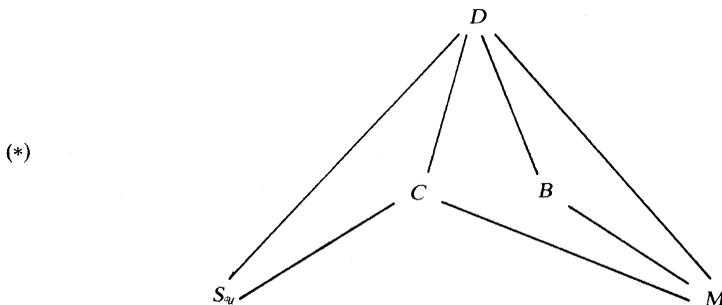
- (i) M is not homeomorphic to B , C , or D .
- (ii) D is not homeomorphic to B , C , or $S_{\mathcal{U}}$.
- (iii) C is not homeomorphic to $S_{\mathcal{U}}$.

Proof. (i) C and D are extremally disconnected whereas M is not. B has a compactification every dense subset of which is separable whereas M has no such compactification.

(ii) D is not resolvable whereas each of the spaces, B , C , and $S_{\mathcal{U}}$ is resolvable.

(iii) C is extremally disconnected whereas $S_{\mathcal{U}}$ is not.

We can summarize Proposition 3.3 in the following graph, the edges of which represent nonhomeomorphism:



4. Completing the graph. The graph (*) is not complete because no edge joins B and C , or B and $S_{\mathcal{U}}$, or M and $S_{\mathcal{U}}$. In this section we show that if \mathcal{U} , B , and M are chosen in particular ways, the graph can be completed. We also show that there are elements of \mathcal{C} and \mathcal{B} which are homeomorphic. We need several facts.

THEOREM (Efimov, [2]). *Any extremally disconnected space of weight at most c (in particular, any countable extremally disconnected space) can be embedded in βN .*

A point $p \in \beta Q - Q$ is called a *remote point* of βQ if p is not in the closure of any discrete subset of Q . Eric van Downen has recently proved the following.

THEOREM (van Downen). *βQ has a dense set of remote points.*

THEOREM (Woods, [12]). *Suppose X is a dense subspace of βQ . Then X is extremally disconnected if and only if every point of X is a remote point of βQ .*

PROPOSITION 4.1. *There are homeomorphic spaces B and C such that $B \in \mathcal{B}$ and $C \in \mathcal{C}$. In fact, every extremally disconnected element of \mathcal{B} is homeomorphic to an element of \mathcal{C} .*

Proof. We first note that there are extremally disconnected elements of \mathcal{B} . For by van Downen's theorem, the set of remote points of βQ is dense in $\beta Q - Q$, and so by Wood's theorem, any countable dense subset of the set of remote points is extremally disconnected. By Efimov's theorem, B is homeomorphic to a subspace C of $\beta N - N$. We show $C \in \mathcal{C}$. The countable space C is C^* -embedded in βN and hence in $C1_{\beta N}C$, that is, $C1_{\beta N}C = \beta C$. Let $f: C \rightarrow B$ be a homeomorphism and $f^*: \beta C \rightarrow \beta Q$ be its Stone extension. Let $g: Q \rightarrow Q \cap [0, 1]$ be a homeomorphism and let $g^*: \beta Q \rightarrow [0, 1]$ be its Stone extension. f^* and g^* are each irreducible ([3], 6.12). It follows that $g^* \circ f^*: C1_{\beta N}C \rightarrow [0, 1]$ is irreducible. Therefore, $C \in \mathcal{C}$.

REMARKS. 1. By Efimov's theorem, every countable extremally disconnected space without points of first countability is embeddable in $\beta N - N$. However, not every such space is homeomorphic to an element of \mathcal{C} . For example, by 3.3 no element of \mathcal{D} is homeomorphic to an element of \mathcal{C} .

2. We do not know if every element of \mathcal{C} is homeomorphic to an element of \mathcal{B} .

Proposition 4.1 says that some elements of \mathcal{B} are homeomorphic to

elements of \mathcal{C} . However, if B is any countable dense subset of $\beta Q - Q$ which contains a nonremote point of βQ , then by Wood's theorem, B is not extremally disconnected and hence not homeomorphic to an element of \mathcal{C} .

PROPOSITION 4.2. *There are elements of \mathcal{B} which are not homeomorphic to any elements of \mathcal{C} .*

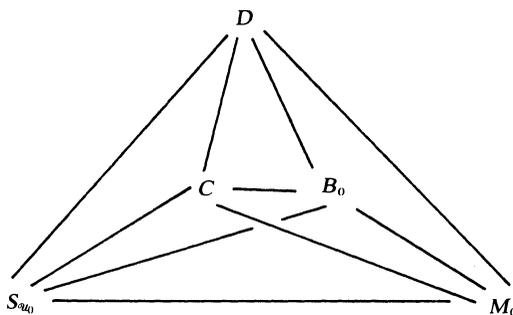
We do not know if there is an ultrafilter $\mathcal{U} \in \beta N - N$ and a space $M \in \mathcal{M}$ such that $S_{\mathcal{U}}$ is homeomorphic to M . There are, however, elements of \mathcal{M} which are not homeomorphic to $S_{\mathcal{U}}$ for any $\mathcal{U} \in \beta N - N$. For example, if M is the set of polynomials with rational coefficients, M has convergent sequences of distinct elements, such as the sequence of constant functions $\{1/n : n \in N\}$. It can be shown that none of the spaces $S_{\mathcal{U}}$ has any nontrivial convergent sequence.

We also do not know if any of the spaces $S_{\mathcal{U}}$ is homeomorphic to any element of \mathcal{B} . However, by 2.1, for a given $B \in \mathcal{B}$ and a given $M \in \mathcal{M}$, there is a $\mathcal{U} \in \beta N - N$ such that $S_{\mathcal{U}}$ is homeomorphic to neither B nor M .

PROPOSITION 4.3. *Suppose \mathcal{C} , \mathcal{D} , \mathcal{B} , and \mathcal{M} are the collections of spaces defined in Proposition 3.3. Then there is a $B_0 \in \mathcal{B}$, $M_0 \in \mathcal{M}$, and $\mathcal{U}_0 \in \beta N - N$ such that if $C \in \mathcal{C}$ and $D \in \mathcal{D}$, no two of the spaces $B_0, C, D, M_0, S_{\mathcal{U}_0}$ are homeomorphic.*

Proof. Choose B_0 as in 4.2, let M_0 be any element of \mathcal{M} , and choose $\mathcal{U}_0 \in \beta N - N$ such that $S_{\mathcal{U}_0}$ is homeomorphic to neither B_0 nor M_0 .

Proposition 4.3 can be summarized in the following complete graph, where again the edges represent nonhomeomorphism.



5. A final observation. A famous theorem of Sierpinski characterizes the rationals among metric spaces. Since a countable first

countable space is metrizable by the Urysohn metrization theorem, Sierpinski's theorem can be stated as follows.

THEOREM (Sierpinski, [10]). *Any countable first countable space without isolated points is homeomorphic to Q .*

By 2.1 (or 3.2 or 3.3 or 4.3), there is no such characterization for countable spaces without points of first countability. However, we note that all such spaces share the property that they have weaker topologies homeomorphic to Q .

PROPOSITION 5.1. *If X is countable and has no isolated point, then there is a continuous, one-to-one function from X onto Q .*

Proof. For each $p, q \in X$, $p \neq q$, let $U_{p,q}$ be a clopen subset of X containing p but not q . The collection $\{U_{p,q} : p, q \in X, p \neq q\} \cup \{X - U_{p,q} : p, q \in X, p \neq q\}$ is a countable subbase for a (regular) topology on set X . Let \hat{X} denote the set X with this new topology. \hat{X} is second countable and has no isolated point (since X has no isolated point) and hence, by Sierpinski's theorem, is homeomorphic to Q . Thus, X has a weaker topology homeomorphic to Q .

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