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TORSION FREE ABELIAN GROUPS QUASI-PROJECTIVE OVER THEIR ENDOMORPHISM RINGS

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Certain classes of torsion free abelian groups which are quasi-projective as modules over their endomorphism rings are characterized. The main results concern completely decomposable and strongly indecomposable groups.

1. Preliminaries. Abelian groups which are quasi-projective over their endomorphism rings have been characterized by Fuchs in the torsion case. His methods have been extended by Longtin to the algebraically compact and cotorsion groups **[5]**. In this paper, we investigate some other classes of groups with this property. Specifically:

DEFINITION. A (left) module M over a ring R is quasi-projective provided the natural map $\mathrm{Hom}_R(M, M) \rightarrow \mathrm{Hom}_R(M, M/K)$ is epic for every submodule K, of M.

An abelian group G will be called E-quasi-projective (Eqp) provided G is quasi-projective as a module over $E = \mathrm{End}(G)$. Henceforth, the word group will denote a torsion free abelian group. Other notation follows Fuchs **[4]**, in particular, $t(G) = \text{type } G$ for any group G of rank 1.

The following simple lemmas will be quite useful.

LEMMA 1.1. *Let G be* Eqp *and K a fully invariant subgroup of G. Then G/K is a quasi-projective E-module.*

Proof. See Proposition 2.1 in Wu and Jans **[9]**.

LEMMA 1.2. *Let G be* Eqp *and K a fully invariant subgroup. Then ZE, the center of E, maps onto* $\mathrm{Hom}_E(G/K, G/K)$.

Proof. Let Π: $G \rightarrow G/K$ be the factor map. Since G is Eqp, for every $\theta \in \mathrm{Hom}_E(G/K, G/K)$, there exists $\alpha \in \mathrm{Hom}_E(G, G) = ZE$ such that $\Pi\alpha = \theta\Pi$.

LEMMA 1.3. *Let G be* Eqp *and K a fully invariant subgroup such that G/K is torsion. Then if ZE is countable, G/K is bounded.*

Proof. If G/K is unbounded and torsion, then $\mathrm{Hom}_E(G/K, G/K)$ is uncountable: it must contain either a copy of Q_p^* (the ring of p-adic integers) for some prime p, or a direct product of an infinite number of cyclic groups. Hence the result follows from Lemma 1.2.

2. Decomposable groups. In this section, some classes of decomposable Eqp groups are characterized, including completely decomposable and homogeneous separable groups.

We begin with completely decomposable groups, those groups G isomorphic to a direct sum of rank one groups.

LEMMA 2.1. *If $G = \bigoplus_{i \in I} \Sigma A_i$ is a direct sum of rank one groups, then G is indecomposable as an E-module if and only if given any two summands A_i and A_j, there is a finite sequence $A_i = A_{i_1}, A_{i_2}, \cdots, A_{i_n} = A_j$ such that $t(A_{i_k})$ is comparable to $t(A_{i_{k+1}})$ for $k = 1, 2, \cdots, n-1$.*

Proof. If S is a subset of the set $\{A_i\}_{i \in I}$, define $I(S) = \{A_l \mid t(A_l)$ is comparable to $t(A_m)$ for some A_m in $S\}$ and $I^n(S) = I(I^{n-1}(S))$. Then it is easy to see that for a fixed A_{i_0}, $\bigoplus \Sigma\{A_i \mid A_i \in \bigcup_{n=1}^{\infty} I^n(A_{i_0})\}$ and $\bigoplus \Sigma\{A_i \mid A_i \notin \bigcup_{n=1}^{\infty} I^n(A_{i_0})$ are E-submodules whose sum is G. The lemma follows immediately.

LEMMA 2.2. *If $G = \bigoplus \Sigma A_i$ is completely decomposable and indecomposable as an $E = \mathrm{End}(G)$ module, then $ZE \subseteq Q$.*

Proof. Maps in ZE must commute with projections and maps $A_i \to A_j$. The fact that G is E-indecomposable and Lemma 2.1 imply that any map in ZE multiplies each A_i by the same rational number.

THEOREM 2.1. *Let $G = \bigoplus_{i \in I} A_i$ be a direct sum of rank one groups such that G is indecomposable as an E-module. Then the following are equivalent:*

1. *G is* Eqp.
2. *The type set $T = \{t_i \mid t_i = t(A_i)$ for some $i\}$ satisfies:*

(a) *If $t_i, t_j \in T$ and $t_i, t_j \leqq t_k$ for some $t_k \in T$, then $t_i, t_j \geqq t_l$ for some $t_l \in T$;*

(b) *Countable descending chains in T are bounded below;*

(c) *If $t_i \in T$ is finite at an infinite set of positions $\{p_j\}$, then $\exists t_k \in T$ such that t_k is 0 at an infinite subset of $\{p_j\}$.*

3. If K is a fully invariant subgroup of G such that G/K is torsion, then G/K is bounded.

Proof. (1) $\Rightarrow$ (2). Let $t_i, t_j, t_k \in T$ such that $t_i, t_j \leqq t_k$. Suppose there is no t_l such that $t_l \leqq t_i, t_j$. Let $K = \bigoplus \Sigma\{A_m \mid t_m \not\leqq t_i$ and $t_m \not\leqq t_j\}$. Then K is a fully invariant subgroup of G, and G/K is a direct sum of two E-modules, $G/K = B_1 \bigoplus B_2$ where $B_1 = \bigoplus \Sigma\{A_m \mid t_m \leqq t_i\} + K$ and $B_2 = \bigoplus \Sigma\{A_m \mid t_m \leqq t_j\} + K$. But multiplication by integers n_1 on B_1 and $n_2 \neq n_1$ on B_2 is an E-map of G/K to G/K which is not induced by a map in ZE. By Lemma 1.2, this is a contradiction.

Now suppose $t_{i_1} \leqq t_{i_2} \lneqq \cdots$ is a countable descending chain of types in T which is not bounded below. Let p be a prime not dividing A_{i_1} and define K to be sum of $\{A_i \mid t_i \not\leqq t_{i_1}\}$ and $\{p^k A_j \mid t_j \leqq t_{i_k}, t_j \not\leqq t_{i_{k+1}}, k = 1, 2, \cdots\}$. Then K is fully invariant and G/K is an unbounded torsion group. Since G is E-indecomposable, by Lemmas 2.2 and 1.3, this is a contradiction.

Finally, assume $t_0 \in T$ is finite at an infinite set of positions $\{p_j\}$, and suppose no $t_i \leqq t_0$ is zero on an infinite subset of $\{p_j\}$. Then for each $t_i \leqq t_0$, choose $x_i \in A_i$ such that p_j-height $x_i \geqq 1$ *for all* p_j. Now let H be the minimal fully invariant subgroup containing the x_i. Since homomorphisms do not decrease height, $(1/p_j)x_0 \not\in H$ for any p_j. Thus $A_0/A_0 \cap H$ contians a copy of $Z(p_j)$ for each p_j. By Lemma 1.3 this is a contradiction.

(2) $\Rightarrow$ (3). Let H be fully invariant in G such that G/H is torsion. Suppose first that for some A_i, $A_i/A_i \cap H$ is unbounded, with nonzero p_k-component for an infinite set $P = \{p_k\}$ of primes. Note that $A_i/A_i \cap H$ contains no $Z(p^\infty)$ since rank $A_i = 1$ and $A_i \cap H$ is fully invariant in A_i. We may, therefore, assume that t_i is finite and positive at all $p_k \in P$. By condition 2(c), there exists $t_j < t_i$ such that t_j is zero at an infinite subset of P. Since H is fully invariant and $t_j < t_i$, $A_i \cap H \subseteq p_k A_i$ implies $A_j \cap H \subseteq p_k A_j$ for all $p_k \in P$. This is impossible since t_j is zero at infinitely many p_k.

Now if G/H is unbounded, choose a countable sequence $A_{i_1}, A_{i_2}, \cdots$ such that $\bigoplus \Sigma_{r=1}^{\infty} A_{i_r}/A_{i_r} \cap H$ is unbounded. By conditions 2(a) and 2(b) there exists a fixed A_i with $t_i \leqq t_{i_r}$ for all $r \geqq 1$. It follows that $A_i/A_i \cap H$ must be unbounded. This is impossible, as above.

(3) $\Rightarrow$ (1). It is easy to show that if G/K is bounded for all fully invariant K with G/K torsion, then $\mathrm{Hom}_E(G, G/K) = \{n\Pi \mid n \in Z\}$ where $\Pi: G \to G/K$ is the natural factor map. It follows that $\mathrm{Hom}_g(G, G/K) = \{n\Pi \mid n \in Z\}$, for any fully invariant K.

The above theorem characterizes the completely decomposable Eqp groups since any completely decomposable group G may be expressed as a direct sum $\bigoplus \Sigma G_i$ of E-indecomposable subgroups which are completely decomposable, and in this decomposition $\mathrm{End}(G_i) = E|_{G_i}$.

COROLLARY 2.1. *Let G be completely decomposable of finite rank with type set T. Then G is* Eqp *iff T satisfies* 2(*a*) *and minimal types in T are idempotent.*

Proof. T is finite so that minimal types are idempotent iff 2(c) holds. Since 2(b) holds vacuously, the result follows.

COROLLARY 2.2. *Let* $G = \bigoplus_{i\in I} \Sigma A_i$ *with* $\{A_i \mid i \in I\}$ *rigid* **[4]**. *Then* G *is* Eqp *iff* $t(A_i)$ *is idempotent for all* $i \in I$.

Proof. If $\{A_i\}$ is rigid, (a) and (b) hold vacuously and (c) holds iff each $t(A_i)$ is idempotent.

REMARK. Since E is commutative if $\{A_i\}$ is rigid, Corollary 2.2 can also be derived from a trivial modification of a result of Arnold (**[1]**, Theorem 1.1).

EXAMPLE. The following is a nontrivial (uncountable E-indecomposable) example of a completely decomposable group satisfying 2(a), 2(b) and 2(c) of 2 in Theorem 2.1.

Define a relation on the set I of all infinite subsets of the natural numbers by $S \leqq T$ iff $S \setminus T$ is finite. Let $\{S_\alpha\}$ be a maximal chain in I. It is easy to see that $\{S_\alpha\}$ is uncountable. For each α, define a type t_α by $t_\alpha = [\langle x_i^\alpha \rangle]$; $x_i^\alpha = 1$, $i \in S_\alpha$; $x_i^\alpha = 0$, $i \notin S_\alpha$. It is easy to see that $\{t_\alpha\}$ satisfies 2(a) and (b) of Theorem 2.1. By the maximality of the chain $\{S_\alpha\}$, $\{t_\alpha\}$ also satisfies 2(c). Let $A = \bigoplus \Sigma_\alpha A_\alpha$, where A_α is of rank one and type t_α. Then A is Eqp by Theorem 2.1.

We next characterize homogeneous separable Eqp groups (**[4]**, §87).

LEMMA 2.3. *Let* G *be homogeneous and separable. Then* $ZE \subseteq Q$.

Proof. This is an easy exercise. (See **[4]**, Problem 12, page 235.)

LEMMA 2.4. *Let* G *be homogeneous and separable. Then, for all nonzero fully invariant* $K \subseteq G$, *we have* G/K *is a torsion group.*

Proof. Let $0 \neq K$ be fully invariant in G. Choose $0 \neq x \in K$. Since G is homogeneous separable we can write $G = \langle x \rangle_* \bigoplus G'$, where $\langle x \rangle_*$ denotes the pure subgroup generated by x. If $G' \subseteq K$, then $G/K = \langle x \rangle_* / \langle x \rangle_* \cap K$ and G/K is torsion. Otherwise, choose $y \in G' \setminus G' \cap K$. Since G' is also homogeneous separable, write $G = \langle x \rangle_* \bigoplus \langle y \rangle_* \bigoplus G''$. Since G is homogeneous, there exists $\alpha \in E$, $n \in Z^+$ with $\alpha(x) = ny$. Thus, $ny \in K$. Since y was an arbitrary element of $G' \setminus G' \cap K$, we have $G/K = \langle x \rangle_* / \langle x \rangle_* \cap K \bigoplus G'/G' \cap K$ is torsion.

REMARK. The claim made in Problem 13, page 235 of **[4]** is incorrect. Any rank one group of nil type will sei e as a counterexample.

A group G is called strongly irreducible iff for all nonzero fully invariant $K \subseteq G$, G/K is bounded. (See **[7]**.)

THEOREM 2.2. *Let G be a homogeneous separable group. Then G is* Eqp *iff G is strongly irreducible.*

Proof. If G is homogeneous, separable and Eqp, Lemmas 1.3, 2.3 and 2.4 show that G is strongly irreducible.

Conversely, let G be strongly irreducible, homogeneous and separable. Let $K \neq (0)$ be fully invariant in G and $\theta \in \operatorname{Hom}(G, G/K)$. Write G/K in its primary decomposition, $G/K = \bigoplus \Sigma_{i=1}^{N} (G/K)_{p_i}$. Say, for some fixed $p_j \in \{p_i \mid i = 1 \cdots N\}$, we have $(G/K)_{p_j} = \bigoplus \Sigma_{\alpha \in A_j} \langle \bar{a}_\alpha \rangle$ with order $(\bar{a}_\alpha) = p_j^{s_\alpha}$, $s_\alpha \leqq s_j$ (Here $\bar{a} = a + K$). Since θ is an E map and G is homogeneous separable, it is easy to show that, for some fixed $m_j \in Z^+$, we must have $\theta(a_\alpha) = m_j \bar{a}_\alpha$ for all $\alpha \in A_j$. Choose $m \in Z^+$ with $m \equiv m_j (p_j^{s_j}), j = 1 \cdots N$. Then $\Pi m = \theta$.

The final results of this section deal with groups G which can be written as a sum of two groups related in a special way. We will need the notions of outer type (OT) and inner type (IT) of a group as defined in Warfield **[8]**.

THEOREM 2.3. *Let $G = A \oplus B$ where $IT(A) > OT(B)$ and let $\bar{E} = \operatorname{End}(B)$. Then G is* Eqp *iff B is* $\bar{\text{E}}$qp *and rank $Z\bar{E} = 1$.*

Proof. ($\Rightarrow$) Let K be an $\bar{E}$-submodule of B. Then $A \oplus K$ is an E-submodule of G since $\operatorname{Hom}(A, B) = 0$. Therefore, any $\bar{E}$-map $\theta\colon B \to B/K$, induces an E-map $0 \oplus \theta\colon A \oplus B \to A \oplus B/A \oplus K$ which must lift to a map in ZE of the form $\alpha \oplus \beta$, where $\alpha\colon A \to A$, $\beta\colon B \to B$. It follows that β is an $\bar{E}$-map which lifts θ.

Now suppose rank $Z\bar{E} > 1$. Choose $\gamma \in Z\bar{E}$ and $b \in B$, such that b, $\gamma(b)$ are independent. Then $0 \oplus \gamma\colon A \oplus B \to (A \oplus B)/A$ is an E map and lifts as above to a map of the form $\alpha \oplus \beta$ in ZE. Since $IT(A) > OT(B)$, there exists $\delta \in \operatorname{Hom}(B, A)$ such that $\delta(b) = 0$ and $\delta(\gamma(b)) \neq 0$. But then $0 = \alpha\delta(b) = \delta\beta(b) = \delta\gamma(b) \neq 0$, a contradiction.

($\Leftarrow$) Let K be a fully invariant subgroup of G, and $\theta\colon G \to G/K$ an E-map. Then $K = K \cap A \oplus K \cap B$ and $\theta(B) \subseteq B/B \cap K$ so that θ restricted to B may be lifted to a map $\alpha \in Z\bar{E} \subseteq Q$. Since $IT(A) > OT(B)$, $A = \bigcup_{f:B \to A} \operatorname{Image} f$. It follows that $\alpha\colon A \to A$ must be a lifting of $\theta|_A$.

REMARK. This theorem may be generalized slightly to the case $IT(A) \geqq OT(B)$.

COROLLARY. *If* $G = D \oplus R$ *where* D *is divisible and* R *is reduced, then* G *is* Eqp *iff* R *is* $E(R)qp$ *and rank* $ZE(R) = 1$.

3. Strongly indecomposable groups. In this section we characterize the strongly indecomposable Eqp groups of finite rank. We start by characterizing the strongly indecomposable, strongly irreducible ones. Recall that a group G is called strongly indecomposable if it admits no nontrivial quasi-decompositions ([**4**], §92).

THEOREM 3.1. *Let* G *be strongly indecomposable, strongly irreducible of finite rank. Then* G *is* Eqp *iff* G/P^kG *is a cyclic* E *module for all nonzero prime ideals* $P \subseteq E$.

Proof. Suppose G is Eqp. Since G is strongly indecomposable and strongly irreducible, we can conclude that E is a subring of an algebraic number field F with $QE = F$. (See [**7**].) Note that E is Noetherian and $P \neq (0)$ prime in E implies P is maximal. (Since $QE = F$, every nonzero ideal $I \subseteq E$ contains a nonzero rational integer. Thus, E/I is finite.)

We show G/P^kG is a cyclic E module for all nonzero prime ideals $P \subseteq E$. If not, let $X = \{\bar{x}_1 \cdots \bar{x}_n\}$ be a minimal set of E generators for G/P^kG, where $\bar{x}_i = x_i + P^kG$. Let H be given by $E\bar{x}_1 \cap \Sigma_{i=2}^n E\bar{x}_i = H/P^kG$. Then H is fully invariant and $G/H = A \oplus B$ with $A = E\bar{\bar{x}}_1$, $B = \Sigma_{i=2}^n E\bar{\bar{x}}_i$, where $\bar{\bar{x}}_i = x_i + H$. This is a nontrivial direct sum decomposition because of the minimality of X.

Let f, g be the projections from G/H onto A, B and $\Pi\colon G \to G/H$ the natural map. Let $\bar{f}, \bar{g} \in E$ be such that $\Pi\bar{f} = f\Pi$, $\Pi\bar{g} = g\Pi$. Finally, let $I = \{\alpha \in E \mid \alpha(G) \subseteq H\}$. Then $P^k \subseteq I$, so $I \subseteq p$ (primes in E are maximal). Clearly $\bar{f}\bar{g} \in I \subseteq P$, so $\bar{f} \in P$ or $\bar{g} \in P$. If $\bar{f} \in P$, $PA = A$. Thus, $P^kA = A$, so $IA = A$. But $IA = (0)$ and $A \neq (0)$ — a contradiction. A similar contradiction arises from the assumption $\bar{g} \in P$. Thus G/P^kG is cyclic.

Conversely, let G be strongly indecomposable strongly irreducible of finite rank with G/P^kG cyclic for all nonzero primes $P \subseteq E$. We show, for all positive rational integers n, G/nG is E cyclic. Let $n \in Z^+$. Since $(0) \neq (n) \subseteq E$ and E is Noetherian we have $(n) \supseteq P_1^{k_1} \cdots P_s^{k_s}$ with the P_i's nonzero prime ideals in E ([**10**], page 200). Now the ideals $P_i^{k_i}$, $i = 1 \cdots s$, are co-maximal in E ([**9**], page 176) and, by assumption, $G/P_i^{k_i}G$ is E-cyclic. It is easy to show (using the Chinese Remainder Theorem in E) that $G/(\Pi P_i^{k_i})G$ is E-cyclic. Thus, G/nG is E-cyclic.

Now let $\theta\colon G \to G/K$ be an E map, $(0) \neq K$ a fully invariant subgroup of G. Since G is strongly irreducible, $nG \subseteq K$ for some positive integer n. Thus G/K is E-cyclic, say $G/K =$

$E(g+K)$. Choose $\alpha \in E$ with $\theta(g) = \alpha(g+K)$. We claim that α is a lifting of θ. To show this, it only remains to show that $\theta(K) = (0)$. Let $G/nG = E(h+nG)$. Then for any $k \in K$, $k = \beta h + nx$ for some $\beta \in E$, $x \in G$. Now $\beta h \in K$, so, since E is commutative, $\beta(G) \subseteq K$. Thus, $\theta\beta(G) = \beta\theta(G) = (0)$ in G/K. Finally, $\theta(k) = \theta\beta(h) + n\theta(x) = 0 + K$. This shows that G is Eqp and completes the proof.

We now consider the general case, and begin with a more general definition of quasi-projectivity which is invariant under quasi-isomorphism.

DEFINITION. If R is a ring, an R-module M is *almost* quasi-projective, if there exists a positive integer t such that the image of $\mathrm{Hom}_R(M, M)$ in $\mathrm{Hom}_R(M, M/N)$ is bounded by t for every submodule N of M.

LEMMA 3.1. *If $M \sim N$ are (quasi-isomorphic) R modules, and M is almost quasi-projective, then N is almost quasi-projective.*

Proof. Without loss of generality, assume $nM \subseteq N \subseteq M$ for some positive integer n. Let K be a submodule of N and $f: N \to N/K$. Then $nf: M \to M/K$ lifts to a map $\bar{f} \in \mathrm{Hom}_R(M, M)$ such that $\Pi\bar{f} = tnf$ where $\Pi: M \to M/K$. Then $n\bar{f} \in \mathrm{Hom}_R(N, N)$ and is a lifting of tn^2f. Hence N is almost quasi-projective.

LEMMA 3.2. *Let G be strongly indecomposable and almost* Eqp. *Then there is a $g \in G$ such that G/Eg is bounded.*

Proof. Choose $\{g_1, \cdots, g_k\}$ of minimal cardinality with respect to $G/Eg_1 + Eg_2 + \cdots + Eg_k$ is bounded. This is possible by Lemma 1.3. If $k > 1$, let $H = Eg_1 \cap \Sigma_{i=2}^k Eg_i$. Then H is fully invariant and $\Sigma_{i=1}^k Eg_i/H = Eg_1 + H \bigoplus \Sigma_{i=2}^k Eg_i + H$. Furthermore, $Eg_1 + H$ is not torsion since $ng_1 \in H \Rightarrow nEg_1 \subseteq H$, contradicting the minimality of k. Since G is strongly indecomposable, any $\alpha \in E$ is either monic or nilpotent (see **[6]**). But if t is a positive integer such that $tG \subseteq \Sigma_{i=1}^k Eg_i$, then t followed by projection onto $\Sigma_{i=2}^k Eg_i + H$ is a map from G to G/H which cannot be lifted, as the lifting could be neither monic nor nilpotent. Thus $k = 1$, proving the lemma.

THEOREM 3.2. *Let G be strongly indecomposable of finite rank. Then G is* Eqp *iff G is strongly irreducible and G/P^kG is a cyclic E module for all nonzero prime ideals $P \subseteq E$.*

Proof. In view of Theorem 3.1, we only need show that strongly indecomposable Eqp groups of finite rank are strongly irreducible.

By the preceding Lemma $nG \subseteq Eg \subseteq G$ for some n and $Eg \cong E/L$ (as E-modules) for some left ideal $L \subseteq J(E)$, the Jacobson radical of E. Therefore by Lemma 3.1, E/L is almost Eqp, with associated integer t, for some $t > 0$.

Now for any $x \in E$ consider

$$\begin{array}{ccc} & E/L & \\ \swarrow^{\alpha} & \downarrow x_l & \\ E/L & \xrightarrow{\pi} E/Lx + L \to 0 & \end{array} \qquad \begin{array}{l} x_l = \text{left multiplication by } x \\ Lx + L \subseteq J(E) \neq E. \end{array}$$

Then $n\alpha$ is an E endomorphism of G, hence in ZE. Furthermore $n\alpha - tx_l\colon E \to Lx + L$, so that $n\alpha - tx \in Lx + L$. Hence $tx \in ZE + Lx + L$. This implies $Ltx \subseteq L + L^2x$, so that $t^2x \in Z(E) + L^2x + L$. Continuing inductively $t^kx \in Z(E) + L^kx + L$. Since L is nilpotent ($L \subseteq J(E)$), for some $m > 0$, $L^m = 0$ and $t^mx \in ZE + L$. Thus $t^mE \subset Z(E) + L$, and $G \sim E/L \sim Z(E) + L/L \cong ZE/L \cap ZE$, a commutative ring with identity. By ([2], Th. 1.4, Cor. 3.6, Th. 1.13) G must be strongly irreducible.

COROLLARY 3.1. *Let G be finite rank strongly indecomposable with rank $E <$* rank G. *Then G is not* Eqp.

Proof. For any $0 \neq g \in G$, Eg is a fully invariant subgroup of G with rank $Eg \leqq \operatorname{rank} E < \operatorname{rank} G$. Thus, G is not strongly irreducible, so G cannot be Eqp.

4. Groups of rank two. In this section we use the results of §§1–3 to survey the Eqp property for groups of rank two. This is most conveniently done by considering the six possibilities for the quasi-endomorphism algebra, $QE(G) = Q \otimes_Z E(G)$. (See [3].) If $QE(G) \cong [Q]_{2\times 2}$ or $QE(G) \cong \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix} \,\middle|\, x, y, z \in Q \right\}$ then G is completely decomposable. In the first case we have $G = A \oplus A$, and in the second case $G = A \oplus B$ with A, B of rank one, $t(A) < t(B)$. In either case Corollary 2.1 applies; G is Eqp iff $t(A)$ is idempotent. If $QE(G) \cong Q \oplus Q$, then G is quasi-decomposable $G \sim A \oplus B$ with $t(A)$, $t(B)$ incomparable. A slight modification of the arguments of Theorem 2.1 prove that G is Eqp iff $t(A)$ and $t(B)$ are idempotent.

We next consider the strongly indecomposable cases. If $QE(G) \cong$

Q or $QE(G) \cong \left\{ \begin{pmatrix} x & 0 \\ y & x \end{pmatrix} \middle| \, x, y \in Q \right\}$, then G is strongly indecomposable but not strongly irreducible, so G is not Eqp by Theorem 3.2. We settle the final possibility, $QE(G) \cong Q(\sqrt{N})$, in the following theorem.

THEOREM 4.1. *Let G be of rank two with $QE(G) \cong Q(\sqrt{N})$. Then g is* Eqp *iff G is strongly irreducible.*

Proof. If G is Eqp, G is strongly irreducible by Theorem 3.2. Conversely, let G be strongly irreducible and K any nonzero fully invariant subgroup of G. Write the finite group G/K in its primary decomposition: $G/K = \bigoplus_{i=1}^{n} (G/K)_{p_i}$. Since rank $G = 2$, K is fully invariant, and $QE(G) = Q(\sqrt{N})$, it is easy to show, for each p_i, either $(G/K_i)_{p_i} = Z(p_i^{s_i})$ for some $s_i \geqq 0$ in Z, or $(G/K_i)_{p_i} = Z(p_i^{t_i}) \oplus Z(p_i^{t_i})$ for some $t_i \geqq 0$ in Z. Moreover, in the latter case we can choose $a \in G$ so that $a + K_i$ and $\sqrt{N}a + K_i$ are generators of $(G/K_i)_{p_i}$. It is now easy to check that G/K is a cyclic E module. Thus, Theorem 3.2 applies and G is Eqp.

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