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ANALYTIC MAPS OF THE OPEN UNIT DISK ONTO A GLEASON PART

KAZUO KISHI

The purpose of this paper is to show that in certain uniform algebras all analytic maps (for the definition see §2) of the open unit disk onto a nontrivial Gleason part are mutually closely related (Theorem 2), and that these maps are isometries of the open unit disk with pseudo-hyperbolic metric onto a nontrivial Gleason part with part metric (Theorem 3).

The results of this paper are contained in §2. Some necessary preliminaries are given in §1.

1. Preliminaries. A *uniform algebra* A on a compact Hausdorff space X is a uniformly closed subalgebra of the algebra $C(X)$ of complex valued continuous functions on X which contains the constants and separates the points of X. Let $\mathscr{M}(A)$ denote the maximal ideal space of A which has the Gelfand topology. Let $\hat{f}$ be the Gelfand transform of f in A and let $\hat{A} = \{\hat{f}: f \in A\}$.

For φ and θ in $\mathscr{M}(A)$ we define

$$(1.1) \qquad G: G(\varphi, \theta) = \sup\{|\varphi(f) - \theta(f)|: f \in A, \|f\| \leqq 1\},$$

$$(1.2) \qquad \sigma: \sigma(\varphi, \theta) = \sup\{|\varphi(f)|: f \in A, \|f\| \leqq 1, \theta(f) = 0\},$$

where $\|f\| = \sup\{|f(x)|: x \in X\}$, and we write $\varphi \sim \theta$ when $G(\varphi, \theta) < 2$ (or, equivalently, $\sigma(\varphi, \theta) < 1$). Then $\sim$ is an equivalence relation in $\mathscr{M}(A)$, and an equivalence class $P(m) = \{\varphi: \varphi \in \mathscr{M}(A), \varphi \sim m\} (\supsetneqq \{m\})$ is called the (*nontrivial*) *Gleason part* of $m \in \mathscr{M}(A)$ (cf. Gleason [2]). $G(\varphi, \theta)$ and $\sigma(\varphi, \theta)$ are metrics on $P(m)$ (cf. König [5]).

If $m \in \mathscr{M}(A)$ has a unique representing measure μ_m, i.e., if m has a unique probability measure μ_m on X such that $m(f) = \int f d\mu_m$ for every $f \in A$, then every φ in $P(m)$ also has a unique representing measure μ_φ. It is also known that φ in M_A belongs to $P(m)$ if and only if there exist mutually absolutely continuous representing measures μ_φ, μ_m for φ, m respectively; indeed, there exists a constant c $(0<c<1)$ such that $c\mu_\varphi \leqq \mu_m$ and $c\mu_m \leqq \mu_\varphi$.

For example, let $A(D)$ denote the disk algebra of all continuous functions on the closed unit disk $\bar{D} = \{s: |s| \leqq 1\}$ in the plane wich are analytic in the open unit disk D. Then $\mathscr{M}(A(D))$ can be identified with $\bar{D}$, and the open unit disk D is one part. For $t, s \in D$, we see that

$$(1.3)\qquad G(t, s) = \sup\{|f(t) - f(s)|: f \in A(D), \|f\| \leqq 1\}\,,$$

$$(1.4)\qquad \sigma(t, s) = \sup\{|f(t)|: f \in A(D), \|f\| \leqq 1, f(s) = 0\} = \left|\frac{t-s}{1-\bar{s}t}\right| \text{ (pseudo-hyperbolic metric)}\,.$$

Throughout the rest of this paper, we do not distinguish in notations $\varphi \in \mathscr{M}(A)$ from its representing measure when φ has a unique representing measure, and we suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m. Let $A_m = \{f \in A; m(f) = 0\}$, the kernel of a complex homomorphism m. Let $H^\infty(m)$, H^∞_m be the weak-star closures of A, A_m in $L^\infty(m)$ respectively, and for $1 \leqq p < \infty$ let $H^p(m)$, H^p_m be the closures of A, A_m in $L^p(m)$ norm respectively. If φ belongs to $P(m)(\supsetneqq \{m\})$, then for $1 \leqq p < \infty$ the spaces $H^p(m)$ and $H^p(\varphi)$ are identical as sets of (equivalence classes of) functions; as Banach spaces, they have distinct but equivalent norms. Under the same hypothesis, the Banach algebras $H^\infty(m)$ and $H^\infty(\varphi)$ are identical.

For a Dirichlet algebra Wermer [7] showed the following theorem, and Hoffman [3] generalized Wermer's result to a logmodular algebra (cf. Browder [1], Chap. IV). Functions in $H^\infty(m)$ of unit modulus are called *inner functions*.

THEOREM 1 (WERMER'S EMBEDDING THEOREM). *Let A be a uniform algebra on a compact space X. Suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. Then we have the following.*

(1) *There is an inner function Z*(= Wermer's embedding function) *such that $ZH^2(m) = H^2_m$.*

(2) *If $\varphi \in P$, define $\hat{Z}(\varphi) = \int Z\, d\varphi$. Then $\hat{Z}$ is a one-one map of the part P onto the open unit disk D in the plane. The inverse map τ of $\hat{Z}$ is a one-one continuous map of D onto P* (with the Gelfand topology).

(3) *For every f in A the composed function $\hat{f}\circ\tau$ is analytic on D.*

Let φ be an element of the Gleason part $P(m)$ of m in $\mathscr{M}(A)$. Then there is a function h in $L^\infty(m)$ such that $\varphi(f) = \int f\, d\varphi = \int fh\,dm$ for all $f \in A$, so φ has a unique extension to a linear functional $\tilde{\varphi}$ on $H^\infty(m)$ which is both multiplicative and weak-star continuous. For any $f \in H^\infty(m)$ $\tilde{\varphi}$ has the form

$$\tilde{\varphi}(f) = \int f\, d\varphi = \int fh\,dm\,.$$

We call $\tilde{\varphi}$ the *measure extension* of φ in $P(m)$.

PROPOSITION. *Let $A, m, P(m)$ and Z be as in Wermer's embedding theorem. Let $\mathscr{P} = \mathscr{P}(m)$ be the set of measure extension $\tilde{\varphi}$ of φ in $P(m)$. Then we have the following.*

(1) *$\mathscr{P}$ is the nontrivial Gleason part of $\tilde{m}$ in $\mathscr{M}(H^\infty(m))$.*

(2) *The map $\hat{Z}|\mathscr{P}$ is a one-one continuous map of the part $\mathscr{P}$* (with the Gelfand topology) *onto the open unit disk D, and thus the inverse map $\tilde{\tau}$ of $\hat{Z}|\mathscr{P}$ is a homeomorphism of D onto $\mathscr{P}$.*

Proof. If the Gelfand transform $\hat{H}^\infty(m) = \hat{H}^\infty$ of $H^\infty(m)$ is restricted to the maximal ideal space Y of $L^\infty(m)$, then $\hat{H}^\infty$ is a logmodular algebra on Y (see Hoffman [3], Theorem 6.4, corollary), and therefore every complex homomorphism φ of $H^\infty(m)$ has a unique representing measure on Y (see [3], Theorem 4.2). In particular, $\tilde{m} \in \mathscr{M}(H^\infty(m))$ has a unique representing (normal) measure $\tilde{m}$ on the hyperstonean space Y. Then for every f in $L^\infty(m)$ we have

$$\int_X f\,dm = \int_Y \hat{f}\,d\tilde{m}\ .$$

And we can identify $L^\infty(\tilde{m})$ with $C(Y) = \hat{L}^\infty(m)$ (cf. Srinivasan-Wang [6], pp. 221–223).

Now let Σ be the Gleason part of $\tilde{m}$. For $\tilde{\varphi}$ in $\mathscr{P}$, we have

$$\tilde{\varphi}(f) = \int_X f\,d\varphi = \int_X fh\,dm = \int_Y \hat{f}\hat{h}\,d\tilde{m}\quad (f \in H^\infty(m))\ , \tag{1.5}$$

where h is a function in $L^\infty(m)$ with $a < h < b$ for some positive constants a and b. From this we see that $\tilde{\varphi}$ is in Σ.

Conversely if λ is an element of Σ, then λ has a unique representing measure $\tilde{\lambda}$ on Y, and we have for every $f \in H^\infty(m)$

$$\lambda(f) = \int_Y \hat{f}\,d\tilde{\lambda} = \int_Y \hat{f}\frac{d\tilde{\lambda}}{d\tilde{m}}\,d\tilde{m}\ .$$

Since $d\tilde{\lambda}/d\tilde{m}$ is an element of $L^\infty(\tilde{m})$, there is a function h in $L^\infty(m)$ such that $d\tilde{\lambda}/d\tilde{m} = \hat{h}$ a.e. $(d\tilde{m})$. Hence we have

$$\lambda(f) = \int_Y \hat{f}\hat{h}\,d\tilde{m} = \int_X fh\,dm$$

and thus $\lambda \in \mathscr{P}$. So we get $\mathscr{P} = \Sigma$. Then the rest part (2) of the proposition follows from Theorem 1.

2. Results.

DEFINITION. Let $P(m)$ be the nontrivial Gleason part of m in the maximal ideal space $\mathscr{M}(A)$ of a uniform algebra A. A one-one continuous map $\rho(t)$ of the open unit disk D onto $P(m)$ (with the Gelfand topolgy) is called an *analytic map* if the composition $\hat{f}(\rho(t))$ is analytic on D, for every f in A.

Now we are in a position to prove the following theorem.

THEOREM 2. *Let A be a uniform algebra on a compact space X. Suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. Let $\tau(t)$ be an analytic map of the open unit disk D onto P which is obtained in Theorem 1. If $\rho(t)$ is an analytic map of D onto P such that $\rho(\alpha) = m$, then we have*

$$\rho(t) = \tau\left(\beta\frac{t-\alpha}{1-\bar{\alpha}t}\right), \tag{2.1}$$

where β is a constant of modulus 1. Furthermore, $\tau(t)$ is a homeomorphism if and only if $\rho(t)$ is a homeomorphism.

Proof. Let $\mathscr{P}$, Z, and $\tilde{\tau}$ be as in Theorem 1 and proposition. For any $t \in D$, $\rho(t)$ has a unique representing measure $h_t dm$, where h_t is an element of $L^\infty(m)$. Let $\tilde{\rho}(t)$ be the measure extension of $\rho(t)$ i.e., $\tilde{\rho}(t)(f) = \int f h_t dm$ for all $f \in H^\infty(m)$. For each $f \in H^\infty(m)$ there exists a sequence $\{f_n\}$ in A such that $\|f_n\| \leqq \|f\|$ for all n and $f_n \to f$ a.e. (dm) (Hoffman-Wermer theorem, see [1], Theorem 4.2.5). Then, by Lebesgue's dominant convergence theorem, $\rho(t)(f_n) = \int f_n h_t dm \to \tilde{\rho}(t)(f)$ for every t in D. Since $\rho(t)(f_n)(n = 1, 2, \cdots)$ are analytic in D and $|\rho(t)(f_n)| \leqq \|f_n\| \leqq \|f\|$, we see that, for every f in $H^\infty(m)$, $\tilde{\rho}(t)(f)$ is analytic in D (Vitali's theorem). Hence we see that $\tilde{\rho}(t)$ is an analytic map of D onto $\mathscr{P}$. If we set $g(t) = (\tilde{\tau}^{-1} \circ \tilde{\rho})(t) = \hat{Z}(\tilde{\rho}(t))$, then $f(t)$ is a one-one holomorphic map of D onto D, and $g(\alpha) = 0$. Hence we see

$$g(t) = \beta\frac{t-\alpha}{1-\bar{\alpha}t},$$

where β is a constant of modulus 1, and thus we have

$$\tilde{\tau}\left(\beta\frac{t-\alpha}{1-\bar{\alpha}t}\right) = \tilde{\rho}(t).$$

Since $\tilde{\tau}(t)|A = \tau(t)$ and $\tilde{\rho}(t)|A = \rho(t)$ we have

$$\tau\left(\beta\frac{t-\alpha}{1-\bar{\alpha}t}\right) = \rho(t).$$

Next we prove that $\tau(t)$ is a homeomorphism if and only if $\rho(t) = \tau(\beta(t-\alpha)/(1-\bar{\alpha}t))$ is a homeomorphism. We put $L_\alpha(t) = (t+\alpha)/(1+\bar{\alpha}t)$ and $\beta = e^{i\theta}$. Then $\tau(t)$ is a homeomorphism of D onto P if and only if $\hat{Z}(\varphi)\left(= \int Z d\varphi\right)$ is a continuous map of P onto D if and only if $L_\alpha \circ e^{-i\theta} \circ \hat{Z}$ is a continuous map of P onto D if and only if

$$(L_\alpha \circ e^{-i\theta} \circ \hat{Z})^{-1}(t) = \tau(e^{i\theta} L_{-\alpha}(t)) = \tau\left(e^{i\theta} \frac{t-\alpha}{1-\bar{\alpha}t}\right) = \rho(t)$$

is a homeomorphism, and the theorem is proved.

Next we shall prove the following theorem which generalizes a formula (6.12) in Hoffman [4], p. 105.

THEOREM 3. *Let A be a uniform algebra on X. Suppose that $m \in \mathscr{M}(A)$ has a unique representing measure m on X, and that the part P of m consists of more than one point. If $\rho(t)$ is an analytic map of the open unit disk D onto $P(m)$, then we have*

$$\sigma(\rho(t), \rho(s)) = \sigma(t, s)\ ,$$
$$G(\rho(t), \rho(s)) = G(t, s)\ .$$

For the definitions of σ, G see (1.1) ~ (1.4).

Proof. Let Z, $\mathscr{P}$, τ and $\tilde{\tau}$ be as Theorem 1 and proposition. Let $\tilde{\tau}(t) = \tilde{\varphi}$, $\tilde{\tau}(s) = \tilde{\theta}$, $\tau(t) = \varphi$ and $\tau(s) = \theta$. From Lemma 4.4.4 in Browder [1], we see that

$$f \in H^\infty_{\tilde{\theta}} = \left\{f : f \in H^\infty(m) = H^\infty(\theta),\ \tilde{\theta}(f) = \int f d\theta = 0\right\}$$

if and only if $f \in (Z-s)H^\infty(m)$, and from this we easily get $H^\infty_{\tilde{\theta}} = \{(Z-s)/(1-\bar{s}Z)\}H^\infty(m)$. So we have

$$\begin{aligned}\sigma(\tilde{\varphi}, \tilde{\theta}) &= \sup\{|\tilde{\varphi}(f)|;\ f \in H^\infty(m),\ \|f\| \leqq 1,\ \tilde{\theta}(f) = 0\} \\ &= \sup\left\{|\tilde{\varphi}(f)| : f \in \frac{Z-s}{1-\bar{s}Z} H^\infty(m),\ \|f\| \leqq 1\right\} \\ &= \sup\left\{\left|\tilde{\varphi}\left(\frac{Z-s}{1-\bar{s}Z}\right)\tilde{\varphi}(g)\right| : g \in H^\infty(m),\ \|g\| \leqq 1\right\} \\ &= \left|\frac{t-s}{1-\bar{s}t}\right| = \sigma(t, s)\ .\end{aligned}$$

Since the closures of A_θ and $H^\infty_{\tilde{\theta}}$ in $L^2(m)$ are the same $H^2_\theta = \left\{f : f \in H^2(m) = H^2(\theta),\ \int f d\theta = 0\right\}$, we have the following equalities from the result which is stated as "the perhaps surprising equality" in

Browder [1], p. 134. (Note that $\tilde{\varphi}(f) = \int_Y \hat{f}\hat{h}d\tilde{m} = \int_Y \hat{f}d\tilde{\varphi}$ (see (1.5)) and $\int_X |f|^2 d\varphi = \int_Y |\hat{f}|^2 d\tilde{\varphi}$, for any $f \in H^\infty_{\tilde{\theta}}$.)

$$\begin{aligned}\sigma(\varphi, \theta) &= \sup\{|\varphi(f)|: f \in A_\theta, \|f\| \leqq 1\} \\ &= \sup\left\{|\varphi(f)|: f \in A_\theta, \int |f|^2 d\varphi \leqq 1\right\} \\ &= \sup\left\{|\varphi(f)|: f \in H^2_\theta, \int |f|^2 d\varphi \leqq 1\right\} \\ &= \sup\left\{|\tilde{\varphi}(f)|: f \in H^\infty_{\tilde{\theta}}, \int |f|^2 d\varphi \leqq 1\right\} \\ &= \sup\{|\tilde{\varphi}(f)|: f \in H^\infty_{\tilde{\theta}}, \|f\| \leqq 1\} \\ &= \sigma(\tilde{\varphi}, \tilde{\theta}) .\end{aligned}$$

Hence we have

$$\sigma(\tau(t), \tau(s)) = \sigma(\tilde{\tau}(t), \tilde{\tau}(s)) = \sigma(t, s) .$$

If $\rho(t)$ is an analytic map of D onto $P(m)$, then by Theorem 2 we have $\rho(t) = \tau(\beta(t-\alpha)/(1-\bar{\alpha}t))$, where β is a constant of modulus 1. Therefore we have

$$\sigma(\rho(t), \rho(s)) = \sigma\left(\beta\frac{t-\alpha}{1-\bar{\alpha}t}, \beta\frac{s-\alpha}{1-\bar{\alpha}s}\right) = \sigma(t, s) .$$

The following equality is proved by König [5], which holds for φ, θ in the maximal ideal space $\mathscr{M}(A)$ of any uniform algebra A.

$$2\log\frac{2+G(\varphi, \theta)}{2-G(\varphi, \theta)} = \log\frac{1+\sigma(\varphi, \theta)}{1-\sigma(\varphi, \theta)} .$$

Using this we get

$$G(\rho(t), \rho(s)) = G(t, s) .$$

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WAKAYAMA UNIVERSITY, WAKAYAMA, JAPAN