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ON LOCAL UNIFORM MEAN CONVERGENCE FOR MARKOV OPERATORS

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The center of a stable Markov operator on $C(X)$ is studied to obtain necessary and sufficient conditions for uniform convergence of the Cesaro means of the iterates on the center of the process. The relation between these results and previous local convergence theorems is also examined.

We denote by $C(X)$ the B-space of all real valued continuous functions on a compact Hausdorff space. The w^*-compact convex set of all Baire Probabilities on X is denoted by $\mathcal{P}(X)$. A (stable) Markov operator T is a bounded linear operator on $C(X)$ which is positive and which satisfies $T1 = 1$. For each x in X and each Baire set E in X we should point out that $T^*\delta(x)E$ can be interpreted as the probability of hitting the set E at time $n = 1$ having started at the point x at time $n = 0$. However we will make no appeal to probabilitistic interpretations or methods here. This note continues the theme that much of the asymptotic behavior of T is intimately connected with the Banach space geometry of the invariant structures of T. We will denote the *invariant function manifold* for T in $C(X)$ by $\mathcal{M} = \mathcal{M}(T) = \{f : Tf = f\}$ and denote the compact convex set of *invariant probabilities* in $\mathcal{P}(X)$ by $\mathcal{K}(T) = \{\mu : T^*\mu = \mu\}$.

In L_2 ergodic theory the strong operator convergence of the Cesaro means of the iterates of a contraction is free (v. Neumann's ergodic theorem). For Markov operators on $C(X)$ this convergence is not free or even true. Thus we have the following definition; T is *uniformly mean stable* (u. m. s.) on $C(X)$ if

$$A_n(T)f = 1/(n+1)(I + T + \cdots + T^n)f$$

is uniformly convergent for all f in $C(X)$. This condition has been studied by Jamison [4], Lloyd [5], Rosenblatt [8] and Sine [9]. We will make strong use of the following *separation property* from [9, p. 161]. A Markov operator T is u.m.s. on $C(X)$ iff $\mathcal{M}(T)$ separates $\mathcal{K}(T)$. We will say a closed set D is *invariant* if $T^*\delta(x)\, D = 1$ for all x in D.

Consider the following very simple example. We take the unit interval $[0,1]$ as X and define $Tf(x) = f(x^2)$. Then $T^n f$ will converge pointwise so $A_n(T)f$ will converge pointwise to the same limit. However this limit function will be continuous iff $f(0) = f(1)$ so T

is not u.m.s. But the doubleton set $M = \{0, 1\}$ is invariant and the restricted operator $T|_M$ is u.m.s. on M. (Indeed $T|_M$ is the identity operator.) It is this sort of local u.m.s. we will characterize.

2. We define the *center* of a Markov operator T by

$$M = \text{closure} \cup \{\text{supp}(\lambda) : \lambda \text{ in } \mathcal{K}(T)\}.$$

This definition is suggested by the center of attraction in dynamic systems (see Nemytskii and Stepanov [**6**, p, 367]). We can obtain M in another way. Let J be defined by

$$J = \{f : A_n(T)|f| \to 0 \text{ [pointwise]}\}.$$

Then firstly the convergence is uniform by Dunford's mean ergodic theorem [**2**, p. 661]. Secondly it is easy to show that J is a norm closed ideal in $C(X)$ which is T invariant as a set of functions. Thus the zero set of the ideal is a closed T invariant set in X. Finally it is easy to show that this zero set is the center M. Thus it can be shown that if f vanishes on M we have $A_n(T)f$ uniformly convergent to zero. (See (See [**11**] for details)

The definition of M is unchanged if the probabilities, λ, are only taken from the extreme points of $\mathcal{K}(T)$. This is an easy consequence of the Krein-Milman theorem; we leave the details to the reader. It is perhaps more surprising that in the case that X is metrizable the definition is unchanged with the omission of the closure operation.

THEOREM 1. *Let K be a sequentially closed convex subset of $\mathcal{P}(T)$ with X compact Hausdorff. Then* $M = \cup\{\text{supp}(\lambda) : \lambda \in K\}$ *is sequentially closed.*

Proof. Suppose $\{x_n\}$ is a sequence of points in M and $x_n \to y$. Then there exists a sequence $\{\lambda_n\}$ in K with x_n in $\text{supp}(\lambda_n)$. If we set $\mu = \Sigma(2^{-n})\lambda_n$ then μ is in K. If W is any neighborhood of y then $\{x_n\}$ is ultimately in W so $\lambda_n(W) > 0$ ultimately. Hence $\mu(W) > 0$ so y is in $\text{supp}(\mu)$. This finishes the argument.

We will consider properties which are too weak or too strong before our main result. We introduced in [**9**] the concept of a *topological ergodic decomposition* for T. This property was shown to imply u.m.s. on a superset of the center but it is much too strong a condition. It required (among other things) a sufficiently rich family of invariant functions to separate the extreme points of $\mathcal{K}(T)$. But our example above has only constants for invariant functions while it is u.m.s. on its center.

We next look at a condition which is too weak. We will say T is *scattered* if each pair of distinct extreme invariant probabilities in $\mathcal{K}(T)$ have disjoint supports.

THEOREM 2. *If D is minimal invariant set for a scattered Markov operator T the $T|_D$ is u.m.s.*

Proof It is clear that D supports exactly one extreme invariant probability. It follows from Krein-Milman that $T|_D$ is uniquely ergodic and it is well known that uniquely ergodic implies u.m.s. (see Oxtoby [7, p. 124]).

REMARK. It follows from the above theorem that $A_n(T)f$ converges pointwise on the union of the minimal invariant sets if T *is scattered. Note that each extreme invariant probability is supported on a minimal set if* T is scattered. If the union of the minimal sets is closed and if the minimal invariant sets form an upper semi-continuous decomposition as well then the scattered Markov operator is u.m.s. on its center. But neither of these topological conditions for a scattered Markov operator need hold. First an example of a scattered operator with union of the minimal sets not closed. We take $X = [0,1]\times[0,1]$ and define the operator by

$$Tf(x,y) = yf(0,y) + (1-y)\int f(s,y)ds.$$

It is straightforward to show that the minimal sets are each uniquely ergodic and consist of the horizontal fibers $I(y) = \{(x,y): 0 \leqq x \leqq 1\}$ for $0 \leqq y < 1$ together with the singleton point $(0, 1)$.

For an example with the union closed but not an upper semi-continuous decomposition take on the same space X

$$Tf(x,y) = yf(x,y) + (1-y)\int f(s,y)ds.$$

The minimal invariant sets consist of the fibers $I(y)$ with $0 \leqq y < 1$ again together with the singleton points $\{(x,1)\}$ for $0 \leqq x \leqq 1$. Finally we remark that these topological conditions are too strong as a u.m.s. operator need not have the union of its minimal invariant sets closed.

We will say T is *continuously scattered* if there is a family of continuous functions each constant on the support of each extreme invariant probability and sufficient to separate the extreme invariant probabilities.

THEOREM 3. *A Markov operator is continuously scattered iff it is u.m.s. on its center.*

Proof. If T is u.m.s. on its center then the invariant functions for $T|_M$ when extended to all of X in any continuous way satisfy the requirements. Conversely suppose there is such a family of functions. We have $Tf = f$ on each minimal invariant set and thus on the union. By continuity $Tf = f$ on M. But in the same way we have $Tg = g$ on M for each function in the norm closed algebra $\mathcal{A}$ generated by the separating family. Let Y be the quotient space $M/\mathcal{A}$. We can drop T to a Markov operator on $C(Y)$; it is, in fact, the identity operator on $C(Y)$. It follows that for each y in Y that the pre-image of y under the quotient map is a closed invariant set of M which supports exactly one invariant probability. Thus T is u.m.s. (by unique ergodicity) on that pre-image. Now let λ_1 and λ_2 be any invariant probabilities. Let f be any function with $(f, \lambda_1) \neq (f, \lambda_2)$. Now $\bar{f} = \lim A_n f$ exists as a pointwise limit on M. This function separates λ_1 and λ_2 and it can be regarded as a Baire function on Y. But then there must be a continuous function on Y separating λ_1 and λ_2. Finally any function of $C(Y)$ is $T|_M$ invariant when regarded as a function on M. Thus the invariant functions of $T|_M$ separate $\mathcal{K}(T)$ so T is u.m.s. on its center.

The following condition was given by Attala [1]. Let $\mathcal{L}_0$ be the norm closed subspace of $C(X)$ defined by

$$\mathcal{L}_0 = \{f : A_n(T)f \to 0 \ [\text{uniformly}]\}.$$

Suppose there is a Markov projection P with the null space of P equal to $\mathcal{L}_0$. Then T is u.m.s. on its center. We will give a converse to this result in the metric case thus further justifying the hypothesis. We will also give an apparent strengthening of the forward theorem in such a way that it is clear that for T to be u.m.s. on M depends only on $\mathcal{K}(T)$ and its orientation in $\mathcal{P}(X)$.

THEOREM 4. *Let T be a Markov operator on X with center M. Suppose there is a u.m.s. Markov operator S on $C(X)$ with $K(S) = K(T)$. Then T is u.m.s. on M.*

Proof. First we note that if R is any stable Markov operator and λ is in $\mathcal{K}(R)$ then $\text{supp}(\lambda)$ is an R-invariant set (see [9, p. 156]). It follows that M is both T and S invariant and we can restrict the processes to M. If λ is an extreme point of $\mathcal{K} = \mathcal{K}(S) = \mathcal{K}(T)$ then $D = \text{supp}(\lambda)$ is both S and T invariant. Moreover since S is uniquely ergodic on D so is T. Now each S invariant function is constant on D. Let f be an S invariant function. Then $Tf = f$ on the support of each extreme since f is constant on each such set. Then by continuity $Tf = f$ on M since as pointed out before M is the closure of the union of the supports of the extremes. Now each S invariant function is T

invariant and each T invariant probability is S invariant. Since S is u.m.s. the S invariant functions separate the extremes of $\mathscr{K}(S) = \mathscr{K}$. Also the T invariant functions separate the extremes of $\mathscr{K} = \mathscr{K}(T)$. We conclude T is u.m.s. on M.

REMARK. If T is a Markov operator on $C(X)$ and T is u.m.s. on M then clearly there is a u.m.s. Markov operator S on $C(X)$ with $\mathscr{K}(S) = \mathscr{K}(T|_M)$. For we just take $S = T|_M$. Now in the above theorem we only need S u.m.s. on M and S need not even be defined on all of X. To that extent the conditions of Theorem 4 (and of Attala's result as well) are necessary and sufficient. But if we ask that S be globally defined we are able to obtain a full converse only in the metric case.

THEOREM 5. *Let T be a Markov operator on $C(X)$ where X is compact metric. Then T is u.m.s. on its center M iff there is a u.m.s. Markov operator S on $C(X)$ with $\mathscr{K}(S) = \mathscr{K}(T)$.*

We will have need of the following.

LEMMA. (Borsuk-Tietze). *Let X be a compact metric space and D a closed nonempty subset. Then there is a Markov projection P with $\mathscr{K}(P) = \mathscr{P}(D)$.*

Borsuk's linear Tietze extension result has far greater generality than this lemma. For the lemma as stated a geometric proof is available based on the strictly convex metrics of Bonsall and Hervé for the w^* topology **[10]**.

Proof of Theorem 5. There is a Markov projection R with $\mathscr{K}(R) = \mathscr{P}(M)$ by the lemma. Let P be the projection on $C(X)$ defined as the limit of $A_n(T|_M)$. For each x in X we have $P^*R^*\delta(x)$ defined as a point in $\mathscr{K}(T)$. If λ is in $\mathscr{K}(T)$ then $P^*R^*\lambda = \lambda$. Thus $S = RP$ is a Markov projection with $\mathscr{K}(S) = \mathscr{K}(T)$ and satisfies the requirements. For f in $C(X)$ we let P act on the restriction to M. Let g be any extension of Pf to all of X. Then $RPf = Rg$ is independent of that extension.

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