RINGS OVER WHICH CERTAIN
MODULES ARE INJECTIVE

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This paper is concerned with rings for which all modules in one of the following classes are injective: simple modules, quasi-injective modules, or proper cyclic modules. Such rings are known as *V*-rings, *QI*-rings, and *PCI*-rings, respectively. First, some conditions are developed under which the properties of being a *V*-ring, *QI*-ring, or *PCI*-ring are left-right symmetric. In the next section, it is shown that a semiprime Goldie ring is a *QI*-ring if and only if all singular quasi-injective modules are injective. An example is constructed to show that the class of *QI*-rings is properly contained in the class of noetherian *V*-rings. Also, it is shown that the global homological dimension of a *QI*-ring cannot be any larger than its Krull dimension. In the final section, it is shown that a *V*-ring is noetherian if and only if it has a Krull dimension. Examples are put forward to show that a noetherian *V*-ring may have arbitrary finite Krull dimension.

1. Introduction and definitions. A ring R is said to be a *right V-ring* provided all simple right R-modules are injective. According to Villamayor [16, Theorem 2.1], this is equivalent to the condition that every right ideal of R is an intersection of maximal right ideals. In particular, the Jacobson radical of any right V-ring is zero, and consequently all right V-rings are semiprime. For further properties and examples of V-rings, we refer the reader to Boyle [1], Cozzens [5], Cozzens-Johnson [6], Farkas-Snider [9], Michler-Villamayor [16], and Osofsky [17].

We recall that a module A is *quasi-injective* provided every homomorphism from a submodule of A into A extends to an endomorphism of A. According to Johnson-Wong [13, Theorem 1.1], this is equivalent to the condition that A be a fully invariant submodule of its injective hull, which we denote by $E(A)$. For example, any *semisimple* module (i.e., a module which is a sum of simple submodules) is quasi-injective. A ring R is a *right QI-ring* provided all quasi-injective right R-modules are injective. Inasmuch as all simple right R-modules are quasi-injective, it follows that R must also be a right V-ring. In addition, since all semisimple right R-modules are quasi-injective and thus injective, we see from Kurshan [15, Theorem 2.4] that R is right noetherian. Therefore: every right QI-ring is a right

noetherian, right V-ring. For further properties and examples of QI-rings, we refer the reader to Boyle [1] and Byrd [2, 3].

The *proper cyclic* right modules over a ring R are those cyclic right R-modules R/I for which $R/I \not\cong R$. We say that R is a *right PCI-ring* provided all proper cyclic right R-modules are injective. If R is not a division ring, then all simple R-modules are proper cyclic, from which we conclude that every right PCI-ring is also a right V-ring. According to Faith [7, Theorems 14, 17], a right PCI-ring is either semisimple or else is a simple, right semihereditary, right Ore domain. In view of this result, we shall only consider right PCI-*domains* in this paper.

The reader is assumed to be familiar with the notions of singular and nonsingular modules, as in [10], for example. Also we shall need the notions of Krull dimension and critical modules as developed in [11]. We remind the reader that a *uniform* module is one in which the intersection of any two nonzero submodules is nonzero. Finally, we use the notation $\text{soc}(A)$ for the socle of a module, and r.gl.dim.(R) for the right global dimension of a ring R.

2. Left-right symmetry.

In this section we consider conditions under which a right V-ring (QI-ring, PCI-domain) is also a left V-ring (QI-ring, PCI-domain). These results are consequences of the existence of a duality (i.e., a contravariant category equivalence) between the categories of finitely generated singular right modules and finitely generated singular left modules. We note that in general a right V-ring need not be a left V-ring, as shown by an example of Michler and Villamayor [16, Remark 4.5]. For QI-rings and PCI-domains, however, the question of left-right symmetry in general remains open.

LEMMA 1. *Let R be a right and left noetherian semiprime ring with classical quotient ring Q, and let $\mathscr{F}_R(_R\mathscr{F})$ denote the category of all finitely generated singular right (left) R-modules. If $(Q/R)_R$ and $_R(Q/R)$ are both injective, then there exists a duality*

$$\text{Ext}_R^1(-, R): \mathscr{F}_R \to {}_R\mathscr{F}.$$

Proof. Inasmuch as Q_R is nonsingular and injective, we observe that $\text{Hom}_R(-, (Q/R)_R)$ and $\text{Ext}_R^1(-, R_R)$ are naturally equivalent on $\mathscr{F}_R$. Using this, it follows readily that $\text{Ext}_R^1(-, R_R)$ and $\text{Ext}_R^1(-, {}_RR)$ define contravariant functors $\mathscr{F}_R \to {}_R\mathscr{F}$ and $_R\mathscr{F} \to \mathscr{F}_R$ respectively, and all that remains is to show that both compositions of these two functors are naturally equivalent to the appropriate identity functors.

In view of the above remarks, $\text{Ext}_R^1(\text{Ext}_R^1(-, R_R), {}_R R)$ is naturally equivalent to $\text{Hom}_R(\text{Ext}_R^1(-, R_R), {}_R(Q/R))$ on $\mathscr{F}_R$. Since ${}_R(Q/R)$ is injective by hypothesis, it follows from Cartan-Eilenberg [**4**, Proposition 5.3, p. 120] that $\text{Hom}_R(\text{Ext}_R^1(-, R_R), {}_R(Q/R))$ is naturally equivalent to $\text{Tor}_1^R(-, \text{Hom}_R({}_R R, {}_R(Q/R)))$ on $\mathscr{F}_R$. Finally, we see by Sandomierski [**18**, Corollary, p. 119] that $\text{Tor}_1^R(-, Q/R)$ is naturally equivalent to the identity functor on $\mathscr{F}_R$.

PROPOSITION 2. *Let R be a right and left noetherian ring with maximal right quotient ring Q, and assume that R is a right V-ring. If $(Q/R)_R$ and ${}_R(Q/R)$ are both injective, then all singular right and left R-modules are semisimple and injective.*

Proof. Since R is a semiprime ring, Q is the classical quotient ring of R. We thus obtain the duality between $\mathscr{F}_R$ and ${}_R\mathscr{F}$ as in Lemma 1. Now $\mathscr{F}_R$ and ${}_R\mathscr{F}$ are noetherian categories, hence it follows from the duality that they are also artinian categories. Thus every object in each of these categories has a composition series. Since R is a right V-ring, we infer from this that every object in $\mathscr{F}_R$ is semisimple, and then it follows from the duality that the same is true in ${}_R\mathscr{F}$. In particular, R/I must be semisimple for any essential right or left ideal I of R. The proposition now follows from Goodearl [**10**, Proposition 3.1].

THEOREM 3. *Suppose R is a right and left noetherian ring with maximal right quotient ring Q, and assume that R is a right V-ring. Then R is right and left hereditary if and only if $(Q/R)_R$ and ${}_R(Q/R)$ are both injective.*

Proof. As in Proposition 2, we see that R is semiprime and that Q is the classical quotient ring of R. Thus Q_R and ${}_R Q$ are both injective, hence if R is right and left hereditary we automatically obtain $(Q/R)_R$ and ${}_R(Q/R)$ injective. Conversely, if Q/R is injective on both sides, Proposition 2 shows that all singular right and left R-modules are injective. According to Goodearl [**10**, Proposition 3.3], R is thus right and left hereditary.

COROLLARY 4. *Suppose that R is a right and left QI-ring with maximal right quotient ring Q. Then R is right and left hereditary if and only if $(Q/R)_R$ and ${}_R(Q/R)$ are both injective.*

Proof. Inasmuch as R is a right and left noetherian, right and left V-ring, this is a direct consequence of Theorem 3.

THEOREM 5. *Let R be a right and left noetherian ring with maximal right quotient ring Q, and assume that $(Q/R)_R$ and $_R(Q/R)$ are both injective. Then the following conditions are equivalent:*

    (a) *R is a right V-ring.*

    (b) *R is a left V-ring.*

    (c) *R is a right QI-ring.*

    (d) *R is a left QI-ring.*

Proof. Under any of the assumptions a, b, c, d, R is a V-ring on one side or the other and hence is semiprime. Thus we may as well assume that R is semiprime to begin with. In this case Q is also the maximal left quotient ring of R, and so our hypotheses are now left-right symmetric.

a $\Rightarrow$ b: According to Faith [8, Theorem 31], R is a finite direct product of simple rings, hence we need only consider the case when R itself is simple. If $\mathrm{soc}(_RR) \neq 0$, then $\mathrm{soc}(_RR) = R$ and R is a semisimple ring, whence (b) is automatic. On the other hand, if $\mathrm{soc}(_RR) = 0$, then all simple left R-modules are singular, hence it follows from Proposition 2 that R is a left V-ring.

b $\Rightarrow$ a: By symmetry.

a $\Rightarrow$ c: Inasmuch as R is right and left hereditary by Theorem 3, this follows from Boyle [1, Theorem 5].

c $\Rightarrow$ a: is automatic, and then b $\Leftrightarrow$ d by symmetry.

COROLLARY 6. *Let R be a right and left noetherian domain. Then R is a right PCI-domain if and only if R is a left PCI-domain.*

Proof. If R is a right noetherian *PCI*-domain, then according to Boyle [1, Theorem 7] R is right hereditary. By Small [19, Corollary 3], R is also left semihereditary and thus left hereditary. Letting Q denote the classical quotient ring of R, we thus see that $(Q/R)_R$ and $_R(Q/R)$ are both injective. Since R is in particular a right V-ring, Theorem 5 now says that R is also a left V-ring. Thus Boyle [1, Corollary 10] shows that R is a left *PCI*-domain.

3. *QI*-rings. This section is concerned with several aspects of the structure of *QI*-rings. We begin by looking at semiprime Goldie rings, in which case we show that R is a *QI*-ring provided only that its singular quasi-injective modules are injective. Second, we give an example to show that not all noetherian V-rings need be *QI*-rings. Finally, we prove several results about the structure of modules over a *QI*-ring R which lead to the inequality r.gl.dim.$R \leqq$ K.dim.R.

LEMMA 7. *Let R be any right nonsingular ring. If R_R is finite-dimensional, then all faithful nonsingular quasi-injective right R-modules are injective.*

Proof. Let A be any faithful nonsingular quasi-injective right R-module. Since A is nonsingular, the annihilator of any subset of A is an $\mathscr{S}$-closed right ideal of R in the sense of [10, p. 14]. According to Goddearl [10, Theorem 1.24], the finite-dimensionality of R_R implies that the $\mathscr{S}$-closed right ideals of R satisfy the descending chain condition. Thus A must have a finite subset $\{a_1, \cdots, a_n\}$ whose annihilator is minimal among the annihilators of finite subsets of A, and then we infer from the faithfulness of A that the annihilator of $\{a_1, \cdots, a_n\}$ is 0. Consequently the element $(a_1, \cdots, a_n) \in A^n$ has zero annihilator, whence A^n contains an isomorphic copy of R_R. Since A^n is quasi-injective by Harada [12, Proposition 2.4], it follows easily from Baer's criterion that A^n must be injective. Therefore A is injective.

Unforunately, the hypothesis of faithfulness cannot be omitted from Lemma 7. For if R is the ring of all lower triangular 2×2 matrices over a field F, then its radical J is a minimal right ideal of R and thus is a nonsingular quasi-injective right R-module. However, since J contains no nonzero idempotents it cannot be injective.

Over a commutative noetherian nonsingular ring, faithfulness can be dropped, as shown by Harada [12, Propositions 2.5, 2.6]. This result carries over to semiprime Goldie rings, as the following theorem shows.

THEOREM 8. *If R is a semiprime right Goldie ring, then all non-singular quasi-injective right R-modules are injective.*

Proof. Let A be any nonsingular quasi-injective right R-module, and let Q denote the classical right quotient ring of R, which is a semisimple ring. Now $E(A)$ is nonsingular and so is a right Q-module, hence there exists a ring decomposition $Q = Q_1 \oplus Q_2$ such that $E(A)Q_1 = 0$ and $E(A)$ is faithful over Q_2. If $I = R \cap Q_1$, then I is a two-sided ideal of R such that $E(A)I = 0$. Since it suffices to show that A is a direct summand of $E(A)$, it follows that we need only prove that A is injective as an (R/I)-module. Inasmuch as R/I is a semiprime right Goldie ring with classical right quotient ring Q_2, we may thus assume, without loss of generality, that $E(A)$ is a faithful right Q-module.

According to Lemma 7, it is enough to show that the annihilator $H = \{r \in R \mid Ar = 0\}$ is 0. Since $E(A)$ is a faithful right module over the semisimple ring Q, it must contain a finite subset $\{x_1, \cdots, x_n\}$ whose

annihilator in Q is zero. There must be an essential right ideal J of R such that $x_i J \leqq A$ for all i, whence $x_i JH = 0$ for all i, and thus $JH = 0$. Now HJ is a nilpotent two-sided ideal of R, hence our semiprime hypothesis implies that $HJ = 0$. Inasmuch as J is essential in R, we conclude that $H = 0$, and so A is indeed faithful.

COROLLARY 9.   *Let R be a semiprime right Goldie ring. Then R is a right QI-ring if and only if all singular quasi-injective right R-modules are injective.*

Proof. Assume that all singular quasi-injective right R-modules are injective, and consider an arbitrary quasi-injective right R-module A. The singular submodule $Z(A)$ is a fully invariant submodule of A and thus is quasi-injective, hence we obtain that $Z(A)$ is injective. Now $A \cong Z(A) \oplus [A/Z(A)]$ and so $A/Z(A)$ is quasi-injective, whence Theorem 8 says that $A/Z(A)$ must be injective. Therefore A is injective.

As we have remarked above, every QI-ring is a noetherian V-ring, and Byrd [2] has raised the converse question of whether every noetherian V-ring must be a QI-ring. The answer is no, as we now show.

EXAMPLE.   There exists a right and left noetherian, right and left V-ring R which is not a right QI-ring.

Proof. Let F be a universal differential field of characteristic 0 with respect to two commuting derivations δ_1 and δ_2 (Kolchin [14, Theorem, p. 771]), and let $R = F[\theta_1, \theta_2]$ be the ring of linear differential operators over F. We recall that the elements of R are noncommutative polynomials in the indeterminates θ_1, θ_2, subject to the relations $\theta_1 \theta_2 = \theta_2 \theta_1$ and $\theta_i \alpha = \alpha \theta_i + \delta_i \alpha$ for all $\alpha \in F$. It is easily seen that R is a right and left noetherian ring. Cozzens and Johnson have shown in [6, Theorem 1] that R is a left V-ring, and the same argument shows that R is also a right V-ring.

In view of the relation $\theta_i \alpha = \alpha \theta_i + \delta_i \alpha$, we can extend δ_i to a derivation of R by setting $\delta_i r = \theta_i r - r \theta_i$. Note that $R = F[\theta_1][\theta_2]$, i.e., R is equal to the ring of linear differential operators over the differential ring $(F[\theta_1], \delta_2)$. Likewise, $R = F[\theta_2][\theta_1]$.

Inasmuch as $R = F[\theta_1][\theta_2] = F[\theta_1] + \theta_2 R$, we have $R/\theta_2 R \cong F[\theta_1]$ as right $F[\theta_1]$-modules. We compute that the right R-module action on $F[\theta_1]$ is given by its right $F[\theta_1]$-module action together with the rule $x * \theta_2 = -\delta_2 x$. Now $F[\theta_1]$ is a right and left noetherian domain, hence

it has a classical quotient division ring Q. We extend δ_2 to a derivation of Q according to the rule $\delta_2(ab^{-1}) = (\delta_2 a)b^{-1} - ab^{-1}(\delta_2 b)b^{-1}$, and then we make Q into a right R-module by using its right R-module action together with the rule $x * \theta_2 = -\delta_2 x$.

Now Q is uniform as a right $F[\theta_1]$-module and thus also as a right R-module. Therefore $E(Q_R)$ is an indecomposable right R-module. Considering $E(Q_R)$ just as a module over the noetherian domain $F[\theta_2]$ (which is a subring of R), it has a torsion submodule which we shall denote by T. We proceed by showing that T is a right R-module which is quasi-injective but not injective. Since T is an $F[\theta_2]$-submodule of $E(Q_R)$, and since $R = F[\theta_2][\theta_1]$, T will be an R-submodule of $E(Q_R)$ provided $T\theta_1 \subseteq T$. Given any $t \in T$, we have $ta = 0$ for some nonzero $a \in F[\theta_2]$. Observing that $\delta_1 a \in F[\theta_2]$, we compute that $t\theta_1 a = t(\delta_1 a) \in T$, and thus $t\theta_1 \in T$. Therefore T is indeed an R-submodule of $E(Q_R)$.

Inasmuch as T is a fully invariant submodule of $E(Q_R)$, T is a quasi-injective right R-module. Observing that $1 * \theta_2 = -\delta_2 1 = 0$, we see that $1 \in T$ and thus $T \neq 0$. Since F is a universal differential field, there must be an element $\alpha \in F$ such that $\delta_2 \alpha = 1$. Then $\delta_2(\alpha + \theta_1) = 1$ also, from which we compute that $\delta_2^n(\alpha + \theta_1)^{-1} = (-1)^n n!(\alpha + \theta_1)^{-n-1}$ for all $n > 0$. We now infer that the elements $(\alpha + \theta_1)^{-1} * \theta_2^n$ in Q are right linearly independent over F, i.e., $(\alpha + \theta_1)^{-1}$ is not annihilated by any nonzero elements of $F[\theta_2]$. Thus $(\alpha + \theta_1)^{-1} \notin T$, and so $T \neq E(Q_R)$.

Now $E(Q_R)$ is indecomposable and T is a nontrivial submodule of $E(Q_R)$, hence T cannot be injective. Since T is quasi-injective, R cannot be a right QI-ring.

PROPOSITION 10. [**8**, Proposition 32]. *Let R be a right QI-ring. If E is any nonzero indecomposable injective right R-module, then $\Lambda =$ $\mathrm{Hom}_R(E, E)$ is a division ring.*

THEOREM 11. *Let R be a right QI-ring. If A is any nonzero finitely generated right R-module with Krull dimension α, then all finitely generated submodules of $E(A)/A$ have Krull dimension strictly less than α.*

Proof. First consider the case where A is critical, and suppose $E(A)/A$ has a finitely generated submodule B/A with K.dim.$(B/A) \geq \alpha$. Since R is a right noetherian, there exist submodules $B_0 = A < B_1 < \cdots < B_n = B$ such that each B_k/B_{k-1} is critical. We must have K.dim.$(B_k/B_{k-1}) \geq \alpha$ for some k. Thus, replacing B by B_k and setting $C = B_{k-1}$, we have modules $A \leq C < B$ such that B/C is β-critical for some $\beta \geq \alpha$.

Now let Q be the sum of all submodules P of $E(A)$ for which $\text{Hom}_R(P, E(B/C)) = 0$. We observe that Q is a fully invariant submodule of $E(A)$, i.e., Q is quasi-injective. Since R is a right QI-ring, Q must be injective. Since A is critical it must be uniform, and thus $E(A)$ is indecomposable, so the only choices for Q are $E(A)$ or 0. Inasmuch as the identity map on B/C extends to a nonzero map $E(A) \to E(A)/C \to E(B/C)$, we obtain $Q \neq E(A)$ and thus $Q = 0$.

Since $Q = 0$, there exists a nonzero map $f: A \to E(B/C)$, which induces a nonzero map from $A_0 = f^{-1}(B/C)$ into B/C. According to Gordon-Robson [11, Proposition 2.3], A_0 is α-critical and all nonzero submodules of B/C are β-critical. Inasmuch as $\beta \geq \alpha$, it follows that f must be a monomorphism, whence A_0 is isomorphic to a nonzero submodule B_0/C of B/C. The map $B_0 \to B_0/C \xrightarrow{\cong} A_0$ extends to a nonzero endomorphism g of $E(A)$, and since $gA = 0$ we see that g cannot be an isomorphism. But then the endomorphism ring of $E(A)$ is not a division ring, which contradicts Proposition 10.

Thus the theorem holds for critical modules. In general, A must have an essential submodule $K = K_1 \oplus \cdots \oplus K_n$, where each K_i is α_i-critical for some $\alpha_i \leq \alpha$. In view of the results above, every finitely generated submodule of $E(K_i)/K_i$ has Krull dimension strictly less than α_i. It follows that every finitely generated submodule of $E(K)/K$ has Krull dimension strictly less than α, from which the theorem follows.

COROLLARY 12. *If R is any right QI-ring, then* r.gl.dim.$R \leq$ K.dim.R.

Proof. We need only consider the case when K.dim.$R = N < \infty$. For any right R-module A, let $\phi(A)$ denote the supremum of the Krull dimensions of all finitely generated submodules of A. Note that $\phi(A) \leq N$, and that $\phi(A) = $ K.dim.A when A is finitely generated. It suffices to show that $id_R(A) \leq N$ for all nonzero right R-modules A, where $id_R(A)$ denotes the injective dimension of A. We proceed by induction to show that $id_R(A) \leq \phi(A)$ for all nonzero A.

If $\phi(A) = 0$, then since R is right noetherian all finitely generated submodules of A must have composition series. Inasmuch as R is a right V-ring, all such submodules of R must be semisimple, and thus A itself is semisimple. Now A is quasi-injective and therefore injective, whence $id_R(A) = \phi(A)$.

Now let $\phi(A) = n > 0$ and assume that $id_R(B) \leq \phi(B)$ for all nonzero modules B with $\phi(B) < n$. Choose an essential submodule of A of the form $K = \oplus K_\alpha$, where each K_α is finitely generated. According to Theorem 11, all finitely generated submodules of

$E(K_\alpha)/K_\alpha$ have Krull dimension strictly less than K.dim.K_α, from which we infer that $\phi(E(K_\alpha)/K_\alpha) < n$. It now follows that $\phi(E(A)/A) < n$, and then $id_R(E(A)/A) < n$ by the induction hypothesis. Therefore $id_R(A) \leqq n$, and the induction works.

4. V-rings. In this section we consider the Krull dimension of V-rings. According to Michler-Villamayor [**16**, Theorem 4.2], a right V-ring with right Krull dimension at most one is right noetherian, right hereditary, and Morita-equivalent to a finite direct sum of simple V-domains. In general, all positive integers are possible as Krull dimensions of V-rings, and we exhibit examples of this. We also show that a V-ring has a Krull dimension if and only if it is already noetherian.

PROPOSITION 13. *Let R be a right V-ring. Then R has right Krull dimension if and only if R is right noetherian.*

Proof. All right noetherian rings have right Krull dimension: Gordon-Robson [**11**, Proposition 1.3]. On the other hand, if R is not right noetherian then by using an argument of Faith in [**7**, Corollary 15B] we infer that there exists a cyclic right R-module E whose socle is an infinite direct sum of simple modules. But then E is not finite-dimensional and so does not have Krull dimension, by Gordon-Robson [**11**, Proposition 1.4]. Since E is cyclic, it follows that R does not have Krull dimension.

COROLLARY 14. *If R is a right PCI-domain, then the following conditions are equivalent:*
   (a) *R is right noetherian.*
   (b) *R has Krull dimension.*
   (c) *R has Krull dimension at most 1.*

Proof. Since R is a right V-ring, a $\Leftrightarrow$ b by Proposition 13. If (a) holds, then R is a simple, right hereditary ring by Faith [**7**, Theorem 14]. Inasmuch as R is a right PCI-domain, we see that all cyclic submodules of any singular right R-module A are direct summands of A. In case A is finitely generated as well, then it must be finite-dimensional since R is right noetherian, and Goodearl [**10**, Proposition 1.22] shows that A is semisimple. In particular, R/I is semisimple for all essential right ideals I of R, whence Goodearl [**10**, Proposition 3.1] shows that all singular right R-modules are semisimple and injective. It now follows from Michler-Villamayor [**16**, Theorem 4.2] that K.dim.$R \leqq 1$.

It is an open question whether all right *PCI*-rings are right noetherian. Corollary 14 might provide a means for attacking this question.

Cozzens and Johnson [6] have constructed examples of noetherian *V*-rings with arbitrary finite global dimension, and we shall show that these examples also have arbitrary finite Krull dimension. Since these examples are differential operator rings, we begin with two general results on the Krull dimension of differential operator rings.

LEMMA 15. *Let R be any right noetherian ring with a derivation δ. Then the ring of linear differential operators $R[\theta]$ is right noetherian, and*

$$\text{K.dim.}R[\theta] \leq \text{K.dim.}R + 1$$

Proof. Given any right ideal J of $R[\theta]$ and any nonnegative integer n, let J_n be the set of all leading coefficients of elements of J of degree n, together with 0. Then the collection $\{J_0, J_1, \cdots\}$ is an ascending sequence of right ideals of R, hence we can define a right ideal in the ordinary polynomial ring $R[x]$ by setting $J_0 + J_1x + J_2x^2 + \cdots$. We thus obtain a monotone map ϕ from the right ideal lattice of $R[\theta]$ into the right ideal lattice of $R[x]$, and as in the Hilbert Basis Theorem an easy induction on degrees shows that ϕ is a strictly monotone map. Now $R[x]$ is certainly right noetherian, and $\text{K.dim.}R[x] = \text{K.dim.}R + 1$ by Gordon-Robson [11, Theorem 9.2], from which the lemma follows.

PROPOSITION 16. *If F is a field with a finite collection $\delta_1, \cdots, \delta_n$ of commuting derivations, then the ring of linear differential operators $R = F[\theta_1, \cdots, \theta_n]$ has Krull dimension n.*

Proof. By induction on Lemma 15, we see that R is a right noetherian ring with $\text{K.dim.}R \leq n$. Now set $J_k = \theta_k R + \cdots + \theta_n R$ for $k = 1, \cdots, n$, and set $J_{n+1} = 0$. We prove by induction on k that $\text{K.dim.}R/J_k \geq k - 1$. Since $R/J_1 \neq 0$, we automatically have $\text{K.dim.}R/J_1 \geq 0$. Now let $1 \leq k \leq n$ and assume that $\text{K.dim.}R/J_k \geq k - 1$. Inasmuch as the θ_i all commute, we see that $\theta_k^m J_{k+1} \leq J_{k+1}$ for all $m > 0$. In fact, we compute that $\{r \in R \mid \theta_k^m r \in J_{k+1}\} = J_{k+1}$, from which it follows that

$$\{r \in R \mid \theta_k^m r \in \theta_k^{m+1}R + J_{k+1}\} = \theta_k R + J_{k+1} = J_k,$$

i.e.,

$$(\theta_k^m R + J_{k+1})/(\theta_k^{m+1}R + J_{k+1}) \cong R/J_k.$$

Since $\mathrm{K.dim.}R/J_k \geqq k - 1$, it follows that $\mathrm{K.dim.}R/J_{k+1} \geqq k$. Thus the induction works, hence we obtain $\mathrm{K.dim.}R \geqq n$.

EXAMPLE. Given any positive integer n, there exists a right and left noetherian, right and left V-ring R_n such that $\mathrm{K.dim.}R_n = n$.

Proof. Let F be a universal differential field with respect to n commuting derivations $\delta_1, \cdots, \delta_n$ (Kolchin [**14**, Theorem, p. 771]), and let R_n be the ring of linear differential operators $F[\theta_1, \cdots, \theta_n]$. Then R_n is a noetherian V-ring by Cozzens-Johnson [**6**, Theorem 1], and $\mathrm{K.dim.}R_n = n$ by Proposition 15.

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