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THE EXCHANGE PROPERTY AND DIRECT SUMS OF INDECOMPOSABLE INJECTIVE MODULES

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This paper contains two main results. The first gives a necessary and sufficient condition for a direct sum of indecomposable injective modules to have the exchange property. It is seen that the class of these modules satisfying the condition is a new one of modules having the exchange property. The second gives a necessary and sufficient condition on a ring for all direct sums of indecomposable injective modules to have the exchange property.

Throughout this paper R will be an associative ring with identity and all modules will be right R-modules.

A module M has *the exchange property* [5] if for any module A and any two direct sum decompositions

$$A = M' \oplus N = \sum_{i \in I} \oplus A_i$$

with $M' \cong M$, there exist submodules $A_i' \subseteq A_i$ such that

$$A = M' \oplus \sum_{i \in I} \oplus A_i' .$$

The module M has *the finite exchange property* if this holds whenever the index set I is finite. As examples of modules which have the exchange property, we know quasi-injective modules and modules whose endomorphism rings are local (see [16], [7], [15] and for the other ones [5]).

It is well known that a finite direct sum $M = \bigoplus_{i=1}^{n} M_i$ has the exchange property if and only if each of the modules M_i has the same property ([5, Lemma 3.10]). In general, however, an infinite direct sum $M = \bigoplus_{i \in I} M_i$ has not the exchange property even if each of M_i's has the same property. On the other hand, Fuller [8] has recently proved that every module over a generalized uniserial ring has the exchange property (c.f., see [9, Theorem 9 and corollary to Lemma 12]).

Therefore, two interesting questions arise:

(1) When does the infinite direct sum $M = \bigoplus_{i \in I} M_i$ of modules $M_i (i \in I)$ with the exchange property have the same property?

(2) What ring R has the property that every module M has the exchange property?

In this paper we consider these two problems for the class of modules M which are direct sums of indecomposable injectives and

completely make answers to them for such a class of modules. In §1 we show a sufficient condition for a direct sum of modules with local endomorphism rings to have the finite exchange property. In §2 we prove the following results (1′) and (2′).

(1′) A module M which is a direct sum of indecomposable injective modules has the exchange property if and only if it has the finite exchange property, and moreover any of these assertions is equivalent to that the Jacobson radical of the endomorphism ring $\mathrm{End}_R(M)$ of M is $\{f \in \mathrm{End}_R(M) \mid \mathrm{Ker}\, f \text{ is essential in } M\}$.

(2′) A ring R satisfies the ascending chain condition for (meet-) irreducible right ideals if and only if every direct sum of indecomposable injective modules has the exchange property.

It is not known whether the exchange and finite exchange properties coincide, so the first equivalence in (1′) is meaningful. Since any direct summand of a module with the exchange property has also the same property as mentioned above, the second equivalence in (1′) trivially includes [2, Corollaire 5] concerning a problem on an indecomposable decomposition of a direct summand of the module which is a direct sum of indecomposable injectives (this is a problem of Matlis). (2′) is a strengthening of [19, Theorem 1] and, as seen in it, such a ring in (2′) has interesting properties concerning the Krull-Remak-Schmidt-Azumaya's theorem and a problem of Matlis. If a module M is quasi-injective, all properties in (1′) are also valid for M, but conversely neither of them implies the quasi-injectivity of M. In §3 we show this fact with an example which means that the class of all modules with the exchange property which are direct sums of indecomposable injectives is a new one of modules with the same property. In §4 we generalize the results of Chamard [3, Théorème 3] and Yamagata [17, Theorem 4] which are obtained from the point of view of a problem of Matlis.

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1. **A semi-T-nilpotent system.** We will recall some definitions and elementary results from [9] and [10]. A family $\{M_i\}_{i\in I}$, with an infinite index set I, which consists of modules M_i whose endomorphism rings are local is called (resp. *semi-*) *T-nilpotent system* if for any family of nonisomorphisms $\{f_{i_n}: M_{i_n} \to M_{i_{n+1}} \mid n \geqq 1\}$ (resp. $i_n \neq i_{n'}$ for $n \neq n'$) and any element $x_{i_1} \in M_{i_1}$, there is an integer m depending on x_{i_1} such that $f_{i_m} f_{i_{m-1}} \cdots f_{i_1}(x_{i_1}) = 0$. If $\mathscr{A}$ is the full subcategory of the category of all right modules whose objects are isomorphic to direct sums of M_i's, then it is said to be the induced category from $\{M_i\}_{i\in I}$ and we denote by $\mathscr{F}$ the class of all morphisms f in $\mathscr{A}$ such that for two objects $X = \bigoplus_{j\in J} X_j$ and $Y = \bigoplus_{k\in K} Y_k$ of $\mathscr{A}$ with $f: X \to$

Y and indecomposable modules X_j and Y_k, each $\pi_k f \kappa_j$ is a nonisomorphism where κ_j is the canonical injection of X_j to X and π_k the projection of Y to Y_k. In [9] we then know the quotient category $\bar{\mathscr{A}} = \mathscr{A}/\mathscr{F}$ is C_3-completely reducible abelian.

For a morphism $f: M \to N$ and a submodule M_0 of M, $f \mid M_0: M_0 \to N$ denotes the restriction of f to M_0. We denote by $\mathrm{End}_R(M)$ an endomorphism ring of a right module M_R over a ring R.

Now we write the proposition, without proof, which will play an important role in our proofs.

PROPOSITION 1.1 ([12], [13]). *Let $\{M_i\}_{i \in I}$ be an infinite family of modules with local endomorphism rings and $M = \bigoplus_{i \in I} M_i$. Then the following conditions are equivalent.*

(i) *$\{M_i\}_{i \in I}$ is a semi-T-nilpotent system.*

(ii) *$\mathscr{F} \cap \mathrm{End}_R(M)$ is the Jacobson radical of $\mathrm{End}_R(M)$.*

In this case, each direct summand of M is also a direct sum of indecomposable modules which are isomorphic to some M_i.

LEMMA 1.2. *For two modules M_1 and M_2, let*

$$M = M_1 \oplus M_2$$

and ρ a projection of M to M_1. Then for a nonzero submodule N of M with $N \cap M_2 = 0$ the restriction $\rho \mid N$ is a monomorphism. If, further, $\rho(N)$ is a direct summand of M, then there exists a submodule N_1 of M_1 such that $M = N \oplus N_1 \oplus M_2$.

Proof. The first assertion is clear. For the rest let $\rho(N)$ be a direct summand of M, $M = \rho(N) \oplus M'$ and ρ a monomorphism on N. By the modular law, we then have

$$M_1 = \rho(N) \oplus N_1$$

with a projection π of M_1 to $\rho(N)$ where $N_1 = M_1 \cap M'$. We consider the decomposition

$$M = \rho(N) \oplus N_1 \oplus M_2 .$$

It is then easy to see that the projection of M to $\rho(N)$ be $\pi\rho$ and the restriction $\pi\rho \mid N$ of $\pi\rho$ to N is an isomorphism by the first part of this lemma. As a consequence, we obtain the desired decomposition

$$M = N \oplus N_1 \oplus M_2 .$$

The following corollaries are essentially proved in [9] but we include proofs for completeness. In them, without proofs, we will

use some properties for completely reducible objects in $\mathscr{A}$ but they are easily proved in the same way as for completely reducible modules (see [9, p. 331–332]).

COROLLARY 1.3. *Let M be a direct sum of indecomposable modules M_i $(i \in I)$, where each M_i has a local endomorphism ring, and $\{N_j\}_{j \in J}$ an independent set of indecomposable submodules of M with local endomorphism rings such that it is a semi-T-nilpotent system. Then, if $\sum_{j \in F} \oplus N_j$ is a direct summand of M for every finite subset $F \subset J$, there exists a subset $K \subset I$ such that*

$$M = \sum_{j \in J} \oplus N_j \oplus \sum_{k \in K} \oplus M_k .$$

REMARK. If J is finite, the finite direct sum $\sum_{j \in J} \oplus N_j$ has the exchange property by [15, Proposition 1] and [5, Lemma 3.10] and is a direct summand of M by hypothesis. Hence there exists a subset $K \subset I$ such that $M = \sum_{j \in J} \oplus N_j \oplus \sum_{k \in K} \oplus M_k$.

Proof. We assume J is infinite. Let $\mathscr{A}$ and $\mathscr{F}$ be as above and $\kappa: N = \sum_{j \in J} \oplus N_j \to M$ an inclusion map. For a morphism f in $\mathscr{A}$ we denote by $\bar{f}$ the induced morphism of f in the quotient category $\bar{\mathscr{A}} = \mathscr{A}/\mathscr{F}$. Since $N_{j_1} \oplus \cdots \oplus N_{j_n}$ is a direct summand for any finite subset $\{j_1, \cdots, j_n\}$ of J by assumption, the restriction of $\bar{\kappa}$ to $\bar{N}_{j_1} \oplus \cdots \oplus \bar{N}_{j_n}$ is then an injection in $\bar{\mathscr{A}}$. This will imply that $\bar{\kappa}$ is an injection in $\bar{\mathscr{A}}$.

To show this we suppose that the kernel $\bar{K} = \operatorname{Ker} \bar{\kappa}$ is not zero. Then there is a finite subset $\{j_1, \cdots, j_n\} \subset J$ such that $\bar{K} \cap (\bar{N}_{j_1} \oplus \cdots \oplus \bar{N}_{j_n}) \neq 0$, because $\bar{\mathscr{A}}$ is a C_3-abelian category and $\bar{N} = \bigoplus_{j \in J} \bar{N}_j$ in $\bar{\mathscr{A}}$ ([9, Theorem 7]). Hence $\bar{\kappa}(\bar{K} \cap \sum_{k=1}^{n} \oplus \bar{N}_{j_k}) \neq 0$ by the fact that $\bar{\kappa} \mid \sum_{k=1}^{n} \oplus \bar{N}_{j_k}$ is injective in $\bar{\mathscr{A}}$, a contradiction.

Then, since the category $\bar{\mathscr{A}}$ is C_3-completely reducible abelian, the morphism $\bar{\kappa}: \bar{N} \to \bar{M}$ splits and by the note just before this corollary there is a subset $K \subset I$ such that

$$M = \sum_{i \in I-K} \oplus M_i \oplus \sum_{k \in K} \oplus M_k \tag{1}$$

and

$$\bar{M} = \bar{N} \oplus \sum_{k \in K} \oplus \bar{M}_k \tag{2}$$

$$= \sum_{i \in I-K} \oplus \bar{M}_i \oplus \sum_{k \in K} \oplus \bar{M}_k . \tag{3}$$

Let the projection of M to $\sum_{j \in I-K} \oplus M_i$ be ρ. Then in (3) the projection of $\bar{M}$ to $\sum_{i \in I-K} \bar{M}_i$ is clearly $\bar{\rho}$ and so $\bar{\rho} \circ \bar{\kappa}$ is a bijection of $\bar{N}$

onto $\sum_{i\in I-K}\oplus \bar{M}_i$ in view of (2) and (3). This means that there is a morphism ϕ of $\sum_{i\in I-K}\oplus M_i$ to N such that $\bar{\phi}\circ\bar{\rho}\circ\bar{\kappa}=1_{\bar{N}}$ and $\bar{\rho}\circ\bar{\kappa}\circ\bar{\phi}=1_{\Sigma_{i\in I-K}\oplus\bar{M}_i}$. Hence we obtain that $\phi\circ(\rho\circ\kappa)-1$ and $(\rho\circ\kappa)\circ\phi-1$ belong to $\mathscr{J}(\mathrm{End}_R(N))=\mathscr{J}\cap\mathrm{End}_R(N)$ and $\mathscr{J}(\mathrm{End}_R(\sum_{i\in I-K}\oplus M_i))=\mathscr{J}\cap\mathrm{End}_R(\sum_{i\in I-K}\oplus M_i)$ respectively. We will show that $\rho\circ\kappa$ is an isomorphism of N to $\sum_{i-I-K\in K}\oplus M_i$.

First, $\phi\circ(\rho\circ\kappa)-1\in\mathscr{J}(\mathrm{End}_R(N))$ implies that $\phi\circ(\rho\circ\kappa)$ is invertible, because $\mathscr{J}(\mathrm{End}_R(N))$ is the Jacobson radical by Proposition 1.1. The morphism $\rho\circ\kappa$ is hence a monomorphism.

Secondly, to show that $\rho\circ\kappa$ is an epimorphism it suffices to show that the family $\{M_i\}_{i\in I-K}$ is a semi-T-nilpotent system by the same reason in the first part. Now since $\bar{N}=\sum_{j\in J}\oplus\bar{N}_j$ is isomorphic to $\sum_{i\in I-K}\oplus\bar{M}_i$, there is a bijection $\sigma: J\to I-K$ such that $\bar{N}_j\cong\bar{M}_{\sigma(j)}$ for every $j\in J$ because $\mathscr{A}$ is a completely reducible C_3-abelian category (see the note before this corollary). It is therefore easy to see that N_j is isomorphic to $M_{\sigma(j)}$ for every $j\in J$ on account of the facts that $\mathscr{J}\cap\mathrm{End}_R(N_j)$ and $\mathscr{J}\cap\mathrm{End}_R(M_{\sigma(j)})$ are the Jacobson radicals of $\mathrm{End}_R(N_j)$ and $\mathrm{End}_R(M_{\sigma(j)})$ respectively. Hence the assumption that $\{N_j\}_{j\in J}$ is a semi-T-nilpotent system implies that the family $\{M_i\}_{i\in I-K}$ is also semi-T-nilpotent, as desired.

Now then, since $(\rho\circ\kappa)(N)=\rho(N)=\sum_{i\in I-K}\oplus M_i$ is a direct summand of M, we can apply Lemma 1.2 to our case and have that

$$M=N\oplus\sum_{k\in K}\oplus M_k\,,$$

which completes the proof of the corollary.

COROLLARY 1.4. *Let M be an infinite direct sum of indecomposable modules M_i $(i\in I)$ with local endomorphism rings. Assume the family $\{M_i\}_{i\in I}$ is a semi-T-nilpotent system and let $\{N_n\}_{n\geqq 1}$ be a family of direct summands of M such that $N_n\subseteqq N_{n+1}$ for all integers $n\geqq 1$. Then the union $\bigcup_{n\geqq 1}N_n$ of the family $\{N_n\}_{n\geqq 1}$ is also a direct summand of M.*

Proof. Since, according to Proposition 1.1, the union $\bigcup_{n\geqq 1}N_n$ is also a direct sum of indecomposable modules with local endomorphism rings, it is an immediate consequence of Corollary 1.3.

For two modules $M=\bigoplus_{i\in I}M_i$ and $N=\bigoplus_{j\in J}N_j$ we can represent every homomorphism f of M to N as a *column summable* matrix (f_{ji}), that is, for the injections κ_i of M_i to M and projections π_j of N to N_j $(i\in I,\ j\in J)$, $f_{ji}=\pi_j f\kappa_i: M_i\to N_j$ and, for any $x\in M$ and $i\in I$ $f_{ji}(\rho_i(x))=0$ for almost all $j\in J$ where ρ_i is the projection of M onto M_i. Hence, in this case we may denote that $f(x)=\sum_j\pi_j f(x)=\sum_{i,j}f_{ji}(\rho_i(x))$ for any $x\in M$ and $f_i=\sum_{j\in J}f_{ji}$ (see [9], p. 332).

A submodule N of M is *essential* in M ($N \subseteqq' M$) if $N \cap L \neq 0$ for all nonzero submodules L of M and M is *uniform* if every nonzero submodule is essential in M. In the following we will denote the kernel of a morphism f by $\mathrm{Ker} f$.

LEMMA 1.5. *Let M be a direct sum of uniform modules M_i ($i \in I$) and $f = (f_{ji}) \in \mathrm{End}_R(M)$. Then $\mathrm{Ker} f$ is essential in M if and only if each $\mathrm{Ker} f_{ji}$ is essential in M_i for all $i, j \in I$.*

Proof. Suppose that $\mathrm{Ker} f_{ji}$ is essential in M_i for all $i, j \in I$. Then to show that $\mathrm{Ker} f \subseteqq' M$ it suffices to show that $\mathrm{Ker} f \cap M_i \subseteqq' M_i$ for all $i \in I$. Now contrary to it, suppose that for some $i \in I$, $\mathrm{Ker} f \cap M_i$ is not essential in M_i or equivalently $\mathrm{Ker} f \cap M_i = 0$ by reason of the uniformity of M_i. Then for $0 \neq x_i \in M_i$ there exists a finite subset $\{j_1, \cdots, j_n\} \subseteqq I$ such that $0 \neq f_i(x_i) = \sum_{k=1}^{n} f_{j_k i}(x_i)$ and $f_{j_i}(x_i) = 0$ for all $j \neq j_k$, where $f_i = \sum_{j \in J} f_{ji}$. Because the restriction $f_i = f \mid M_i : M_i \to M$ is a monomorphism. On the other hand, by hypothesis, $(\bigcap_{k=1}^{n} \mathrm{Ker} f_{j_k i}) \cap x_i R \neq 0$ and $f_i((\bigcap_{k=1}^{n} \mathrm{Ker} f_{j_k i}) \cap x_i R) = (\sum_{k=1}^{n} f_{j_k i})((\bigcap_{k=1}^{n} \mathrm{Ker} f_{j_k i}) \cap x_i R) = 0$, a contradiction. Thus $\mathrm{Ker} f \cap M_i$ is essential in M_i for every $i \in I$.

Conversely, we assume that $\mathrm{Ker} f \subseteqq' M$. Clearly this implies that $\mathrm{Ker} f \cap M_i \subseteqq' M_i$ by the uniformity of M_i ($i \in I$). On the other hand, since $f_i(x_i) = \sum_{j \in J} f_{ji}(x_i)$ and $f_{ji}(x_i) \in M_j$ for every $x_i \in M_i$, that $f_i(x_i) = 0$ implies that $f_{ji}(x_i) = 0$ for all $j \in I$. Therefore, $\mathrm{Ker} f_{ji} \neq 0$ for all $i, j \in I$, because $\mathrm{Ker} f_i = \mathrm{Ker} f \cap M_i \neq 0$. As a consequence, $\mathrm{Ker} f_{ji} \subseteqq' M_i$ for $i, j \in I$.

LEMMA 1.6 ([9], [10]). *Let $\{M_i\}_{i \in I}$ be a family of a semi-T-nilpotent system of modules with local endomorphism rings and $M = \bigoplus_{i \in I} M_i$. Then S/J is a regular ring in the sense of von Neumann and an idempotent of S/J can be lifted to S, where S is an endomorphism ring of M and J its Jacobson radical.*

This follows from Proposition 1.1, [9, Theorem 7] and [10, Theorem 3].

PROPOSITION 1.7. *Let $\{M_i\}_{i \in I}$ be a family of a semi-T-nilpotent system of modules with local endomorphism rings. Then $M = \bigoplus_{i \in I} M_i$ has the finite exchange property.*

Proof. Let $S = \mathrm{End}_R(M)$ and J the Jacobson radical of S. Then S/J is a regular ring and every idempotent is lifted to S by Lemma 1.6. Hence, for every element $s \in S$ there exists an idempotent $e \in S$ such that $sS + J = eS + J$. This shows that S has the exchange

property as a S-module and so M_R has the finite exchange property by [17, Theorems 3 and 4].

2. **The exchange property.** In this section we prove our main theorems being concerned with modules which are direct sums of indecomposable injectives.

First we will continue to consider a general case of modules with local endomorphism rings instead of indecomposable injectives.

LEMMA 2.1. *Let M, N, and A_i $(i \in I)$ be submodules of a module A such that*

$$A = \sum_{i \in I} \oplus A_i = M \oplus N$$

and, furthermore, let M be a direct sum of indecomposable submodules M_j $(j \in J)$ with local endomorphism rings. If $M \cap \sum_{i \in F} \oplus A_i \neq 0$ for some finite subset F of I, then there exist elements $i_0 \in F$ and $j_0 \in J$ such that

$$A = M_{j_0} \oplus A'_{i_0} \oplus \sum_{i \in I - \{i_0\}} \oplus A_i$$

for a suitable submodule A'_{i_0} of A_{i_0}.

Proof. First we remark that, since each M_j $(j \in J)$ has a local endomorphism ring, it has the exchange property by [15, Proposition 1], so that any finite direct sum of M_j's has also the exchange property ([5, Lemma 3.10]).

Now by hypothesis there exists a finite subset J_0 of J such that $\sum_{j \in J_0} \oplus M_j \cap \sum_{i \in F} \oplus A_i \neq 0$. Hence applying the exchange property of $\sum_{j \in J_0} \oplus M_j$ to the given decomposition $A = \sum_{i \in I} \oplus A_i$, we have decompositions such that

$$A_i = B_i \oplus C_i \ (i \in I) \ ,$$

$$A = \sum_{i \in I} \oplus B_i \oplus \sum_{i \in I} \oplus C_i \tag{1}$$

and

$$= \sum_{j \in J_0} \oplus M_j \oplus \sum_{i \in I} \oplus C_i \ . \tag{2}$$

Here there exists at least one element i_0 of F such that $B_{i_0} \neq 0$. For, if the contrary were true, $\sum_{i \in F} \oplus B_i = 0$ and hence $\sum_{i \in F} \oplus A_i = \sum_{i \in F} \oplus C_i$. So $\sum_{j \in J_0} \oplus M_j \cap \sum_{i \in F} \oplus C_i = \sum_{j \in J_0} \oplus M_j \cap \sum_{i \in F} \oplus A_i \neq 0$ by the definition of J_0, which contradicts the decomposition (2).

Now it is clear that $M' = \sum_{j \in J_0} \oplus M_j$ is isomorphic to $\sum_{i \in I} \oplus B_i$ via the restriction $\pi \mid M'$ of π to M', where π is the projection of A onto $\sum_{i \in I} \oplus B_i$ in the formula (1). It follows that

$$(3) \qquad \pi(M') = \sum_{j \in J_0} \pi(M_j) = B_{i_0} \oplus \sum_{i \in I - \{i_0\}} \oplus B_i \ .$$

Since each $\pi(M_j)$ for $j \in J_0$ is isomorphic to M_j, it has a local endomorphism ring. We can thus apply the Krull-Remak-Schmidt-Azumaya's theorem [1, Theorem 1] to this module $\pi(M')$ and the projection ξ of $\pi(M')$ onto B_{i_0} in the formula (3). As a consequence, there exists an element $j_0 \in J_0$ such that the restriction $\xi \mid \pi(M_{j_0})$ is a monomorphism and $\xi\pi(M_{j_0})$ is a direct summand of $\pi(M')$ and hence of B_{i_0}. On the other hand, a simple computation shows that the projection of A to B_{i_0} in the decomposition (1) is $\xi\pi$. Thus from these facts and Lemma 1.2 there is a submodule D_{i_0} of B_{i_0} such that

$$A = M_{j_0} \oplus D_{i_0} \oplus \sum_{i \in I - \{i_0\}} \oplus B_i \oplus \sum_{i \in I} \oplus C_i$$

because the restriction $\xi\pi \mid M_{j_0}$ is clearly a monomorphism. Setting $A'_{i_0} = C_{i_0} \oplus D_{i_0}$, we finally have a desired decomposition

$$A = M_{j_0} \oplus A'_{i_0} \oplus \sum_{i \in I - \{i_0\}} \oplus A_i \ .$$

From now on we will consider indecomposable injectives.

LEMMA 2.2. *Every indecomposable injective module is uniform and has a local endomorphism ring.*

This is well known (c.f., see [6, § 5 Proposition 8]).

Assume M_1 and M_2 are indecomposable injectives and f a morphism of M_1 to M_2. If f is a nonmonomorphism, then its kernel $\operatorname{Ker} f$ is essential in M_1 by Lemma 2.2 and the converse is, of course, true. This shows that f is a nonisomorphism if and only if $\operatorname{Ker} f$ is essential in M_1. Under this observation we have

PROPOSITION 2.3. *Let $\{M_i\}_{i \in I}$ be an infinite family of indecomposable injective modules and $M = \bigoplus_{i \in I} M_i$. Let S be an endomorphism ring of M_R and J the Jacobson radical of S. Then $J = \{f \in S \mid \operatorname{Ker} f \subseteqq' M\}$ if and only if the family $\{M_i\}_{i \in I}$ is a semi-T-nilpotent system.*

Proof. We will represent every endomorphism f of M as a column summable matrix: $f = (f_{ji})$, where $f_{ji} = \pi_j f \kappa_i$ for the projections π_j of M onto M_j and injections κ_i of M_i into $M (i, j \in I)$. Then,

in accordance with our earlier notations (see § 1), by the above remark we have

$$\mathfrak{F} \cap S = \{f = (f_{ji}) \in S \mid \operatorname{Ker} f_{ji} \subseteqq' M_i\}$$

and by Lemma 1.5

$$\{f = (f_{ji}) \in S \mid \operatorname{Ker} f_{ji} \subseteqq' M_i\} = \{f \in S \mid \operatorname{Ker} f \subseteqq' M\} .$$

On the other hand, we know by Proposition 1.1 that the family $\{M_i\}_{i \in I}$ is a semi-T-nilpotent system if and only if $J = \mathfrak{F} \cap S$. It follows from them that $\{M_i\}_{i \in I}$ is a semi-T-nilpotent system if and only if $J = \{f \in S \mid \operatorname{Ker} f \subseteqq' M\}$, which proves the proposition.

We need more lemmas for the main theorems.

LEMMA 2.4. *A module M has the exchange property if for any modules $A_i (i \in I)$ which are isomorphic to submodules of M and any decomposition $A = \bigoplus_{i \in I} A_i = M' \oplus N$ where $M' \cong M$, there exist submodules $A_i' \subseteqq A_i$ such that $A = M' \oplus \sum_{i \in I} \oplus A_i'$.*

This is well known in [5, Theorem 8.2] and its proof will be omitted.

LEMMA 2.5. *Let $G = M \oplus N$ for submodules M and N of a given module G. We moreover assume $M = \sum_{i \in I} \oplus M_i$, where $\{M_i\}_{i \in I}$ is an infinite family of indecomposable injective submodules of G and a semi-T-nilpotent system. Then if a module A is isomorphic to a submodule of M and contains an injective submodule, there exists a maximal submodule A_0 of A with the property that A_0 is a direct sum of indecomposable injective submodules. In this case such a module A_0 is a direct summand of A.*

Proof. Let the monomorphism of A to M be f and E an injective submodule of A. Then by [1, Theorem 1] and Lemma 2.2, $f(E)$ contains an indecomposable injective submodule isomorphic to some M_i in view of that $f(E)$ is a direct summand of M. This implies that A contains a submodule isomorphic to some M_i. Now then we can take a family $\{A_n\}_{n \geqq 1}$ of submodules of A such that each A_n is a direct sum of indecomposable injectives and $A_n \subseteqq A_{n+1}$ for any $n \geqq 1$. Then, by Zorn's lemma, we will be done if we can show that the union $A_0 = \bigcup_n A_n$ is also a direct sum of indecomposable injectives and, furthermore, a direct summand of A.

Since f is a monomorphism, the image $f(A_n)$ of A_n by f is also a direct sum of indecomposable injectives and hence $f(A_n)$ is a direct

summand of M for any $j \in J$ by Corollary 1.3 and Lemma 2.2. Thus the union $\bigcup_n f(A_n)$ is also a direct summand of M and a direct sum of indecomposable injectives by Corollary 1.4. Taking account of $f(A_0) = \bigcup_n f(A_n)$, we have $M = f(A_0) \oplus N$ for a submodule N of M and A_0 is a direct sum of indecomposable injectives since $A_0 \cong f(A_0)$. By the modular law, $f(A) = f(A_0) \oplus f(A) \cap N$. We therefore have $A = A_0 \oplus f^{-1}(f(A) \cap N)$, where $f^{-1}(f(A) \cap N)$ is the inverse image of $f(A) \cap N$ by f, which proves the lemma.

It is clear that the exchange property implies the finite exchange property, but it is not known whether the converse is true in general. However, in our case that modules are direct sums of indecomposable injectives we can conclude this question affirmatively.

THEOREM 2.6. *Let M be a module which is a direct sum of indecomposable injective modules and let S be an endomorphism ring of M_R. Then the following assertions are equivalent.*

(i) *M has the exchange property.*

(ii) *M has the finite exchange property.*

(iii) *The Jacobson radical of S is $\{f \in S \mid \operatorname{Ker} f \subseteq' M\}$.*

Proof. Let $M = \sum_{i \in I} \oplus M_i$, where every submodule M_i is indecomposable injective. If the index set I is finite, then M is clearly injective, so all of the above assertions (i), (ii), and (iii) are true. It therefore suffices to show the theorem for only the case with the infinite index set I.

Now let I be an infinite index set. By Proposition 2.3 the assertion (iii) is then equivalent to

(iii′) *The family $\{M_i\}_{i \in I}$ is a semi-T-nilpotent system.*

Thus we will consider (iii′) instead of (iii) in the following.

The implication (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii′). The idea of the proof is due to [9, Lemma 9]. Assume that M has the finite exchange property. Take an arbitrary countable subfamily of $\{M_i\}_{i \in I}$, say $\{M_n\}_{n \geqq 1}$, and nonisomorphisms $f_n: M_n \to M_{n+1}$ $(n \geqq 1)$. For every $x \in M_1$ we will find an integer $n(x)$ depending on x such that $f_{n(x)} f_{n(x)-1} \cdots f_1(x) = 0$.

For this put $M_n' = \{x + f_n(x) \mid x \in M_n\}$. It is then clear that $M_n' \oplus M_{n+1} = M_n \oplus M_{n+1}$ for $n \geqq 1$. Since each M_i is indecomposable injective, every nonisomorphism f_n is only nonmonomorphism, i.e., $\operatorname{Ker} f_n \neq 0$. This implies that $M_n' \cap M_n \neq 0$ for every $n \geqq 1$.

It is clear that

$$(1)\quad \sum_{i=1}^{\infty} \oplus M_i = M_1' \oplus M_2 \oplus M_3' \oplus M_4 \oplus \cdots \oplus M_{2n-1}' \oplus M_{2n} \oplus \cdots$$

$$(2)\quad = M_1 \oplus M_2' \oplus M_3 \oplus M_4' \oplus \cdots \oplus M_{2n-1} \oplus M_{2n}' \oplus \cdots$$

and we put

$$N = M_1' \oplus M_3' \oplus \cdots \oplus M_{2n-1}' \oplus \cdots .$$

Then, applying the fact that N has also the finite exchange property ([5, Lemma 3.10]) to the decomposition (2), we have that $\sum_{i=1}^{\infty} \oplus M_i = N \oplus X \oplus Y$ for some submodules X and Y of $\sum_{n=1}^{\infty} \oplus M_{2n-1}$ and $\sum_{n=1}^{\infty} \oplus M_{2n}'$ respectively. Here, in fact, it will hold $X = 0$.

To show this, suppose that $X \neq 0$ contrary. Then by Lemma 2.1 there exists M_{2m-1} such that

$$\sum_{i=1}^{\infty} \oplus M_i = N \oplus M_{2m-1} \oplus X' \oplus Y$$

for some submodule X' of X.

This however contradicts that $0 \neq M_{2m-1} \cap M_{2m-1}' \subseteqq M_{2m-1} \cap N$. Thus it holds

$$\sum_{i=1}^{\infty} \oplus M_i = N \oplus Y .$$

Now we take an arbitrary nonzero element $x \in M_1$ and we let $x = y + z$ with $y \in N$ and $z \in Y$. Considering these y and z in the decompositions $N = \sum_{n=1}^{\infty} \oplus M_{2n-1}'$ and $\sum_{n=1}^{\infty} \oplus M_{2n}'$ respectively, we have

$$y = \sum_{i=1}^{s} (x_{2i-1} + f_{2i-1}(x_{2i-1}))$$

and

$$z = \sum_{i=1}^{s} (x_{2i} + f_{2i}(x_{2i})) ,$$

and substituting these expressions for y and z, we have

$$x = \sum_{i=1}^{s} (x_{2i-1} + f_{2i-1}(x_{2i-1})) + \sum_{i=1}^{s} (x_{2i} + f_{2i}(x_{2i}))$$

$$= x_1 + \sum_{i=1}^{2s-1} (f_i(x_i) + x_{i+1}) + f_{2s}(x_{2s}) .$$

Therefore, $x = x_1$, $f_i(x_i) + x_{i+1} = 0$ $(1 \leqq i \leqq 2s - 1)$ and $f_{2s}(x_{2s}) = 0$, that is, $x_1 = x$, $x_2 = -f_1(x)$, $\cdots$, $x_{2s} = f_{2s}(x_{2s-1})$ and $f_{2s-1}(x_{2s}) = 0$. By successive substitutions, we obtain $x_{2s} = (-1)^{2s-1} f_{2s-1} \cdots f_1(x)$ and, finally, $f_{2s} f_{2s-1} \cdots f_1(x) = 0$. Thus we can put $n(x) = 2s$, which completes the proof of (ii) $\Rightarrow$ (iii').

(iii') $\Rightarrow$ (i). We assume the family $\{M_i\}_{i\in I}$ is a semi-T-nilpotent system. Suppose $A = \sum_{j\in J} \oplus A_j = M' \oplus N$, where $M' \cong M$ and each A_j is isomorphic to a submodule of M'. Then, taking account of Lemma 2.4, we will be done if we can find submodules A'_j of A_j $(j \in J)$ such that $A = M' \oplus \sum_{j\in J} \oplus A'_j$.

For this, we will first refine the given decomposition $A = \sum_{j\in J} \oplus A_j$. We should note that M' is also a direct sum of indecomposable injective submodules $M'_i (i \in I)$. By Lemma 2.1 there exists at least one element $j_0 \in J$ such that A_{j_0} has a nonzero submodule isomorphic to some M'_i. Let the subset of J of such elements $j_0 \in J$ be J_0. By Lemma 2.5 there exist maximal submodules B_j of $A_j (j \in J_0)$ such that each B_j is a direct sum of indecomposable injective submodules of A_j, in which case every B_j is a direct summand of A_j, say $A_j = B_j \oplus C_j$ for a submodule $C_j \subset A_j$ for $j \in J_0$. Consequently, we have such a refinement of $A = \sum_{j\in J} \oplus A_j$ that

$$(1) \qquad A = \sum_{j\in J_0} \oplus B_j \oplus \sum_{j\in J_0} \oplus C_j \oplus \sum_{j\in J-J_0} A_j ,$$

where $J - J_0$ is the complement of J_0 in J and if $J - J_0$ is empty, we put A_j in the formula (1) to be zero submodule of A for convenience.

Next we will have that

$$(2) \qquad M' \cap (\sum_{j\in J_0} \oplus C_j \oplus \sum_{j\in J-J_0} \oplus A_j) = 0 .$$

For this we suppose that $M' \cap (\sum_{j\in J_0} \oplus C_j \oplus \sum_{j\in J-J_0} \oplus A_j) \neq 0$. Then by Lemma 2.1 and the choice of J_0 there exists M'_{i_0} such that for a submodule $X_{j_0} \subseteqq C_{j_0}$,

$$A = M'_{i_0} \oplus X_{j_0} \oplus B_{j_0} \oplus \sum_{j\in J-\{j_0\}} \oplus A_j .$$

This implies there exists an injective submodule C'_{j_0} of C_{j_0} which is isomorphic to M'_{i_0}. However, in this case we have that $B_{j_0} \oplus C'_{j_0}$ is a direct summand of A_{j_0} and a direct sum of indecomposable injective submodules, which contradicts the maximality of B_{j_0}.

Now we can exchange the complement N of M' for a direct sum of submodules of $A_j (j \in J)$. For this let the projection of A onto $\sum_{j\in J_0} \oplus B_j$ in (1) be ρ. The family $\{M'_i\}_{i\in I}$ is semi-T-nilpotent by hypothesis, and so is $\{\rho(M'_i)\}_{i\in I}$ because the restriction $\rho \mid M'$ of ρ to M' is a monomorphism by (2) and Lemma 1.2. Using Corollary 1.3 the image $\rho(M')$ therefore is a direct summand of $\sum_{j\in J_0} \oplus B_j$ and there is a subset $K \subset J_0$ such that $\sum_{j\in J_0} \oplus B_j = \rho(M') \oplus \sum_{k\in K} \oplus B_k$ and, consequently we have $A = \rho(M') \oplus \sum_{k\in K} \oplus B_k \oplus \sum_{j\in J_0} \oplus C_j \oplus$

$\sum_{j\in J-J_0}\oplus A_j$. Computing the projection of A to $\rho(M')$ and by Lemma 1.2, we therefore have a decomposition

$$A = M' \oplus \sum_{k\in K}\oplus B_k \oplus \sum_{j\in J_0}\oplus C_j \oplus \sum_{j\in J-J_0}\oplus A_j ,$$

which completes the proof of the implication (iii') $\Rightarrow$ (i). Thus we conclude the theorem.

The original definition of the exchange property given in the introduction is due to Crawly and Jónsson [5]. However, we will consider the following weaker exchange property, too ([10]).

DEFINITION. A direct summand M of a module A has *the exchange property in* A if for any direct sum decomposition $A = \sum_{i\in I}\oplus A_i$, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus \sum_{i\in I} A_i'$.

We recall that for a ring R a right ideal I is *(meet-) irreducible* provided $I \neq R$ and $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$ for all right ideals I_1 and I_2 or R.

THEOREM 2.7. *The following conditions are equivalent.*

(i) *A ring R satisfies the ascending chain condition for irreducible right ideals.*

(ii) *Any direct sum of indecomposable injective modules has the exchange property.*

(iii) *Any direct sum of indecomposable injective modules has the finite exchange property.*

(iv) *Any direct summand of the module M which is a direct sum of indecomposable injective modules has the exchange property in M.*

(v) *For any direct sum M of indecomposable injective modules, the Jacobson radical of the endomorphism ring* $\mathrm{End}_R(M)$ *is* $\{f \in \mathrm{End}_R(M) \mid \mathrm{Ker}\, f \subseteq' M\}$.

Proof. The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v) are trivial from Theorem 2.6, and (ii) $\Rightarrow$ (iv) follows from [5, Lemma 3.10]. The implication (iv) $\Rightarrow$ (i) is contained in [19, Theorem 1].

(i) $\Rightarrow$ (ii): Let $M = \sum_{i\in I}\oplus M_i$, where M_i is indecomposable injective for any $i \in I$. If I is finite, then M is clearly injective, so it has the exchange property ([16, Lemma 2]). If I is infinite, the family $\{M_i\}_{i\in I}$ is a semi-T-nilpotent system by [19, Theorem 1 and Lemma 2]. Therefore, M has the exchange property by Proposition 2.3 and Theorem 2.6.

3. **Example.** Here we show the existence of modules which are not quasi-injective but isomorphic to direct sums of indecomposable

injectives and have the exchange property.

We first note that a quasi-injective module M over a ring R is injective by the criterion of Fuchs [7, Lemma 2] provided that M has the property that some finite direct sum of copies of M contains an element with a zero annihilator right ideal or, equivalently, contains a submodule isomorphic to the ring R.

The ring R regarded as a right (left) module over itself will be written $R_R({}_RR)$.

LEMMA 3.1. *For a ring R the following conditions are equivalent.*

(i) *R is right perfect and its injective hull $E(R_R)$ is projective, Σ-(quasi-) injective.*

(ii) *R is left perfect and its injective hull $E({}_RR)$ is projective, Σ-(quasi-) injective.*

REMARK. By the above note the "Σ-quasi-injective" and "Σ-injective" are coincident in Lemma 3.1.

Proof. We will only prove that (i) implies (ii) as the converse follows by symmetry.

Assume (i). Since R is right perfect, $E(R_R)$ has an indecomposable direct sum decomposition, $E(R_R) = \sum_{i=1}^{m} \oplus P_i$, where each P_i is injective projective right module. Let $R = e_1R \oplus \cdots \oplus e_nR$ for primitive idempotents e_i. Then there is an integer $\kappa(i)$ such that $P_i \cong e_{\kappa(i)}R$ for any $1 \leqq i \leqq m$. Let $\{P_j\}_{j=1}^{s}$ be a subclass of mutually nonisomorphic projective modules of $\{P_i\}_{i=1}^{m}$ such that each $P_i (1 \leqq i \leqq m)$ is isomorphic to some $P_j (1 \leqq j \leqq s)$ (here, if need, the indecies are renumbered) and we put $M = P_1 \oplus \cdots \oplus P_s$, then a right ideal $I = e_{\kappa(1)}R \oplus \cdots \oplus e_{\kappa(s)}R$ is isomorphic to M. Since M is clearly Σ-injective and faithful, so is then also I. Thus, by [4, Theorem 1.3], $E({}_RR)$ is projective, and R is left perfect and contains faithful, Σ-injective left ideal $\sum_{i=1}^{l} \oplus E(S_i)$, where $\{S_i\}_{i=1}^{l}$ is the representative class of simple left ideals which are nonisomorphic mutually and $E(S_i)$ an injective hull contained in R. As a consequence, $E({}_RR)$ is Σ-injective because $E({}_RR)$ is isomorphic to a submodule of a finite direct sum of copies of $\sum_{i=1}^{l} \oplus E(S_i)$. This completes the proof.

Now then, we suppose R is a (left and right) perfect ring such that $E(R_R)$ is projective and $E({}_RR)$ is not projective (for the existence of such a ring, see Müller [14] and Colby and Rutter [4]). Then, $E(R_R) = \sum_{i=1}^{m} \oplus P_i$, where each P_i is indecomposable injective for $1 \leqq i \leqq m$ and, since the radical of every projective right module over a right perfect ring is small, any infinite family of modules each of which is isomorphic to some P_i is a T-nilpotent system ([12, Theorem 3]). On the other hand, an infinite direct sum $M = \bigoplus_{i \in I} M_i$ with

$M_i \cong E(R_R)$ is not quasi-injective by Lemma 3.1. Thus M is the desirable module having the exchange property by Proposition 2.3 and Theorem 2.6.

4. **Applications.** We will generalize the theorems of Chamard [3, Théorème 3] and Yamagata [18, Theorem 4].

We recall definitions. A submodule N of a module M is said to be *closed* if it has no proper essential extension in M, that is, if $N \subseteqq' X$ for any submodule X of M, then $N = X$. A module M is said to be *well-complemented* in case any finite intersection of closed submodules of M is also closed.

LEMMA 4.1. *Let M be a direct sum of indecomposable injective modules $M_i (i \in I)$ and N a direct summand of M. If N is well complemented, then N is also a direct sum of indecomposable injective submodules.*

Proof. By [1, Theorem 1] it is clear N has a nonzero indecomposable injective submodule, so we can choose a maximal independent set $\{N_j\}_{j \in J}$ of indecomposable injective submodules of N. Put $N_0 = \sum_{j \in J} \oplus N_j$.

We will show $N = N_0$. To show this take an arbitrary nonzero element $x \in N$. Then there exists an injective hull $E(xR)$ of xR in N by [18, Lemma 2] and it is a finite direct sum of indecomposable injectives by [1, Theorem 1], say $E(xR) = E_1 \oplus \cdots \oplus E_n$. By the maximality of $\{N_j\}_{j \in J}$, it is evident that $N_0 \cap E_i \neq 0$ for $1 \leqq i \leqq n$. Then, since N is well-complemented by hypothesis, this will imply $E_i \subseteqq N_0$ for $1 \leqq i \leqq n$ and so $x \in E(xR) \subseteqq N_0$, which means $N = N_0$.

Because there exists a finite subset $\{j_1, \cdots, j_m\} \subseteqq J$ such that $\sum_{k=1}^{m} \oplus N_{j_k} \cap E_i \neq 0$ for $1 \leqq i \leqq n$. Since $\sum_{k=1}^{m} \oplus N_{j_k}$ and E_i are injective, they are closed in N and so is $\sum_{k=1}^{m} \oplus N_{j_k} \cap E_i$ by hypothesis of N for any $1 \leqq i \leqq n$. Then, since E_i is an essential extension of $\sum_{k=1}^{m} \oplus N_{j_k} \cap E_i$ by Lemma 2.2, it must be that $E_i = \sum_{k=1}^{m} \oplus N_{j_k} \cap E_i$ and therefore $E_i \subseteqq \sum_{k=1}^{m} \oplus N_{j_k}$ for any i. Consequently $x \in E(xR) \subseteqq \sum_{k=1}^{m} \oplus N_{j_k} \subseteqq N$, which concludes the lemma.

Under the same assumptions as in Lemma 4.1, we remark that N has no proper essential submodule which is a direct sum of indecomposable injectives from the proof of Lemma 4.1. This is first shown by Chamard [3, Lemma 4.1].

PROPOSITION 4.2. *Let M be a direct sum of indecomposable injective modules $M_i (i \in I)$ and N a direct summand of M; $M = N \oplus N'$. If N is well-complemented, then N has the exchange property*

and N and N' are also direct sums of indecomposable injective submodules.

Proof. By Lemma 4.1, N is a direct sum of indecomposable injective submodules $N_j (j \in J)$. To show that N has the exchange property we will check the property (iii) in Theorem 2.6.

Let S be an endomorphism ring of N_R and J its Jacobson radical. We must show that $J = \{f \in S \mid \operatorname{Ker} f \subseteqq' N\}$. The inclusion $J \subseteqq \{f \in S \mid \operatorname{Ker} f \subseteqq' N\}$ is known in [2, p. 564]. Conversely take an arbitrary element $f \in S$ with $\operatorname{Ker} f \subseteqq' N$. To show that $f \in J$, it is enough to show that $1 - f$ is an isomorphism.

First we will prove that $1 - f$ is a monomorphism. If $\operatorname{Ker}(1 - f) \neq 0$, $xR \cap \operatorname{Ker} f \neq 0$ for any nonzero element $x \in \operatorname{Ker}(1 - f)$ since $\operatorname{Ker} f \subseteqq' N$. There is hence a nonzero element y of xR with $f(y) = 0$ and so $y = (1 - f)(y)$ which must imply $y = 0$, because $y \in \operatorname{Ker}(1 - f)$, a contradiction.

Next we will prove that $1 - f$ is an epimorphism. Since $1 - f$ is a monomorphism, $(1 - f)(N)$ is also a direct sum of indecomposable injectives. Take an arbitrary nonzero element $x \in N$. Then $xR \cap \operatorname{Ker} f \neq 0$, that is, there is a nonzero element $y \in xR \cap \operatorname{Ker} f$. We therefore have $xR \cap (1 - f)(N) \neq 0$, because $y = (1 - f)(y) \in xR \cap (1 - f)(N)$. This shows that $(1 - f)(N)$ is essential in N, so that $N = (1 - f)(N)$. Because N has no proper essential submodule which is a direct sum of indecomposable injectives by the remark just before this proposition.

Thus we have shown that N has the exchange property. We can then exchange N' for $\sum_{k \in K} \oplus M_k$ for some subset $K \subset I$, $M = N \oplus \sum_{k \in K} \oplus M_k$. This implies that $N' \cong \sum_{k \in K} \oplus M_k$, which completes the proof of the proposition.

Let M_R be any *nonsingular* module over a ring R, that is, $M \neq 0$ and if $xI = 0$ for $x \in M$ and essential right ideal I of R, then $x = 0$. It is then well known that the lattice of all closed submodules of M is complete and so M is clearly well-complemented (c.f., see [6, Corollary 8, p. 61]). Thus we can sharpen [18, Theorem 4] and [11, Proposition 4].

COROLLARY 4.3. *Let M, N, and N' be as above. If N is nonsingular, then it has the exchange property and so N and N' are also direct sums of indecomposable injective submodules.*

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