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ON DOMINANT AND CODOMINANT DIMENSION OF $QF-3$ RINGS

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In this paper the concept of codominant dimension is defined and studied for modules over a ring. When the ring R is artinian, a left R module M has codominant dimension at least n in case there exists a projective resolution

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

with P_i injective. It is proved that every left R-module has the above property if and only if R has dominant dimension at least n. The concept of codominant dimension is also used to study semi-perfect $QF-3$ rings.

Let R be an associative ring with an identity 1. Denote by ${}_{\mathfrak{R}}R$ (resp. $R_{\mathfrak{R}}$) the left (resp. right) R-module R. Using the terminology of [5], we have the following definitions:

(1) R is left $QF-3$, if R has a faithful projective injective left ideal.

(2) R is left $QF-3^+$ if the injective hull $E({}_{\mathfrak{R}}R)$ is projective.

(3) R is left $QF-3'$ if $E({}_{\mathfrak{R}}R)$ is torsionless, i.e., there exists a set A such that $E(R) \leqq \prod_A R$.

In general $(1) \Rightarrow (3)$. For perfect rings the three conditions are equivalent for left and right $QF-3$ rings. (See [5].)

The dominant dimension of a left (resp. right) R-module M, denoted by dom. dim $({}_{\mathfrak{R}}M)$ (resp. dom. dim $(M_{\mathfrak{R}})$) is at least n, if there exists an exact sequence

$$0 \longrightarrow M \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n$$

of left (resp. right) R-module where each X_i is torsionless and injective for $i = 1, \cdots, n$. See [3] for details.

Note that this says when dom. dim $({}_{\mathfrak{R}}R) \geqq 1$ and R is left-artinian that $E(Re_i)$ for $i = 1, \cdots, n$ is projective where $\{e_i\}, i = 1, \cdots, n$ is a complete set of orthogonal idempotents, and that each X_i is projective.

We define codominant dimension as follows:

Let M be a left R-module. The codom. dim of M is at least n in case there exists an exact sequence

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

where P_i is torsionless and injective for $i = 1, \cdots, n$.

Following the notation of [3], we say that if such an exact sequence exists for $1 \leqq i \leqq n$, but no such sequence exists for $1 \leqq i \leqq n+1$, then codom. dim $({}_{\mathfrak{R}}M) = n$. If such a sequence exists for all n then codom. dim $({}_{\mathfrak{R}}M) = \infty$. If no such sequence exists codom. dim $({}_{\mathfrak{R}}M) = 0$.

An R-module U is defined to be a cogenerator if for any module M we can embed it in a product of copies of U. We have:

LEMMA. *Let U, V be left injective cogenerators then the* codom. dim (U) = codom. dim (V).

The proof follows easily from properties of injective cogenerators and shall omit it.

Let U be a left injective cogenerator. If the codom. dim $(U) = n$, we say that R has l. codom. dim $({}_{\mathfrak{R}}R) = n$. In a similar manner one defines r. codom. dim $(R_{\mathfrak{R}})$. Note that if ${}_{\mathfrak{R}}R$ is artinian, products of projectives are projective and direct sums of injectives are injective. Hence l. codom. dim $({}_{\mathfrak{R}}R) = n$ is equivalent to the existence of a resolution

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow U \longrightarrow 0$$

where P_i is projective and injective and $U = E(S_1) \oplus \cdots \oplus E(S_n)$ where S_i: $i = 1, \cdots, n$ is a copy of each simple left R-module.

In § 1 we characterize semi-perfect $QF-3^+$ rings in terms of their finitely generated projective, injectives.

In § 2 we show that l. dom. dim $({}_{\mathfrak{R}}R)$ and l. codom. dim $({}_{\mathfrak{R}}R)$ are the same for artinian rings. Hence, if R is artinian $QF-3$ then the l.-dom. dim (r-dom. dim) l. codom. dim (r-codom. dim) are the same.

For notation we use J to donote the Jacobson radical, and $R^{(A)}(R^A)$ denotes a direct sum (resp. direct product) of A-copies of R. Also $E(M)$ will be used to denote the injective hull of an R-module M and $P(M)$ will denote the projective cover of M when M has a projective cover. For a left R-module M, we let $\ell_{\mathfrak{R}}(M) = \{x \in R \mid x \cdot M = 0\}$, and $r_{\mathfrak{M}}(I) = \{x \in M \mid I \cdot x = 0\}$ where $I \subseteqq R$. We will use $T(M)$ to denote $M/J(M)$ where $J(M)$ is the Jacobson radical of M.

1. $QF-3$ **Rings.** Recall that if ${}_{\mathfrak{R}}R$ is noetherian $rt \cdot QF-3 \Leftrightarrow rt \cdot QF-3^+$. (See [1] and [6].)

To begin with we shall prove that under those hypotheses

$$rt \cdot QF-3^+ \Longleftrightarrow rt \cdot QF-3' .$$

PROPOSITION 1.1. *Let ${}_{\mathfrak{R}}R$ be noetherian. If $E(R_{\mathfrak{R}})$ is torsionless then $E(R_{\mathfrak{R}})$ is projective.*

Proof. Given that $0 \longrightarrow E \overset{\theta}{\longrightarrow} R^A$ is monic, where A is an indexing set. We show that there exists a finite number of R_α's, $\alpha \in A$ say $R_{\alpha_i}, \cdots, R_{\alpha_m}$ such that $\pi\theta|_R = \tilde{\theta}$ where π is the projection $R^A \longrightarrow \bigoplus \sum_{i=1}^m R_{\alpha_i}$ is monic. Let S be the set of all finite intersections of right ideals $\{K_\alpha\}_{\alpha \in A}$ where $K_\alpha = \ker(\pi_\alpha \circ \theta|_R)$. Note that $\bigcap_{i=1}^n K_{\alpha_i}$ induces a natural embedding of

$$0 \longrightarrow R \Big/ \bigcap_{i=1}^n K_{\alpha_i} \longrightarrow R^{(n)} .$$

Thus $R/\bigcap_{i=1}^n K_{\alpha_i}$ is torsionless. Hence by [2, Thm. I, p. 350]

$$\bigcap_{i=1}^n K_{\alpha_i} = r_{\mathfrak{R}}\ell_{\mathfrak{R}}\Big(\bigcap_{i=1}^n K_{\alpha_i}\Big) .$$

Now since ${}_{\mathfrak{R}}R$ noetherian, the set $\{\ell_{\mathfrak{R}}(\bigcap_{i=1}^n K_{\alpha_i})\}$ has a maximal element $\ell_{\mathfrak{R}}(\bigcap_{i=1}^m K_{\alpha_i})$ where $\bigcap_{i=1}^{n} K_{\alpha_i} \in S$. Thus $r_{\mathfrak{R}}\ell_{\mathfrak{R}}(\bigcap_{i=1}^m K_{\alpha_i}) = \bigcap_{i=1}^m K_{\alpha_i}$ is a minimal right ideal in S. But then $x \in \bigcap_{i=1}^m K_{\alpha_i} \Rightarrow x \in \bigcap_{\alpha \in A} K_\alpha$. Thus $\bigcap_{i=1}^m K_{\alpha_i} = 0$. This implies that $\tilde{\theta}$ is monic. But then $\pi\theta$ is monic since $\ker(\pi\theta) \cap R \neq 0$ if $\ker(\pi\theta) \neq 0$. This shows E is projective.

We next show that $QF-3^+ \Rightarrow QF-3$ for semi-perfect rings.

First we need the following lemma.

LEMMA 1.2. *Let K be finitely generated. Suppose there exists an exact sequence*

$$0 \longrightarrow K \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n$$

where $E(K) = E_1$, $E_{i+1} = E(E_i)$ for $1 \leqq i \leqq n-1$ and each E_i is projective. Then $E_1, \cdots, E_n$ are all finitely generated.

Proof. This follows easily from the proof of [4, Lemma 1].

PROPOSITION 1.3. *Suppose R is semi-perfect. If R is left $QF-3^+$ then R is left $QF-3$.*

Proof. By Lemma 1.2 $E(R)$ is finitely generated. Since R is semi-perfect $E(R) \cong \bigoplus \sum_{i=1}^n Re_i$, where each e_i is an indecomposable idempotent.

Let $Re_1, \cdots, Re_k$ be a subset of $Re_1, \cdots, Re_n$, where the set $\{Re_1, \cdots, Re_k\}$ is a complete set of isomorphism classes of $\{Re_1, \cdots, Re_n\}$. Then $U = Re_1 \oplus \cdots \oplus Re_k$ is a minimal projective injective.

Now we come to the main theorem of this section.

THEOREM 1.4. *Let R be semi-perfect. The following are equivalent*:

(a) *R is left $QF-3^+$.*

(b) *$E({}_{\mathfrak{R}}R)$ is finitely generated and every finitely generated left injective has an injective projective cover.*

(c) *Every finitely generated left projective has a projective injective hull.*

Proof. (b) $\Rightarrow$ (a): Consider

$$P(E(R)) \longrightarrow E(R) \longrightarrow 0 .$$

Embed $R \xrightarrow{i_{\mathfrak{R}}} E(R)$ then by the projectivity of R there exists a map $\theta' : R \rightarrow P(E(R))$ such that θ' is monic.

Consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & R & \xrightarrow{i_{\mathfrak{R}}} & E(R) \\ & & {\scriptstyle\theta'}\downarrow & \swarrow{\scriptstyle\theta''} & \\ & & P(E(R)) \, . & & \end{array}$$

Here $\theta''(r) = \theta'(r)$ for all $r \in R$. Also θ'' is monic. The injectivity of $E(R)$ forces $E(R)$ to be a direct summand of $P(E(R))$, hence projective.

(a) $\Leftrightarrow$ (c): Consider $R^{(n)}$, $R^{(n)} \leqq E(R)^{(n)}$. Thus $E(P) \leqq E(R)^n$, where $P \oplus P' = R^{(n)}$, as a direct summand. Hence $E(P)$ is projective. The converse is trivial.

(a) $\Rightarrow$ (b): By Lemma 1.2 $E(R)$ is finitely generated.

Consider $P(E) \xrightarrow{\theta} E \rightarrow 0$ where $P(E)$ is finitely generated injective. Let $R^{(n)} \xrightarrow{\rho} E \rightarrow 0$. Combining the above maps we have the following diagrams:

$$\begin{array}{ccccc} 0 & \longrightarrow & R^{(n)} & \xrightarrow{i_{\mathfrak{R}}^{(n)}} & E(R)^{(n)} \\ & & {\scriptstyle\rho}\downarrow & \swarrow{\scriptstyle\rho'} & \\ & & E \, . & & \end{array}$$

So we have ρ' epic and $\rho' \circ i_{\mathfrak{R}}^{(n)} = \rho$. Further we have

$$\begin{array}{ccccc} & & E(R)^{(n)} & & \\ & \swarrow{\scriptstyle\rho''} & \downarrow{\scriptstyle\rho'} & & \\ P(E) & \xrightarrow{\theta} & E & \longrightarrow & 0 \end{array}$$

Noting that ρ'' is epic and $P(E)$ is projective, $P(E)$ is a direct summand of $E(R)^{(n)}$. Hence injective.

A ring is perfect in case every module has a projective cover. We show that $QF-3^+$ rings can be characterized in terms of the projective cover of $E({}_{\mathfrak{R}}R)$.

THEOREM 1.5. *Let R be perfect. Then every indecomposable summand of $P(E({}_{\mathfrak{R}}R))$ is injective if and only if R is left $QF-3^+$.*

Proof. $\Rightarrow$ Consider the following diagram:

$$\begin{array}{ccc} & & {}_{\mathfrak{R}}R \\ & \swarrow^{f} & \downarrow i \\ P(E({}_{\mathfrak{R}}R)) & \xrightarrow{\pi} & E({}_{\mathfrak{R}}R) \longrightarrow 0\,. \end{array}$$

Here i is a monomorphism and π is epic. Since R is projective there exists on f such that $\pi f = i$. Clearly f is monic. Since R is perfect $P(E({}_{\mathfrak{R}}R)) \cong \sum_{\alpha\in A} Re_\alpha$, where e_α are primitive idempotents of R. Now $\mathrm{Im}(f)$ is contained in $\sum_{\alpha=1}^{n} Re_\alpha$, for n a positive integer, since ${}_{\mathfrak{R}}R$ is cyclic.Thus using the hypothesis, $E({}_{\mathfrak{R}}R)$ is projective and R is left $QF-3^+$. $\Leftarrow$ This is trivial.

2. Codominant dimension of rings. We begin with a lemma which holds the key to the main results of this section.

LEMMA 2.1. *Let R be a ring. The following conditions are equivalent.*

(1) *For every projective left R-module P, there exists an exact sequence*

$$0 \longrightarrow P \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n$$

where E_i, $1 \leqq i \leqq n$, are injective and projective.

(2) *For every injective left R-module Q, there exists an exact sequence*

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow Q \longrightarrow 0$$

where P_i, $1 \leqq i \leqq n$, are injective and projective.

Proof. (1) $\Rightarrow$ (2). For $n=1$ a modification for the proof of Theorem 1.4 will suffice. We assume the lemma is true for the nth case and prove the $n+1$ case. So consider the following exact sequences.

$$(1) \qquad 0 \longrightarrow P_{n+1} \xrightarrow{J_1} E_1 \xrightarrow{J_2} E_2 \longrightarrow \cdots \xrightarrow{J_{n+1}} E_{n+1}$$

$$(2) \qquad P_{n+1} \xrightarrow{\theta_1} P_n \xrightarrow{i_n} \cdots \longrightarrow P_1 \xrightarrow{i_1} Q \longrightarrow 0\,.$$

Here Q is an arbitrary injective module and

$$P_1, \cdots, P_n, E_1, \cdots, E_{n+1}$$

are both projective and injective and P_{n+1} is projective.

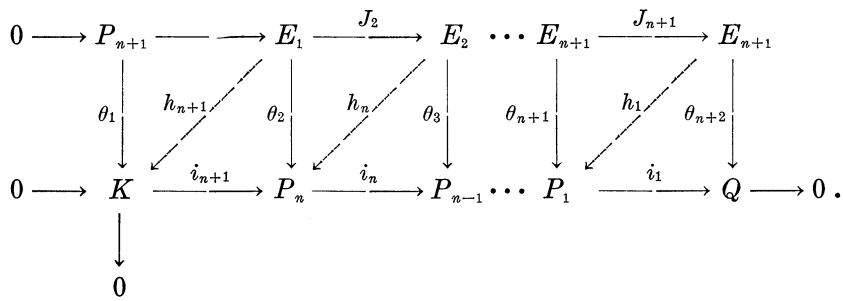
Also E_k is the injective hull of Cok (J_k).

Denote by K the image of θ_1. Using the injectivity of P_n, there is a map $\theta_2\colon E_1 \to P_n$ such $\theta_2 J_1 = i_{n+1}\theta_1$ where i_{n+1} is the embedding of K into P_n. The injectivity of P_{n-1} and the exact sequence $0 \to E_1/P_{n+1} \to E_2$ induce a map $\theta_3\colon E_2 \to P_{n-1}$ which one can easily check has the property $\theta_3 J_2 = i_n \theta_2$.

In like manner we can define $\theta_k\colon E_{k-1} \to P_{n+2-k}$ such that

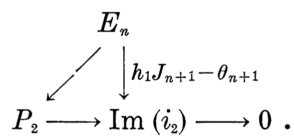
$$\theta_k J_{k-1} = i_{n+3-k}\theta_{k-1}\ , \quad k = 2, \cdots, n+2\ .$$

This information is summed up in the following diagram:



Having constructed θ_{n+2}, the projectivity of E_{n+1} induces a map $h_1\colon E_{n+1} \to P_1$ such $i_1 h_1 = \theta_{n+2}$. Now consider the map $h_1 J_{n+1} - \theta_{n+1}\colon E_n \to P_1$. We have $i_1(h_1 J_{n+1} - \theta_{n+1}) = \theta_{n+2}J_{n+1} - i_1\theta_{n+1} = 0$. So $\operatorname{Im}(h_1 J_{n+1} - \theta_{n+1}) \leqq \ker(i_1)$.

Now consider the following diagram:



We can construct h_2 using the projectivity of E_n. By a similar argument we can show that $\operatorname{Im}(h_2 J_n - \theta_n) \leqq \ker(i_2)$. By a recursive argument we can construct $h_k J_{n+2-k} - \theta_{n+2-k}$ for $k = 1, \cdots, n$ in like manner. In particular we have $h_n J_2 - \theta_2\colon E_1 \to P_n$ where $\operatorname{Im}(h_n J_2 - \theta_2) \leqq K$. We need only show equality to complete the proof. Let $k \in K$. Then there exists an $x \in P_{n+1}$ such that $\theta_1(x) = k$. Thus $(h_n J_2 - \theta_2)(J_1(-x)) = \theta_2 J_1(x) = \theta_1(x) = k$. Thus $h_n J_2 - \theta_2$ maps on to K. The proof $(2) \Rightarrow (1)$ is similar. This completes the proof.

Noting that for left artinian rings products of projectives are projective, and direct sums of injectives are injective one can easily show that dom. dim $(R) \geqq n$ implies dom. dim. $(P) \geqq n$ for all projective P.

Likewise letting $I = \bigoplus \sum E_\alpha(S_\alpha)$ be the minimal injective cogenerator of R, we find that codom. dim$(I) \geqq n$ implies codom. dim $(Q) \geqq n$ for all injectives Q. Thus we have:

THEOREM 2.2. *Let R be left artinian then the following are equivalent*:

(1) *The* $\inf \{m \in Z \mid \text{dom. dim } (P) = m$ *for all* P *projectives*$\} = n$.

(2) *The* $\inf \{m \in Z \mid \text{dom. dim } (Q) = m$ *for all* Q *injectives*$\} = n$.

(3) l. dom. dim $({}_{\mathfrak{R}}R) = n$.

(4) l. codom. dim $({}_{\mathfrak{R}}R) = n$.

If no such n exists we say l. dom. dim $(R) = \infty$

Proof. $(3) \Rightarrow (1)$, $(4) \Rightarrow (2)$ by our previous discussion. $(1) \Rightarrow (3)$: There exists a projective module P such dom. dim $(P) = n$.

Now $P \cong \bigoplus \sum_{\mathscr{A}} Re_\alpha$, $\{e_\alpha\}$ primitive idempotents such that for some e_β dom. dim $(Re_\beta) < n+1$ where $e_\beta \in \{e_\alpha\}$. Since $Re_\beta < R$, $n+1 >$ dom. dim $(R) \geqq n$. This yields the desired result. $(2) \Rightarrow (4)$ is similar. $(1) \Rightarrow (2)$: By Lemma 2.1 $\inf \{m \in Z \mid \text{codom. dim } (Q) = m\} \geqq n$. If inf of the above set is strictly greater than n, another application of the lemma forces $\inf \{m \in Z \mid m = \text{dom. dim } (P),\ P \text{ projective}\} > n$ which is impossible. $(2) \Rightarrow (1)$ is similar.

Let R be left artinian and both left and right $QF-3$. Then by [4, Thm. 10] l. dom. dim $({}_{\mathfrak{R}}R)$ = r. dom. dim $(R_{\mathfrak{R}})$. Thus in view of 2.2 we have:

PROPOSITION 2.3. *Let ${}_{\mathfrak{R}}R$ be artinian and $QF-3$. Then* l. domdim $({}_{\mathfrak{R}}R)$ = r. domdin $(R_{\mathfrak{R}})$ = l. codomdin $({}_{\mathfrak{R}}R)$ = r. codomdim $(R_{\mathfrak{R}}) = n$.

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