PACIFIC JOURNAL OF MATHEMATICS
Vol. 47, No. 2, 1973

THE PRODUCT OF F-SPACES WITH P-SPACES

NEIL HINDMAN

A condition on a basically disconnected space X is known which is necessary and sufficient for the product space $X \times Y$ to be basically disconnected for every P-space Y. This same condition, when applied to an F'-space X, guarantees that $X \times Y$ is an F'-space whenever Y is a P-space and is necessary for this result. The principal result of this paper establishes that this condition is not sufficient when applied to F-spaces. A condition which is sufficient but not necessary is also derived.

1. Introduction. The notation and general point of view are those of the Gillman and Jerison textbook [5]. In particular, all hypothesized spaces are completely regular Hausdorff. The reader should recall from [4] the following characterizations. A space X is: a P-space if and only if each cozero set is closed; a basically disconnected space if and only if each cozero set has open closure; a U-space if and only if disjoint cozero sets can be separated by an open-and-closed set; an F-space if and only if disjoint cozero sets can be completely separated; and an F'-space if and only if disjoint cozero sets have disjoint closures. It is clear from these characterizations that the conditions named grow progressively weaker.

In [3] Gillman asked for a necessary and sufficient condition that a product of two spaces be an F-space and, parenthetically, for a necessary and sufficient condition that a product of two spaces be a basically disconnected space. Curtis had shown [2] that if $X \times Y$ is an F'-space then either X or Y must be a P-space. It is easily seen that if $X \times Y$ has any of the properties listed above so must both X and Y for X and Y nonempty. Observing also that the product of a space X with a discrete space Y has any of the above mentioned properties which X has, one can rephrase the question in the form: For which spaces X with property A does the product $X \times Y$ have property A for every P-space Y?

This question was answered for the properties F' and basically disconnected in [1]. The condition was that the space be countably locally weakly Lindelöf (appreviated CLWL). That is, for every countable collection $\{\Gamma_n\}_{n=1}^{\infty}$ of open covers of X and each point x of X there must be a neighborhood V of x and, for each n, a countable subfamily Δ_n of Γ_n such that $V \subseteq \mathrm{cl} \bigcup \Delta_n$.

Since F-spaces are F'-spaces the condition that X be CLWL is clearly necessary for $X \times Y$ to be an F-space for each P-space Y. The obvious question, since basically disconnected spaces are F-spaces, is whether that condition is also sufficient [1, 4.7]. It is shown in §3 the answer is no. That is, there are a CLWL F-space X and a P-space Y such that $X \times Y$ is not an F-space.

In §2 sufficient conditions for the product of two spaces to be an F-space are derived. The same conditions suffice when "F-space" is replaced by "U-space" throughout.

2. Conditions guaranteeing that a product space is an F-space. We shall have need of the following lemma from [1, 3.2].

LEMMA 2.1. *Let $f \in C^*(X \times Y)$, where X is CLWL and Y is a P-space. If $(x_0, y_0) \in X \times Y$ then there is a neighborhood $U \times V$ of (x_0, y_0) such that $f(x, y) = f(x, y_0)$ whenever $(x, y) \in U \times V$.*

It is shown in [6] that this in fact characterizes CLWL spaces in the sense that if X is not CLWL then there is some P-space Y such that the conclusion of Lemma 2.1 fails. (The proof is a slight modification of the "necessity" proof in [1, 3.3].)

DEFINITION 2.2. A point x of X is a basically disconnected point of X if whenever U is a cozero set of X and $x \in \operatorname{cl} U$ then in fact $x \in \operatorname{int} \operatorname{cl} U$.

It is clear that X is basically disconnected if and only if every point of X is a basically disconnected point. The proof of the following lemma can be taken verbatim from [1, 3.4].

LEMMA 2.3. *If X is CLWL and x is a basically disconnected point of X and Y is a P-space then (x, y) is a basically disconnected point of $X \times Y$ for every y in Y.*

The reader should recall that a space X is weakly Lindelöf if each open cover of X has a countable subfamily whose union is dense in X.

THEOREM 2.4. *If X is a CLWL F-space (respectively U-space) and there is a weakly Lindelöf subspace D of X such that every point of $X \backslash D$ is a basically disconnected point and if Y is a P-space then $X \times Y$ is an F-space (respectively U-space).*

Proof. Let $f \in C^*(X \times Y)$. By Theorem 3.3 of [1] $X \times Y$ is an F'-space so $\operatorname{cl} \operatorname{pos} f \cap \operatorname{cl} \operatorname{neg} f = \varnothing$. (Here $\operatorname{pos} f = \{(x, y) : f(x, y) > 0\}$ and $\operatorname{neg} f = \{(x, y) : f(x, y) < 0\}$.) To show that $X \times Y$ is an F-space it suffices to show that $\operatorname{pos} f$ and $\operatorname{neg} f$ can be completely separated.

Define an equivalence relation on Y by agreeing that $y_1 \sim y_2$ if and only if the following three conditions hold for every x in D: (1) $f(x, y_1) = f(x, y_2)$; (2) $(x, y_1) \in \mathrm{cl\,pos}\, f$ if and only if $(x, y_2) \in \mathrm{cl\,pos}\, f$; and (3) $(x, y_1) \in \mathrm{cl\,neg}\, f$ if and only if $(x, y_2) \in \mathrm{cl\,neg}\, f$. It is clear that $\sim$ is an equivalence relation. Let Γ be the set of $\sim$ equivalence classes.

We claim that each element V of Γ is open. To see this let $V \in \Gamma$ and $y_0 \in V$. For each x in D there is a neighborhood $U_x \times V_x$ of (x, y_0) such that $f(x', y') = f(x', y_0)$ whenever $(x', y') \in U_x \times V_x$, by Lemma 2.1. Further, since X and Y are completely regular, U_x and V_x may be chosen to be cozero sets in X and Y. Now $\{U_x : x \in D\}$ is an open cover of D so there exists a countable subset $\{x(n)\}_{n=1}^{\infty}$ of D such that $D \subseteqq \mathrm{cl} \bigcup_{n=1}^{\infty} U_{x(n)}$. Let $V_0 = \bigcap_{n=1}^{\infty} V_{x(n)}$. Since Y is a P-space V_0 is a neighborhood of y_0. We claim that $V_0 \subseteqq V$, and hence that V is open as desired. To see this, let $y_1 \in V_0$. We will show that $y_1 \sim y_0$. To see that condition (1) holds suppose instead that $f(x, y_0) \neq f(x, y_1)$ for some x in D. Without loss of generality we may assume that $f(x, y_0) < f(x, y_1)$ so that there exist neighborhoods $U' \times V'$ of (x, y_0) and $U'' \times V''$ of (x, y_1) such that $f(x', y') < f(x'', y'')$ whenever $(x', y') \in U' \times V'$ and $(x'', y'') \in U'' \times V''$. Now $U' \cap U''$ is a neighborhood of x, a point of D, so there is some n and some $\bar{x}$ such that $\bar{x} \in U_{x(n)} \cap U' \cap U''$. Now $(\bar{x}, y_1) \in U_{x(n)} \times V_{x(n)}$ and $(\bar{x}, y_0) \in U_{x(n)} \times V_{x(n)}$ so $f(\bar{x}, y_1) = f(\bar{x}, y_0)$. But $(\bar{x}, y_0) \in U' \times V'$ and $(\bar{x}, y_1) \in U'' \times V''$ so $f(\bar{x}, y_0) < f(\bar{x}, y_1)$, a contradiction.

To see that condition (2) holds suppose instead that there is some x in D such that either $(x, y_1) \in \mathrm{cl\,pos}\, f$ and $(x, y_0) \notin \mathrm{cl\,pos}\, f$ or $(x, y_1) \notin \mathrm{cl\,pos}\, f$ and $(x, y_0) \in \mathrm{cl\,pos}\, f$. Suppose that the former case holds. Then there is a neighborhood $U' \times V'$ of (x, y_0), where U' and V' are cozero sets, such that $(U' \times V') \cap \mathrm{pos}\, f = \varnothing$. For each n in N let $U_n = U_{x(n)} \cap U'$. Then U_n is a cozero set in X so $\bigcup_{n=1}^{\infty} U_n$ is a cozero set of X. Also, since Y is a P-space, $\bigcap_{n=1}^{\infty} V_{x(n)} = V_0$ is a cozero set. Therefore, $\bigcup_{n=1}^{\infty} U_n \times V_0$ is the cozero set of some continuous function on $X \times Y$, say $\bigcup_{n=1}^{\infty} U_n \times V_0 = \mathrm{coz}\, g$. Further, if $(x', y') \in \bigcup_{n=1}^{\infty} U_n \times V_0$ then $f(x', y') = f(x', y_0)$, since $(x', y') \in U_{x(n)} \times V_{x(n)}$ for some n, and $f(x', y_0) \leqq 0$ since $(x', y_0) \in U' \times V'$. Thus $f(x', y') \leqq 0$ and so $\mathrm{coz}\, g$ and $\mathrm{pos}\, f$ are disjoint cozero sets. Consequently $\mathrm{cl\,coz}\, g \cap \mathrm{cl\,pos}\, f = \varnothing$ and so there is a neighborhood $U'' \times V''$ of (x, y_1) which misses $\mathrm{coz}\, g$. But $U'' \cap U'$ is a neighborhood of x, an element of D, so that there is some n and some $\bar{x}$ such that $\bar{x} \in U_{x(n)} \cap U' \cap U'' = U_n \cap U''$. Now $(\bar{x}, y_1) \in U_n \times V_0$ so $(\bar{x}, y_1) \in \mathrm{coz}\, g$ while $(\bar{x}, y_1) \in U'' \times V''$ so $(\bar{x}, y_1) \notin \mathrm{coz}\, g$, a contradiction. By interchanging y_1 and y_0 in the above argument one sees that it is also impossible to have (x, y_0) in $\mathrm{cl\,pos}\, f$ while $(x, y_1) \notin \mathrm{cl\,pos}\, f$.

One also sees in an identical fashion that condition (3) holds. Thus

$y_1 \sim y_0$ as desired.

Now choose y_V in V for every V in Γ and define f_V in $C^*(X)$ by the rule $f_V(x) = f(x, y_V)$. Now X is an F-space so, for each V in Γ, there exists g_V in $C^*(X)$ such that $g_V = 0$ on neg f_V, $g_V = 1$ on pos f_V and $0 \leqq g_V \leqq 1$. Define g in $C^*(X \times Y)$ by the rule $g(x, y) = g_V(x)$ where V is that element of Γ in which y lies. (The function g is continuous since each V in Γ is open.) Let h_1 and h_2 be the characteristic functions of cl pos f and $(X \times Y)\backslash$cl neg f respectively. Define a function k on $X \times Y$ by $k = (g \vee h_1) \wedge h_2$. Then $k = 1$ on cl pos f and $k = 0$ on cl neg f so to complete the proof it remains only to show that k is continuous.

Let $(x, y) \in X \times Y$. If $x \notin D$ then x is a basically disconnected point of X and so, by Lemma 2.3, (x, y) is a basically disconnected point of $X \times Y$. Consequently each of g, h_1, and h_2 are continuous at (x, y), and so k is continuous at (x, y). If $x \in D$ and $(x, y) \notin$ cl pos $f \cup$ cl neg f then there is a neighborhood of (x, y) on which k agrees with the continuous function g so that k is continuous at (x, y). If $x \in D$ and $(x, y) \in$ cl pos f then $(x, y_V) \in$ cl pos f where V is the member of Γ in which y lies. Therefore, $g(x, y) = g_V(x) = 1$. Let $\varepsilon > 0$ be given. Then there is a neighborhood U of x on which $g_V > 1 - \varepsilon$. Let $U' \times V'$ be a neighborhood of (x, y) which misses cl neg f. Then $(U \cap U') \times (V \cap V')$ is a neighborhood of (x, y) on which $k > 1 - \varepsilon$ and hence k is continuous at (x, y). Similarly if $x \in D$ and $(x, y) \in$ cl neg f and $\varepsilon > 0$ one can find a neighborhood of (x, y) on which $k < \varepsilon$. Thus k is continuous and hence $X \times Y$ is an F-space.

To prove the parenthetical theorem it is only necessary to note that if X is a U-space one can choose the functions g_V in the above argument to assume only the values 0 and 1. (The characteristic function of an open-and-closed set is continuous.) Consequently the function k assumes only the values 0 and 1 and the set $A = \{(x, y): k(x, y) = 0\}$ is an open-and-closed set containing neg f and missing pos f. Thus $X \times Y$ is a U-space.

Corollary 2.5 is the strongest result we have been able to obtain. It is shown in Example 3.2 that the conditions given on X are still not necessary in order for its product with every P-space to be an F-space.

COROLLARY 2.5. *If X is a CLWL F-space (respectively U-space), and there are a subset D of X and a partition Δ of X into open-and-closed sets such that every point of $X\backslash D$ is a basically disconnected point and $U \cap D$ is weakly Lindelöf for each U in Δ, and if Y is a P-space, then $X \times Y$ is an F-space (respectively U-space).*

Proof. Let $f \in C^*(X \times Y)$ and for each U in Δ let $k_U \in C^*(U \times Y)$ such that $k_U = 1$ on pos $f \cap (U \times Y)$ and $k_U = 0$ on neg $f \cap (U \times Y)$. Define k in $C^*(X \times Y)$ by the rule $k(x, y) = k_U(x, y)$ where $x \in U$. The parenthetical statement is similarly proved.

Corollary 2.6 appears in [6] and Corollary 2.7 appears in [8].

COROLLARY 2.6. *If X is a weakly Lindelöf F-space (respectively U-space) and Y is a P-space then $X \times Y$ is an F-space (respectively U-space).*

COROLLARY 2.7. *If X is a compact F-space and Y is a P-space then $X \times Y$ is an F-space.*

3. EXAMPLES. The first example establishes that the condition that a U-space be CLWL is not sufficient to guarantee that its product with each P-space is an F-space.

EXAMPLES 3.1. A CLWL U-space X and a P-space Y such that $X \times Y$ is not an F-space.

Let $\omega_2 + 1$ have the order topology and let $D = \{\sigma \in \omega_2 + 1: \sigma$ is not the supremum of countably many predecessors$\}$ with the relative topology from $\omega_2 + 1$. (The space D differs from the space of [5, 9L] only by the inclusion of the endpoint, ω_2.) Since we have deleted all non P-points of $\omega_2 + 1$ we have that D is a P-space. Following the hints in [5, 9L] one easily sees that elements of $C^*(D\backslash\{\omega_2\})$ are constant on a tail.

Let p be a free ultrafilter on N, the set of natural numbers. Let $E = N \cup (\omega_3 + 1)$ where every point of E is isolated except ω_3. Let basic neighborhoods of ω_3 be of the form $Z \cup]\gamma, \omega_3]$ where $Z \in p$ and $\gamma < \omega_3$. (We shall use the interval notation to indicate subsets of $\omega_2 + 1$ and $\omega_3 + 1$. Thus the interval $[0, \gamma[$ in E is $\{\sigma \in \omega_3 + 1: 0 \leqq \sigma < \gamma\}$ and does not include points of N.)

Let $X = (E \times D)\backslash((N \cup \{\omega_3\}) \times \{\omega_2\})$ and let X have the relative topology. The reader will observe that the space X bears a strong resemblance to the space constructed in [4, 8.14]. Both E and D are Hausdorff spaces with bases of open-and-closed sets so X is a completely regular Hausdorff space. It is easily verified that E is CLWL, that the product of a CLWL space with a P-space is CLWL and that open subspaces of CLWL spaces are CLWL. Consequently, since D is a P-space, one has that X is CLWL.

Note that E satisfies the hypotheses of Theorem 2.4 and so $E \times D$ is a U-space. Consequently, to show that X is a U-space it suffices to show that X is C^*-embedded in $E \times D$. To this end let $f \in C^*(X)$. For each n in N there exists $\gamma_n < \omega_2$ such that f is constant on $\{n\} \times (]\gamma_n, \omega_2[\cap D)$. (We have observed that continuous functions on $D \backslash \{\omega_2\}$ are constant on a tail.) Define the extension f^* of f to have this constant value at (n, ω_2). Similarly there is some $\gamma_0 < \omega_2$ such that f is constant on $\{\omega_3\} \times (]\gamma_0, \omega_2[\cap D)$ and we may define f^* to have this constant value at (ω_3, ω_2). The extension f^* of f is clearly continuous at every point of $E \times D$ except possibly (ω_3, ω_2).

For each σ in $D \backslash \{\omega_2\}$ there is an $\alpha_\sigma < \omega_3$ such that f is constant on $]\alpha_\sigma, \omega_3] \times \{\gamma\}$ (since ω_3 is a P-point of $E \backslash N$). Let $\gamma = \sup \{\gamma_n : n \in N \cup \{0\}\}$ and let $\alpha = \sup \{\alpha_\sigma : \sigma \in D \backslash \{\omega_2\}\}$.

Let $\varepsilon > 0$ be given and let $n \in N$ such that $|f(m, \gamma + 1) - f(\omega_3, \gamma + 1)| < \varepsilon$ whenever $m \in N$ and $m > n$. Then on $(]\alpha, \omega_3] \cup \{m : m \in N$ and $m > n\}) \times]\gamma, \omega_2]$ f^* differs from $f^*(\omega_3, \omega_2)$ by less than ε. Consequently f^* is continuous as desired.

Now, let $Y = \omega_2 + 1$, where every point is isolated except ω_2, whose basic neighborhoods are as in the interval topology. Since ω_2 is not the supremum of countably many predecessors we have that Y is a P-space.

We claim that $X \times Y$ is not an F-space. To see this define f in $C^*(X \times Y)$ by the rule $f((n, \tau), \gamma) = 1/n$ if $n \in N, \gamma$ is even and $\tau > \gamma, f((n, \tau), \gamma) = -1/n$ if $n \in N, \gamma$ is odd and $\tau > \gamma$ and $f = 0$ elsewhere. (An ordinal is even if it is a limit ordinal or the sum of a limit ordinal and an even finite ordinal.)

For each $\gamma < \omega_2$ f is clearly continuous on the open subset $X \times \{\gamma\}$ of $X \times Y$. Also, for each $\tau < \omega_2$ f is identically 0 on the open subset $(E \times ([0, \tau[\cap D)) \times]\tau, \omega_2[$of $X \times Y$. Finally, for each $\delta < \omega_3$, f is identically 0 on the open subset $(\{\delta\} \times D) \times Y$ of $X \times Y$. Thus f is continuous on all of $X \times Y$.

Now let U and V be open sets with $\mathrm{cl}\,\mathrm{pos}\, f \subseteqq U$ and $\mathrm{cl}\,\mathrm{neg}\, f \subseteqq U$. We claim that $\mathrm{cl}\, U \cap \mathrm{cl}\, V \neq \varnothing$ and consequently that $\mathrm{pos}\, f$ and $\mathrm{neg}\, f$ are not completely separated. Let γ be even, with $\gamma < \omega_2$. For each $\tau \in D$ such that $\tau > \gamma$ one has $((\omega_3, \tau), \gamma) \in \mathrm{cl}\,\mathrm{pos}\, f$ so there is some $\eta_\tau < \omega_3$ such that $(]\eta_\tau, \omega_3] \times \{\tau\}) \times \{\gamma\} \subseteqq U$. Let $\mu_\gamma = \sup \{\eta_\tau : \tau \in]\gamma, \omega_2[\cap D\}$. Then $(]\mu_\gamma, \omega_3[\times \{\omega_2\}) \times \{\gamma\} \subseteqq \mathrm{cl}\, U$. Similarly, for each odd $\gamma < \omega_2$ there is some $\mu_\gamma < \omega_3$ such that $(]\mu_\gamma, \omega_3[\times \{\omega_2\}) \times \{\gamma\} \subseteqq \mathrm{cl}\, V$.

Let $\mu = \sup \{\mu_\gamma: \gamma < \omega_2\}$. Then $\mu < \omega_3$ and $((\mu + 1, \omega_2), \omega_2) \in \mathrm{cl}\, U \cap \mathrm{cl}\, V$ as desired.

The following example shows that the sufficient condition obtained in Corollary 2.5 is not necessary.

EXAMPLE 3.2. A U-space which does not satisfy the hypotheses of Corollary 2.5 but whose product with each P-space is a U-space.

Let p be a free ultrafilter on N. Let $B = N \cup (\omega_2 + 1)$ with every point of B isolated except ω_2 whose basic neighborhoods are of the form $Z \cup \{\sigma: \gamma < \sigma \leqq \omega_2\}$ where $Z \in p$ and $\sigma < \omega_2$. (This is the space T of [7].) Note that B is a CLWL U-space with only one non basically disconnected point. Consequently by Theorem 2.4, its product with any P-space is a U-space.

Let $C = \omega_2 + 1$ with every point of C isolated except ω_2 and with basic neighborhoods of ω_2 as in the interval topology. Then C is a P-space. Let $X = B \times C$. Then X is a U-space. If Y is any P-space then $X \times Y$ is homeomorphic to $B \times (C \times Y)$ and $C \times Y$ is a P-space so $X \times Y$ is a U-space, by Theorem 2.4.

Suppose X satisfies the hypotheses of Corollary 2.5 and let D and Δ be as given there. There is some member U of Δ such that $(\omega_2, \omega_2) \in U$. Note also that $D \supseteqq \{\omega_2\} \times C$ since (ω_2, γ) is a non basically disconnected point of X whenever $\gamma \in C$. Since U is open there is some δ in C such that $\delta < \omega_2$ and $\{\omega_2\} \times \{\gamma \in C: \delta < \gamma\} \subseteqq U \cap D$. Let $\mu \in C$ such that $\mu < \omega_2$ and $\{\gamma \in C: \delta < \gamma < \mu\}$ is uncountable. Let $\Gamma = \{B \times \{\gamma\}: \gamma \leqq \mu\} \cup \{B \times \{\gamma \in C: \gamma > \mu\}\}$. Let $\Pi = \{V \cap (U \cap D): V \in \Gamma\}$. Then Π is an open over of $U \cap D$, no countable subfamily of which has dense union in $U \cap D$. This is a contradiction since $U \cap D$ is weakly Lindelöf.

The author wishes to express his gratitude to the referee for his constructive and thoughtful criticism.

REFERENCES

1. W. W. Comfort, Neil Hindman, and S. Negrepontis, *F′-spaces and their product with P-spaces*, Pacific J. Math., **28** (1969), 489-502.

2. Philip C. Curtis, Jr., *A note concerning certain product spaces*, Archiv der Math., **11** (1960), 50-52.

3. Leonard Gillman, *A P-space and an extremely disconnected space whose product is not an F-space*, Archiv der Math., **11** (1960), 53-55.

4. Leonard Gillman and Melvin Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc., **82** (1956), 366-391.

5. Leonard Gillman and Meyer Jerison, *Rings of Continuous Functions*, D. Van Nostrand Company, Inc., Princeton, N. J., 1960.
6. Neil Hindman, *On P-like Spaces and their Product with P-Spaces*, Thesis, Wesleyan University, 1969.
7. Carl W. Kohls, *Hereditary properties of some special spaces*, Archiv der Math., **12** (1960), 199-133.
8. S. Negrepontis, *On the product of F-spaces*, Trans. Amer. Math. Soc., **135** (1969), 443-457.

Received June 30, 1972.

CALIFORNIA STATE UNIVERSITY, LOS ANGELES