PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 1, 1972

OPERATOR-VALUED INNER FUNCTIONS ANALYTIC ON THE CLOSED DISC

STEPHEN L. CAMPBELL

It is shown that the class of operator valued inner functions analytic on the closed disc is sufficiently large for the invariant subspace problem. These inner functions are then transferred to the upper half-plane and studied via the differential equation $U' = iAU$. The relationship between A and U is investigated. Necessary and sufficient conditions are given on A for U to be the Potapov inner function of a normal operator.

1. First we establish our notation. H is a fixed separable, infinite dimensional Hilbert space. H_H^2 is the usual Hardy spaces of the circle or upper half-plane with values in H. Which H_H^2 will be clear from the context. We will always denote a real variable by x, variables in the disc by u or w, and variables in the upper half-plane by z. S denotes multiplication by w. S^* is the adjoint of S restricted to H_H^2. An operator-valued function $U(w)$, defined on the circle, is inner if $Uf \in H_H^2$ for every $f \in H_H^2$ and $U(w)$, $|w| = 1$, is unitary. U has an analytic extension into the unit disc which we shall identify with U. The map $z = i(1-w)/(1+w)$, $w = (z-i)/(z+i)$, transfers $U(w)$ to $U(z)$, an inner function on H_H^2 of the upper half-plane. U' will always denote differentiation with respect to z, or x if we are restricting U to the real axis. If T is a bounded operator on H, $r(T)$ is its spectral radius, $N(T)$ is its null space. If M, R are subspaces of a Hilbert space, then $M \ominus R = M \cap R^{\perp}$ and $\oplus$ is an orthogonal sum.

2. If T is an operator on H, $\|T\| \leqq 1$, and $T^n \to 0$ strongly, then associated to T is an inner function $V_T(w)$ called the Potapov inner function for T. If $K = H_H^2 \ominus V_T H_H^2$, then S^* restricted to K is similar to T. The question then of invariant subspaces for T reduces to finding invariant subspaces for S^* restricted to K. This is equivalent to whether V_T factors into two nonconstant inner functions. For a more complete discussion of these ideas see [6, 8].

3. Two special classes already proposed are the (IN)-operators of Herrero [7] and the scalar inner operators of Sherman [11]. Both of these classes are too restrictive for the invariant subspace problem. Sherman's calculations can be made to show V_T is scalar if and only if T is normal. In this case the invariant subspace problem

is solved. But if one is looking for invariant subspaces, no generality is lost in assuming $\|T\| \leqq 1$, $r(T) < 1$. Then $V_T(w)$ is analytic on the closed disc. We will see later that $\|V_T'(x)\|$ is then integrable and hence $V_T(w)$ is a finite Blaschke product if it is (IN) [2].

4. The preceding paragraph suggests a natural class of operator-valued inner functions, those which are analytic on the closed unit disc. We shall refer to these as analytic inner functions, or the class (AI). This includes all Potapov inner functions for T such that $r(T) < 1$, $\|T\| \leqq 1$, and hence is big enough for the invariant subspace problem. (AI) also has the advantage that one is not bothered by irregular behavior on the boundary of the disc.

A slightly more general class was originally studied by Helson [5]. He worked on the upper half-plane and utilized the differential equation $U'(x) = iA(x)U(x)$ that all norm-differentiable inner functions on the real axis satisfy. $A(x)$ is a self-adjoint, positive, norm-continuous operator-valued function. If U is norm-differentiable on the axis it is analytic on the real axis [5]. Some of his results were generalized in [1].

5. (AI) is only slightly larger than the class of Potapov inner functions as Theorem 2 will show. The motivation for Theorem 2 is contained in a theorem of Jackson [8, p. 29]. Our proof essentially follows here.

THEOREM 2. *If $U(w)$ is an analytic inner function such that $N(U_w(1)) = \{0\}$, then*

(1) *$U(w) = cV_T\tau U_0$ where U_0 is an arbitrary constant unitary operator, τ is an isometry onto the range of $(I - TT^*)^{1/2}$, c^* is an isometry onto the range of $(I - T^*T)^{1/2}$, $\|T\| \leqq 1$, $r(T) < 1$,*

(2) *$A(x) = cA_T(x)c^*$ and*

(3) *if $\|T\varphi\| < \|\varphi\|$ for all φ in H, then $U(w)$ differs from a Potapov inner function by a constant unitary operator on the right.*

Proof. Let $K = H_H^2 \ominus M$, where $M = UH_H^2$. Identify H with the constant functions in H_H^2.

(i) $M \cap H = \{0\}$. For if not, let $H_1 = M \cap H$. Then $H_{H_1}^2 \subseteqq M$ and $U = U_0 \oplus U_1$ where U_0 is constant on H_1. But then $N(U_w(1)) \supseteqq U^*(1)H_1$ which is a contradiction of the assumption that $N(U_w(1)) = \{0\}$.

(ii) K is infinite dimensional. We note that if $f \in K$, then f is analytic on the closed disc since U is [6, p. 76]. Thus $\langle f(w), \varphi\rangle = (f(0), \varphi)$ where $\langle , \rangle$ is the inner product in H_H^2 and $(,)$ is the inner product in H. If K were finite dimensional, there would exist φ in H such that $\varphi \perp K$, and this contradicts i.

(iii) Following [8], we let B: $H \to K$ be unitary and define T by $S^*B = BT$. Then $S^{*n}B = BT^n$ so that $B = B_0(I - wT)^{-1}$ and $(I - T^*T) = B_0^*B_0$. Thus $B_0 = c(I - T^*T)^{1/2}$ where c is a partial isometry with $N(c) = N(B_0) = N(I - T^*T)$. But by (ii), B_0 has dense range and hence c^* is an isometry onto $(I - T^*T)^{1/2}H$. Thus $UH_H^2 = K^\perp = cV_TH_H^2$. $r(S^*|K) < 1$ by [5, p. 76].

(iv) $V_T = -T^* + w(I - T^*T)^{1/2}(I - wT)^{-1}(I - TT^*)^{1/2}$. If φ is in $N(I - TT^*)$, then $V_T\varphi = -T^*\varphi$ and $cV_T\varphi = 0$. On the other hand, $V_T(w)$, $|w| = 1$, maps the closure of $(I - TT^*)^{1/2}H$ onto the closure of $(I - T^*T)^{1/2}H$. Let τ be as in the statement of the theorem. Then $cV_T\tau H_H^2 = cV_TH_H^2 = UH_H^2$. For $|w| = 1$, $cV_T(w)\tau$ is an isometry of H onto H and hence is an inner function. (1) follows.

(v) To prove (2) we calculate $A(x) = -iU'(x)U^*(x)$ and use the fact that $\tau\tau^*V_T^*c^* = V_T^*c^*$.

If $||T\varphi|| < ||\varphi||$ for all φ in H, then $N[(I - T^*T)^{1/2}] = \{0\}$ and c is unitary. The identity $cV_T = V_{cTc*}c$, valid for unitary c, implies (3).

The next theorem connects Helson's class of inner functions with ours.

THEOREM 3. *If $U(x)$ is a norm-differentiable inner function acting on H_H^2 of the upper half-plane and $U'(x) = iA(x)U(x)$, then $U(w)$ is analytic on the closed disc if and only if $\int_{-\infty}^{\infty} ||A(x)||\, dx < \infty$.*

Proof. $A(x)$ integrable gives us that $U(x)$ has a finite limit as x approaches infinity [3, p. 43]. But U is bounded and analytic on the upper half-plane, so it approaches this limit uniformly in the upper half-plane and in the lower-plane for sufficiently large $|z|$ [4, p. 162]. Thus U is in (AI).

To prove the converse, we observe that $U(w)$ is analytic on the closed disc so $U(z)$ is analytic at infinity. Differentiating the Laurent series for $U(z)$ we get that $||A(x)||$ is integrable.

6. The preceding sections suggest that it might be useful to find out the relationship between the properties of innner functions $U(x)$ and those of $A(x)$. This relationship has been studied in [1, 2, 5]. Unlike [2], we will not restrict our attention to (IN)-operators. [1] will be used for purposes of simplification. Our results are of a different nature then those in [5].

Some results are well known. If $V_T(w)$ is analytic on the closed disc, then there is a compact set σ of the open upper half-plane such that $A_T(x)$ has an analytic extension everywhere into the upper half-plane except on σ. σ is the image of the spectrum of T under the map $z = i(1-w)/(1+w)$. We also know that $N(A(x))$ is constant [1].

Our next result is

THEOREM 4. *If* $\|T\| \leqq 1$, $r(T) < 1$, *then*
(1) $A_T(x)$ *is one-to-one for all* x *if and only if* $(I - T^*T)$ *is and*
(2) $A_T(x)$ *is invertible for all* x *if and only if* $(I - T^*T)$ *is.*

Proof. We observe that the conditions on $(I - T^*T)$ occur if and only if they occur for $(I - T^*T)^{1/2}$. Secondly we note that $A_T(x)$ is one to one or invertible precisely when $V_T'(x)$ is. But up to a nonzero scalar function, $V_T'(x) = (I - T^*T)^{1/2}(I - wT)^{-1}(1 - wT)^{-1}(I - TT^*)^{1/2}$. $(I - TT^*)^{1/2}$ is invertible or one to one if and only if $(I - T^*T)^{1/2}$ is.

COROLLARY 1. $A_T(x)$ *is invertible for all* x *if and only if* $\|T\| < 1$.

Proof. One way is obvious. On the other hand if $(I - T^*T)$ is invertible, then $\inf_\varphi | \|\varphi\|^2 - \|T\varphi\|^2 | > 0$ and $\|T\| < 1$.

REMARK. The condition $r(T) < 1$ may be replaced by $T^n \longrightarrow 0$ strongly.

REMARK. The formula $V_T(w) = (I - T^*T)^{-1/2}(w - T^*)(I - wT)^{-1}(I - TT^*)^{1/2}$ which is used by some authors under the condition $T^n \longrightarrow 0$ strongly or $r(T) < 1$ is not valid for $(I - T^*T)^{1/2}$ may fail to exist. $T = \begin{bmatrix} 01 \\ 00 \end{bmatrix}$ is an example.

As Theorem 3 shows, the integrability of $A(x)$ is one of the analytically distinguishing characteristics of analytic inner functions. $A(x)$ can not be too small, however.

THEOREM 5. *Suppose* $U(x)$ *is a norm-differentiable inner function and* $U'(x) = iA(x)U(x)$. *Then* $U(x)$ *is identically constant if and only if* $\int_{-\infty}^{\infty} \|A(x)\| dx < 1$.

Proof. If U is constant, then clearly $A(x) = 0$. So assume $\int_{-\infty}^{\infty} \|A(x)\| dx = d < I$. Let $V(x) = U(x)U^*(0)$, so that $V'(x) = iA(x)V(x)$ and $V(0) = I$. E is the identity operator on H_H^2. V is $V(x)$ thought of as an operator on H_H^2. Now $\|V(x) - I\| \leqq \|\int_0^x V'(y)dy\| \leqq \int_0^x \|A(y)\| dy \leqq d$. Thus $\|V - E\| \leqq d < 1$ as operators in H_H^2. Hence V is invertible, that is, $VH_H^2 = H_H^2$ and V is constant. So U is constant.

This theorem can be used to test whether one norm-differentiable inner function can divide another.

COROLLARY 2. *If $U = VW$ where U, V, and W are inner operator functions and U is norm-differentiable, then $\int_{-\infty}^{\infty} \|A(x) - M(x)\| dx < 1$ implies W is constant where $V'(x) = iM(x)V(x)$ and $U'(x) = iA(x)U(x)$.*

Proof. Helson has shown that if U is norm-differentiable, then so are V and W [5, p. 318]. Let $W'(x) = iD(x)W(x)$. Then $A(x) = M(x) + V(x)D(x)V^*(x)$. $\int_{-\infty}^{\infty} \|D(x)\| dx = \int_{-\infty}^{\infty} \|A(x) - M(x)\| dx < 1$ implies that W is constant by Theorem 5.

7. We conclude with a theorem motivated by some results of Sherman [11].

THEOREM 6. *If $\|T\| < 1$, then T is normal if and only if $A_T(x)$ commutes with $A_T(y)$ for every real x, y.*

Proof. If T is normal, then clearly $V_T(x)$ and $V_T(y)$ commute for all x and y. Suppose that $V_T(x)V_T(y) = V_T(y)V_T(x)$. Then $V_T(x)V_T^*(-x) = V_T^*(-x)V_T(x)$. But $V_T^*(-x) = V_{T^*}(x)$. Thus V_T and V_{T^*} commute. Hence T is normal [11, p. 395]. It suffices then to show that $V_T(x)$ commutes with $V_T(y)$ for all x and y if and only if $A_T(x)$ commutes with $A_T(y)$ for all x and y.

Suppose that $V_T(x)V_T(y) = V_T(y)V_T(x)$. Then differentiation with respect to x and the commutivity of the V_T's enables us to get $A_T(x)V_T(y) = V_T(y)A_T(x)$. Differentiate this equation with respect to y and simplify to get $A_T(x)A_T(y) = A_T(y)A_T(x)$ as desired.

Now suppose that $A_T(x)A_T(y) = A_T(y)A_T(x)$. Multiply this equation on the right and left by $[I - T^*T]^{-1/2}$. Invert both sides to get

$$\begin{aligned}(uI - T^*)(I - uT)[I - T^*T]^{-1}(wI - T^*)(I - wT)\\ = (wI - T^*)(I - wT)[I - T^*T]^{-1}(uI - T^*)(I - uT)\,.\end{aligned}$$

Here $u = (i - y)/(i + y)$. Both sides are entire functions in u and w. Equating w coefficients gives $T^*[I - T^*T]^{-1}[I + T^*T] = [I - T^*T]^{-1}[I + T^*T]T^*$. But $\sigma(T^*T)$ is in $[0, 1]$. Hence $T^*(T^*T) = (T^*T)T^*$. The equation now becomes $(uI - T^*)(I - uT)(wI - T^*)(I - wT) = (wI - T^*)(I - wT)(uI - T^*)(I - uT)$. Equating w^2 coefficients gives $T^*T = TT^*$. Hence T is normal.

Theorem 6 is not valid without the assumption that the norm of T is less than one.

EXAMPLE. Let $T = \begin{bmatrix} 01 \\ 00 \end{bmatrix}$. Then $V_T(w) = \begin{bmatrix} 0 & w^2 \\ -1 & 0 \end{bmatrix}$ and $A_T(x) = (4/|(1 + x^2)) \begin{bmatrix} 10 \\ 00 \end{bmatrix}$. Thus $A_T(x)$ commutes with $A_T(y)$ for all x and y, but $V_T(x)$ does not commute with $V_T(y)$ unless $w^2 = u^2$.

8. With the exception of the example, we have assumed throughout this paper that H was infinite dimensional. This was done, in large part, to set up a theory with the invariant subspace problem in mind. The finite dimensional case has been considered by [6,9,10], though they were interested in different aspects of inner functions than considered here.

REFERENCES

1. Stephen L. Campbell, *Inner functions analytic at a point*, Illinois J. Math., to appear.
2. ———, *Norm-differentiable Inner Functions with Integrable Derivatives*, preprint, Marquette University.
3. Lamberto Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, Springer-Verlag (Academic Press), New York, 1963.
4. Earl A. Coddington and Norman Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
5. Henry Helson, *The differential equation of an inner function*, Studia Math., **35** (1970), 311-321.
6. ———, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.
7. Domingo Antonio Herrero, *Inner Function-Operators*, preprint, University of Chicago.
8. Shirley L. Jackson, *Operators and Inner Functions*, Dissertation, University of Calif., Berkeley, 1967.
9. V. P. Potapov, *The multiplicative structure of J-contractive matrix functions*, Amer. Math. Soc. Transl., **15** (1960), 131-243.
10. Malcolm J. Sherman, *Operators and inner functions*, Pacific J. Math., **22** (1967), 159-170.
11. ———, *A spectral theory for inner functions*, Trans. Amer. Math. Soc., **135** (1969), 387-398.

Received January 29, 1971. This research was done as part of a doctoral thesis being prepared for Northwestern University, under the supervision of Professor Allen Devinatz.

MARQUETTE UNIVERSITY