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A CLASS OF OPERATORS ON HILBERT SPACE

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If T is an operator (bounded endormorphism) on the complex Hilbert space H, then $T \in \mathscr{R}$ if and only if $\|(T - zI)^{-1}\| = 1/d(z, W(T))$ for all $z \notin \mathrm{Cl}\ W(T)$, where Cl $W(T)$ is the closure of the numerical range of T and $d(z, W(T)) = \inf\{|z - u|: u \in W(T)\}$. The main results of this paper are: (1) $T \in \mathscr{R}$ if and only if the boundary of the numerical range of T is a subset of $\sigma(T)$, the spectrum of T; and (2) $\mathscr{R}$ is an arc-wise connected, closed nowhere dense subset of the set of all operators on H (norm topology) when dim $H \geqq 2$.

Introduction. If T is an operator (bounded endomorphism) on the complex Hilbert space H, then

$$1/d(z, \sigma(T)) \leqq \|(T - zI)^{-1}\| \text{ and } \|(T - zI)^{-1}\| \leqq 1/d(z, W(T)),$$

where the first inequality holds for all $z \notin \sigma(T)$, the spectrum of T; and the second inequality holds for all $z \notin \mathrm{Cl}\ W(T)$, the closure of the numerical range of T [3]. Here $d(z, S)$ denotes the distance from z to the set S. In the current literature, much space has been devoted to the study of those operators T such that $\|(T - zI)^{-1}\|$ is equal to its smallest possible value for all $z \notin \sigma(T)$. In this paper, the properties of operators T with $\|(T - zI)^{-1}\|$ equal to its largest possible value, for all $z \notin \mathrm{Cl}\ W(T)$, are investigated. Let $\mathscr{R}$ denote this set of operators.

We first characterize the operators in $\mathscr{R}$ in terms of the boundary of the numerical range and spectrum of the operator. If S is a set of complex numbers, then let co S denote the convex hull of S, let ∂S denote the boundary of S, and let Cl S denote the closure of S. For $z \notin \sigma(T)$, let $R(T, z) = (T - zI)^{-1}$.

THEOREM 1. *$T \in \mathscr{R}$ if and only if $\partial W(T) \subseteqq \sigma(T)$.*

Proof. First suppose that $\partial W(T) \subseteqq \sigma(T)$. Let $z \notin \mathrm{Cl}\ W(T)$. Then $d(z, \sigma(T)) = d(z, W(T))$ so that

$$1/d(z, W(T)) = 1/d(z, \sigma(T)) \leqq \|R(T, z)\| \leqq 1/d(z, W(T)).$$

Therefore $T \in \mathscr{R}$.

Suppose $T \in \mathscr{R}$ and let $z_0 \in \partial W(T)$. Then there exists a sequence $\{z_n\}$ approaching z_0 such that $|z_n - z_0| = d(z_n, W(T)) > 0$ for all $n = 1, 2, 3, \cdots$. Then $\|R(T, z_n)\| = 1/d(z_n, W(T)) \to \infty$. Therefore $z_0 \in \sigma(T)$.

COROLLARY 1. *If* $T \in \mathscr{R}$, *then* $\operatorname{co} \sigma(T) = \operatorname{Cl} W(T)$.

COROLLARY 2. *If* $T \in \mathscr{R}$ *and* $\sigma(T)$ *is a finite set, then there exists a complex number* α *such that* $T = \alpha I$.

Corollary 1 follows immediately from Theorem 1. To prove Corollary 2 we do the following: Since $W(T)$ is a convex set, $\partial W(T) \subseteqq \sigma(T)$ implies that the finite set $\sigma(T)$ must contain exactly one point, say α. Then $\sigma(T) = W(T) = \{\alpha\}$ so that $((T - \alpha I)x, x) = 0$ for all $x \in H$. Therefore $T = \alpha I$.

It follows from Corollary 2 that $\mathscr{R}$ is just the set of all scalar multiples of the identity operator when $\dim H < \infty$.

To see that all operators satisfying $\operatorname{co} \sigma(T) = \operatorname{Cl} W(T)$ are not in $\mathscr{R}$, simply let N be a normal operator whose spectrum is a finite set with more than one point. Then by [3, problem 171], $\operatorname{co} \sigma(N) = \operatorname{Cl} W(N)$, and by Corollary 2, $N \notin \mathscr{R}$.

Let $B(H)$ denote the set of all operators on the complex Hilbert space H and give $B(H)$ the norm topology.

THEOREM 2. $\mathscr{R}$ *is an arc-wise connected, closed, nowhere dense subset of* $B(H)$ *when* $\dim H \geqq 2$.

Proof. If $T \in \mathscr{R}$, then $\alpha T \in \mathscr{R}$ for every complex number α. Therefore the ray through T in $B(H)$ is contained in $\mathscr{R}$, and thus $\mathscr{R}$ is arc-wise connected.

To see that $\mathscr{R}$ is closed, we let $\{T_n\}$ be a sequence of operators in $\mathscr{R}$ approaching $T \in B(H)$ in norm. Then $W(T_n) \to W(T)$ in the Hausdorff metric [3, p. 176]. Let $z \notin \operatorname{Cl} W(T)$. Then there exists a positive integer N such that for all $n \geqq N$, $z \notin W(T_n)$. Then

$$\| R(T_n, z) \| = 1/d(z, W(T_n)) \longrightarrow 1/d(z, W(T)) .$$

Now choose $M \geqq N$ so that for all $n \geqq M$, $\| (T - T_n) R(T_n, z) \| < 1$. Then [2, p. 52]

$$R(T, z) = (I - (T - T_n) R(T_n, z))^{-1} R(T_n, z) .$$

Since $(T - T_n) R(T_n, z) \to 0$ as $n \to \infty$,

$$\| R(T, z) \| = \lim_{n \to \infty} \| R(T_n, z) \| = 1/d(z, W(T)) .$$

Therefore $T \in \mathscr{R}$ and hence $\mathscr{R}$ is closed.

If $\mathscr{C}$ is the set of all $T \in B(H)$ such that $\operatorname{co} \sigma(T) = \operatorname{Cl} W(T)$, then by Corollary 1 $\mathscr{R} \subseteqq \mathscr{C}$. By [4] $\mathscr{C}$ is a nowhere dense subset of $B(H)$ when $\dim H \geqq 2$. Therefore $\mathscr{R}$ is also a nowhere dense subset of $B(H)$ when $\dim H \geqq 2$.

The set of all operators on H ($\dim H = \infty$) that satisfy property (G_1) locally is a larger set than the set of all operators on H that satisfy property (G_1) [see 5]. This situation does not occur for $\mathscr{R}$. To see this, suppose T is an operator such that

$$\|R(T, z)\| = 1/d(z, W(T))$$

for all $z \in U - (\mathrm{Cl}\ W(T))$ where U is an open set containing $\mathrm{Cl}\ W(T)$. We now show that $T \in \mathscr{R}$, i.e., the above relationship holds for all $z \notin \mathrm{Cl}\ W(T)$. Let $z_0 \in \partial W(T)$. Then there exists a sequence $\{z_n\} \subseteqq U - (\mathrm{Cl}\ W(T))$ such that $z_n \to z_0$ and $|z_n - z_0| = d(z_n, W(T))$. Then

$$\|R(T, z_n)\| = 1/d(z_n, W(T)) \longrightarrow \infty .$$

Thus $z_0 \in \sigma(T)$. Therefore, by Theorem 1, $T \in \mathscr{R}$.

We now give a method to construct nontrivial examples of operators in $\mathscr{R}$.

THEOREM 3. *If A is an operator on H, then $A \oplus N \in \mathscr{R}$ on $H \oplus K$ whenever N is a normal operator on K with $\sigma(N) \supseteqq \partial W(A)$.*

Proof. Let N be as above and let $T = A \oplus N$. Since $W(A) \subseteqq W(N)$,

$$W(T) = \mathrm{co}\,(W(A) \cup W(N)) = W(N) .$$

For $z \notin \sigma(T) = \sigma(A) \cup \sigma(N)$,

$$\|R(N, z)\| = 1/d(z, \sigma(N)) = 1/d(z, W(T)) .$$

Thus, since $\|R(A, z)\| \leqq 1/d(z, W(A)) \leqq 1/d(z, W(T))$,

$$\begin{aligned} \|R(T, z)\| &= \mathrm{Max}\,\{\|R(A, z)\|, \|R(N, z)\|\} \\ &= \mathrm{Max}\,\{\|R(A, z)\|, 1/d(z, W(T))\} \\ &= 1/d(z, W(T)) . \end{aligned}$$

Therefore $T \in \mathscr{R}$.

There are a number of nice properties that not all operators in $\mathscr{R}$ enjoy. Let $R_{sp}(T)$ denote the spectral radius of T.

THEOREM 4. *There exists $T \in \mathscr{R}$ such that*

(i) $T^2 \notin \mathscr{R}$,

(ii) $R_{sp}(T) < \|T\|$, *and*

(iii) $T^{-1} \notin \mathscr{R}$.

Proof. Let $A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$, and let N be a normal operator with $\sigma(N) = W(A)$. Let $T = A \oplus N$. Then by Theorem 3, $T \in \mathscr{R}$. By [1] $W(A)$ is the closed disc of radius 1/2 about $z = 1$, and $W(A^2)$ is the closed disc of radius 1 about $z = 1$. Therefore

$$0 \in W(A^2) \subseteqq W(T^2) \text{ and } 0 \notin \operatorname{co}(\sigma(T)^2) = \operatorname{co}\sigma(T^2) .$$

Therefore, $\operatorname{co}\sigma(T^2) \neq \operatorname{Cl} W(T^2)$ and so $T^2 \notin \mathscr{R}$. A computation yields $\|T\| = (3/2 + \sqrt{5/2})^{1/2}$. Thus $\|T\| > 3/2 = R_{sp}(T)$. If T^{-1} were in $\mathscr{R}$, then

$$\|T\| = \|R(T^{-1}, 0)\| = 1/d(0, W(T^{-1})) .$$

But $1/d(0, W(T^{-1})) = 2 > \|T\|$. Therefore $T^{-1} \notin \mathscr{R}$.

References

1. William F. Donoghue, *On the Numerical Range of a Bounded Operator*, Mich. Math. J., **4** (1957), 261-263.
2. P. R. Halmos, *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, Chelsea Publishing Company, New York, 1957.
3. ———, *A Hilbert Space Problem Book*, Van Nostrand Company, Princeton, New Jersey, 1967.
4. Glenn R. Luecke, *Topological Properties of Paranormal Operators on Hilbert Space*, (to appear).
5. ———, *Operators Satisfying Condition* (G_1) *Locally*, (to appear).

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