PERTURBATIONS OF SPECTRAL OPERATORS, AND APPLICATIONS

I. BOUNDED PERTURBATIONS

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1. Introduction. A principal theorem on self-adjoint boundary-value problems is the existence of a complete orthonormal set of eigenfunctions. This corresponds to the diagonal reduction of a hermitian matrix, and to the spectral theorem for self-adjoint operators in Hilbert space. How much remains true if we drop the fundamental condition of self-adjointness? Infinite dimensional examples show that, in general, we cannot expect even the existence of a single eigenvector.

Nevertheless, there does exist a class of operators which behave in a "regular" fashion from this spectral theoretic point of view, namely, the spectral operators introduced in [**4**, p. 560]. The paper [**4**], while extensively developing the theory of these operators, still leaves open a very significant question. Are many (or any) of the nonsymmetric integral, differential, and so on, operators arising in the more "classical" branches of analysis spectral? The main result of the present paper is a positive answer to the foregoing question.

The principal indication that a positive answer is to be expected comes from a classical series of papers [**1**; **2**; **3**; **11**; **13**;], in which it is demonstrated that for certain general types of boundary-value problems involving nonsymmetric linear differential operators, expansions in eigenfunctions exist and converge in much the same way as ordinary Fourier series. The method in all of these papers is "analytic;" that is, it operates with asymptotic estimates of the solutions of the various differential equations and of the partial sums of the various series arising. The method in the present paper is abstract, and is phrased in terms of Banach spaces, linear operators, and so on. This has the advantage of greater simplicity in proof, and greater generality in applications. For instance, we shall be able to prove results on certain types of partial differential operators which appear difficult to prove by an analytic method.

The general idea of our abstract method is the following. Let T be a spectral

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operator. Let B be an operator which is, in some sense, small relative to T. Then $T+B$ will be a spectral operator. A less stringent restriction on B will yield a weaker conclusion on the spectral nature of $T+B$. In particular, there are many cases in which it can be asserted that the set of generalized eigenvectors of $T+B$ spans our Banach space, but not that $T+B$ is spectral.

2. Preliminaries. Let $\mathfrak{X}$ be a (complex) reflexive Banach space. A bounded operator in $\mathfrak{X}$ is an everywhere-defined continuous linear mapping of $\mathfrak{X}$ into itself. An unbounded operator is a linear mapping of a dense linear subspace of $\mathfrak{X}$ into $\mathfrak{X}$. The set on which the operator T is defined is its domain, denoted by $\mathfrak{D}(T)$. The open set of λ in the complex plane, for which

$$(T-\lambda I)^{-1} = (T-\lambda)^{-1}$$

is everywhere defined and bounded, is the resolvent of T. Its closed complement, which is bounded for bounded operators, is the spectrum $\sigma(T)$ of T.

DEFINITION 1. An operator T is *regular* if its spectrum $\sigma(T)$ is not the entire complex plane, and if $(T-\lambda)^{-1}$ is compact for some $\lambda \notin \sigma(T)$.

REMARK. Except in the trivial case where $\mathfrak{X}$ is finite dimensional, a regular T cannot be bounded. For, if T is bounded,

$$I = (T-\lambda)(T-\lambda)^{-1}$$

is compact; and this implies immediately that $\mathfrak{X}$ is finite dimensional.

LEMMA 1. *If T is regular, then:*

(a) *Its spectrum is a denumerable set of points with no finite limit point.*

(b) *$(T-\lambda)^{-1}$ is compact for every $\lambda \notin \sigma(T)$.*

(c) *Every $\lambda_0 \in \sigma(T)$ is a pole of finite order $\nu(\lambda_0)$ of the resolvent $R_\lambda = (T-\lambda)^{-1}$. If a vector f satisfies*

$$(T-\lambda_0)^k f = 0,$$

then f satisfies

$$(T-\lambda_0)^{\nu(\lambda_0)} f = 0.$$

The set of all such vectors f makes up a finite dimensional linear space, called the space of generalized eigenvectors of T corresponding to the eigenvalue λ_0. If $E(\lambda_0)$ is the idempotent function of T corresponding to the analytic function *which is one on λ_0 and zero elsewhere on the spectrum of T, then $E(\lambda_0)$ projects $\mathfrak{X}$ onto the space of generalized eigenvectors corresponding to λ_0.*

Proof. We can suppose, without loss of generality, that $0 \notin \sigma(T)$, and that T^{-1} is compact. If we then make use of the identity

$$\lambda(T^{-1} - \lambda)^{-1} T^{-1} = (\lambda^{-1} - T)^{-1},$$

parts (a), (b), and the first statement in (c), of our result follow readily from the corresponding statements in the ordinary Fredholm theory of compact operators. (For this theory, see, for instance, [**7**, Chap. VII.].) We have

$$(T - \lambda_0)^k f = 0$$

if and only if

$$(\lambda_0^{-1} - T^{-1})^k f = 0,$$

so that the second and third parts of the lemma also follow by a simple application of the corresponding result for compact operators.

To prove the last part of the lemma, we may argue as follows: If C is a small closed curve surrounding the point λ_0 and traversed once in the positive sense, then by definition

$$E(\lambda_0) = \frac{1}{2\pi i}\int_C (\lambda - T)^{-1} d\lambda$$

$$= \frac{1}{2\pi i}\int_C T^{-1}(T^{-1} - \lambda)^{-1}\lambda^{-1} d\lambda$$

$$= \frac{1}{2\pi i}\int_{C'} \mu^{-1} T^{-1}(\mu - T^{-1})^{-1} d\mu,$$

where C' is a small curve surrounding λ_0^{-1}, and traversed in the positive sense. This last integral can easily be evaluated in terms of the functional calculus for bounded operators (cf [**4**]), and turns out to be the idempotent analytic function $\overline{E}(\lambda_0^{-1})$ of T^{-1} corresponding to the analytic function which is one on λ_0^{-1} and zero elsewhere on $\sigma(T^{-1})$, and now the desired result for T follows readily from the corresponding result for T^{-1}.

REMARK. It is to be noted that we have actually proved a little more than is stated in Lemma 1. We have, in fact, proved that the points of $\sigma(T)$ and the nonzero points of $\sigma(T^{-1})$ are in one-to-one correspondence through the map

$$\lambda \leftrightarrow \lambda^{-1},$$

and, that if we call $E(\lambda_0)(\overline{E}(\lambda_0))$ the spectral measure of the point λ_0 corresponding to the operator T (the operator T^{-1}), then

$$E(\lambda_0) = \overline{E}(\lambda_0^{-1}).$$

This result is, of course, merely a particular case of the "unbounded" analogue of the general "Spectral Mapping Theorem" of Dunford [**4**].

Now, by [**6**, Theorem 20], it follows that if S is a compact spectral operator, and $E(e)$ is its spectral resolution, then $E(\lambda_0)$ is the projection associated above with the point λ_0 (for $\lambda_0 \in \sigma(S)$; for $\lambda_0 \notin \sigma(S)$, $E(\lambda_0) = 0$). Conversely if S is a compact operator, and $E(\lambda_0)$ is the spectral measure of the point λ_0, then S is spectral if and only if there is a uniform bound for all sums $\sum_{i=1}^{k} E(\lambda_i)$ taken over finite subsets $\lambda_1, \lambda_i, \cdots, \lambda_k$ of $\sigma(S)$; that is, if and only if the various projections $E(\lambda_0)$, $\lambda_0 \in \sigma(T)$, generate a uniformly bounded Boolean algebra of projections. We can carry this result over to unbounded operators in a trivial way, making use of the following:

LEMMA 2. *Let T be a regular unbounded operator.*

(a) *If $(\lambda_0 - T)^{-1}$ is spectral for some $\lambda_0 \notin \sigma(T)$, then $(\lambda - T)^{-1}$ is spectral for all $\lambda \notin \sigma(T)$. In this case we say that T is an unbounded spectral operator.*

(b) *The regular operator T is spectral if and only if the spectral measures $E(\lambda_0)$ of the various points $\lambda_0 \in \sigma(T)$ generate a uniformly bounded Boolean algebra.*

Proof. Suppose that T^{-1} is spectral. Then the spectral measures $\overline{E}(\lambda_0)$ of the points $\lambda_0 \in \sigma(T^{-1})$ generate a uniformly bounded Boolean algebra. Since

$$\overline{E}(\lambda_0^{-1}) = E(\lambda_0),$$

the projections $E(\lambda_0)$ generate a uniformly bounded Boolean algebra. The converse argument to this argument evidently goes through. Moreover, since the spectral measure $E_1(\lambda_0)$ corresponding to the operator $T + c$ is evidently $E(\lambda_0 - c)$, it is evident that T has the property of part (b) if and only if $T + c$ does. But this immediately implies part (a).

3. Bounded perturbations. We now come to the main point of the paper.

THEOREM 1. *Let T be a regular spectral operator, and suppose that λ_n is an enumeration of its spectrum. Let d_n denote the distance from λ_n to the rest of* *the spectrum. Suppose that for all but a finite number of* n, $E(\lambda_n)$ *projects onto a one dimensional subspace; suppose that*

$$\sum_{i=1}^{\infty} E(\lambda_i) = I.^{1}$$

Let B *be a bounded operator.*

(a) *If* $\sum_{n=1}^{\infty} d_n^{-1} < \infty$, *then* $T + B$ *is spectral.*

(b) *If* $\mathfrak{X}$ *is Hilbert space and* T *is normal, and* $\sum_{n=1}^{\infty} d_n^{-2} < \infty$, *then* $T + B$ *is spectral.*[2]

Proof. Put $R_\lambda = (\lambda - T)^{-1}$ for $\lambda \notin \sigma(T)$. Then we have

$$(1) \qquad (\lambda - T - B)^{-1} = (I - R_\lambda B)^{-1} R_\lambda,$$

whenever $(I - R_\lambda B)^{-1}$ exists. Now, by Lemma 3 below, there exists a constant $K > 0$ such that

$$|R_\lambda| \le K[\operatorname{dist}(\lambda, \sigma(T))]^{-1}.$$

Hence no λ at a greater distance than KB from the spectrum of T is in the spectrum of $T + B$, since, for such λ, $|R_\lambda B| < 1$. It follows also that $T + B$ is regular.

From (1) it follows that

$$\overline{R}_\lambda = (\lambda - T - B)^{-1} = \{I + R_\lambda B(I - R_\lambda B)^{-1}\} R_\lambda$$

$$= R_\lambda + R_\lambda B(I - R_\lambda B)^{-1} R_\lambda.$$

That is,

$$\overline{R}_\lambda - R_\lambda = R_\lambda B(I - R_\lambda B)^{-1} R_\lambda.$$

Let C_n be a circle about λ_n of radius $d_{n/2}$. Then, for $\lambda \in C_n$, we have $|R_\lambda| \le 2Kd_n^{-1}$, and thus when n is large enough to ensure $2Kd_n^{-1} < 1$, we have

$$|(I - R_\lambda B)^{-1}| \le (1 - 2Kd_n^{-1})^{-1}.$$

Since $d_n \longrightarrow \infty$, we may replace this estimate, at least for all but a finite number

[1] The series $\sum_{i=1}^{\infty} E(\lambda_i)$ converges in the strong operator topology.

[2] Of course, $T + B$ is also regular. This is proved in the course of the following argument; but c.f. also Lemma 17 below.

of C_n, by

$$|(I - R_\lambda B)^{-1}| \leq 2.$$

It then follows that

$$|\overline{R}_\lambda - R_\lambda| \leq 8K^2 |B| d_n^{-2}.$$

If we integrate this inequality around C_n in the positive sense, we obtain the inequality

$$|E(\lambda_n) - E_n| \leq 8K^2 |B| d_n^{-1},$$

where $E(\lambda_n)$ is the spectral measure of λ_n corresponding to the operator T, and where E_n is the sum of the spectral measures $E'(\lambda)$ corresponding to $T + B$ of the points λ of the $\sigma(T + B)$ lying within C_n.

Lemma 4 below then implies that for n sufficiently large, E_n has a one-dimensional range. It follows immediately that there must be exactly one point λ'_n of $\sigma(T + B)$ in C_n, and that $E_n = E'(\lambda'_n)$. That is,

$$|E(\lambda_n) - E'(\lambda'_n)| \leq 8K^2 |B| d_n^{-1}$$

for all but a finite number of n. From the above, case (a) of our theorem follows immediately.

To prove case (b), we have only to refine our estimates slightly. We have, from (1),

$$\overline{R}_\lambda = \{I + R_\lambda B + (R_\lambda B)^2 (I - R_\lambda B)^{-1}\} R_\lambda.$$

We then obtain the expression

$$\overline{R}_\lambda - R_\lambda - R_\lambda B R_\lambda = (R_\lambda B)^2 (I - R_\lambda B)^{-1} R_\lambda.$$

so that for $\lambda \in C_n$, and n sufficiently large,

$$|\overline{R}_\lambda - R_\lambda - R_\lambda B R_\lambda| \leq 16K^3 |B|^2 d_n^{-3}.$$

The question now is, what is the integrated form of this inequality? The only problem is to find

$$F_n = \frac{1}{2\pi i} \int_{C_n} R_\lambda B R_\lambda d\lambda,$$

and this is easily done.

Indeed, R_λ has the Laurent expansion

$$R_\lambda = (\lambda - \lambda_n)^{-1} E(\lambda_n) + R^0(\lambda_n) + c_1(\lambda - \lambda_n) + \cdots$$

around λ_n. In this expression $R^0(\lambda_n)$ is a "partial resolvent" of T; that is, we have

$$R^0(\lambda_n) = \lim_{\lambda \to \lambda_n} (I - E(\lambda_n)) R_\lambda .$$

Thus, $R^0(\lambda_n)$ is that analytic function of T which corresponds to the analytic function $f(z)$ which is equal to $(z - \lambda_n)^{-1}$ everywhere on $\sigma(T)$ but in the immediate neighborhood of λ_n, where we put $f(z) = 0$. In terms of this Laurent expansion, we readily find that

$$F_n = E(\lambda_n) B R^0(\lambda_n) + R^0(\lambda_n) B E(\lambda_n).$$

Having majorized

$$| E'(\lambda_n') - E(\lambda_n) - F_n |$$

by the terms $16K^3 |B|^2 d_n^{-2}$ of an absolutely convergent series, we have only to prove that a uniform bound exists for finite sums $\sum_{i=1}^R F_{n_i}$ of the terms F_n. Since a term of the form $E(\lambda_n) B R^0(\lambda_n)$ can be treated as an adjoint of a term of the form $R^0(\lambda_n) B E(\lambda_n)$, we have only to show that a uniform bound exists for finite sums

$$\sum_{i=1}^{R} R^0(\lambda_{n_i}) B E(\lambda_{n_i})$$

of these latter terms. It follows from Lemma 3$'$ below that a constant K' exists such that

$$| R^0(\lambda_n) | \le K' d_n^{-1}.$$

Thus

$$\left| \sum_{i=1}^{l} R^0(\lambda_{n_i}) B E(\lambda_{n_i}) f \right| \le |B| K' \sum_{i=1}^{l} d_{n_i}^{-1} | E(\lambda_{n_i}) f |$$

$$\leq |B|K'\left\{\sum_{i=1}^{l} d_{n_i}^{-2}\right\}^{1/2}\left\{\sum_{i=1}^{l} |E(\lambda_{n_i})f|^2\right\}^{1/2} \leq |B|K'\left\{\sum_{i=1}^{\infty} d_i^{-2}\right\}|f|,$$

since the normality of T implies that the projections $E(\lambda_i)$ are orthogonal perpendicular projections in the Hilbert space $\mathfrak{X}$. Thus both parts of our theorem are proved.

Before continuing with the main line of our discussion, we shall state and prove the lemmas referred to in the foregoing proof.

Lemma 3, below, depends on the functional calculus for our unbounded operators; before proceeding to the proof of this lemma, we must discuss the functional calculus. We consider a regular unbounded operator S with a denumerable spectrum $\{\lambda_n\}$. We shall allow a finite set $\lambda_1, \lambda_2, \cdots, \lambda_N$ of the eigenvalues to be multiple poles of the resolvent, but shall require that all the remaining eigenvalues are simple poles of the resolvent. In addition, we require that

$$\sum_{i=1}^{\infty} E(\lambda_i) = I.$$

In this situation, we can set up the functional calculus for T by setting

$$f(T) = \sum_{i=1}^{N} \sum_{j=0}^{\nu(\lambda_i)} \frac{f^{(j)}(\lambda_i)}{j!} (T - \lambda_i)^j E(\lambda_i) + \sum_{j=N+1}^{\infty} f(\lambda_i)E(\lambda_i)$$

for every function f which is uniformly bounded on the spectrum $\sigma(S)$ and which belongs to the class $C^{\nu(\lambda_i)}$ near the spectral point λ_i ($1 \leq i \leq N$). It may be remarked that, here and in all that follows, the finite number of multiple poles $\lambda_1, \lambda_2, \cdots, \lambda_N$ of the resolvent function $(\lambda - S)^{-1}$ contribute only a finite number of terms, whose influence on any of our arguments it will be trivial to determine by inspection. Thus, to avoid notational complications, we shall assume, without loss of generality, that all the λ_i are simple poles of the resolvent; that is, that $N = 0$. In this case, our proposed expression for the functional calculus is

$$f(T) = \sum_{i=1}^{\infty} f(\lambda_i)E(\lambda_i),$$

where $f(\lambda)$ is any function uniformly bounded on the spectrum.

Functional calculi of this sort are discussed in [**6**], in a much more general situation. In particular, it follows from [**6**, Lemma 6] that the series defining $f(T)$ converges in the strong topology, and that there exists an absolute constant $K = K(T)$ such that we have

$$|f(T)| \leq K \cdot \max_{\lambda \in \sigma(T)} |f(\lambda)|.$$

From this fact, we have:

LEMMA 3. *If S is a regular spectral operator all but a finite set of whose eigenvalues λ_n are simple poles of the resolvent, and S also satisfies*

$$\sum_{i=1}^{\infty} E(\lambda_i) = I,$$

then there exists an absolute constant K such that

$$|(\lambda - S)^{-1}| \leq K \operatorname{dist}(\lambda, \sigma(S))^{-1}$$

for all λ not within a fixed radius ϵ of any multiple pole of the resolvent.

Lemma 3′ involves the operator $R^0(\lambda_n)$ defined as the constant term in the Laurent expansion

$$(\lambda - S)^{-1} = \frac{E(\lambda_n)}{\lambda - \lambda_n} + R^0(\lambda_n) + \cdots$$

of the resolvent function around λ_n. Since

$$(\lambda - S)^{-1} = \frac{E(\lambda_n)}{\lambda - \lambda_n} + \sum_{i \neq n} (\lambda - \lambda_i)^{-1} E(\lambda_i),$$

it is evident that

$$R^0(\lambda_n) = \sum_{i \neq n} (\lambda_n - \lambda_i)^{-1} E(\lambda_i). \tag{2}$$

We obtain, as an immediate consequence of this formula:

LEMMA 3′. *If S is a regular spectral operator having the properties described in Lemma 3, then there exists an absolute constant K' such that if $\lambda_n \in \sigma(S)$ and*

$$d_n = \min_{i \neq n} \operatorname{dist}(\lambda_n, \lambda_i),$$

then for the operator $R^0(\lambda_n)$ defined by formula (2) we have

$$|R^0(\lambda_n)| \leq K' d_n^{-1}.$$

LEMMA 4.[3] *Let E be a projection of $\mathfrak{X}$ onto an n-dimensional space. E' is a projection in $\mathfrak{X}$ satisfying*

$$|E - E'| = \frac{1}{2}|E|^{-1},$$

then E' also projects $\mathfrak{X}$ onto an n-dimensional space.

Proof. We have

$$|E - EE'| < \frac{1}{2}|E||E|^{-1} < 1$$

and

$$|E'| \leq |E| + \frac{1}{2}|E|^{-1} < 2|E|,$$

so that

$$|E'E - E'| < 2|E| \cdot \frac{1}{2}|E|^{-1} = 1.$$

If we then consider EE' as a mapping of $E(\mathfrak{X})$ into itself, it follows that EE' has an inverse. Thus E' maps $\mathfrak{X}$ onto a space of dimension n at least. Applying the same argument to $E'E$, we see that E' maps $\mathfrak{X}$ onto a space of dimension n at most. It follows that the dimension of $E'(\mathfrak{X})$ is exactly n.

Part (b) of Theorem 1 is capable of some improvement. Inspection of the proof of this result reveals that the only thing essential is that the spectral measures $E(\lambda_i)$ should be orthogonal projections. But, by a theorem of Lorch and Mackey (proved in [**17**]), any uniformly bounded Boolean algebra $\{E\}$ of projections in Hilbert space can be reduced to a Boolean algebra of orthogonal projections by an inner automorphism

$$E \longrightarrow D^{-1}ED,$$

where D is a bounded operator in Hilbert space with a bounded inverse. Since such an inner automorphism evidently preserves all operator theoretic properties of the sort involved in our proof, we may state:

Corollary 1b′.[4] *If T is a regular spectral operator in Hilbert space, if all but*

[3] A similar lemma is found in [**18**, remark after Corollary 2.5].

[4] This improvement of Theorem 1b was pointed out to the author in conversation with N. Dunford.

a finite number of its eigenvalues λ_n are simple poles of the resolvent and correspond to one-dimensional eigenspaces, if

$$\sum_{i=1}^{\infty} E(\lambda_i) = I,$$

and if, putting

$$d_n = \min_{i \neq n} \operatorname{dist}(\lambda_n, \lambda_i),$$

we have $\sum d_n^{-2} < \infty$, then $T + B$ is a spectral operator for any bounded B.

4. Two counterexamples. It would be useful to be able to prove Theorem 1 without the restriction to simple eigenvalues. Unfortunately, the appropriate generalization is not true, even if the eigenvalues are restricted to be simple poles of the resolvent, and even if the eigenvalues go to infinity very rapidly The following example shows this to be the case:

EXAMPLE 1. We take two infinite sequences ϕ_n^+ and ϕ_n^- of vectors to be, together, an orthonormal basis for Hilbert space $\mathfrak{X}$. We let T be the self-adjoint unbounded operator defined by

$$T\phi_n^+ = n!\phi_n^+, \quad T\phi_n^- = n!\phi_n^-.$$

Then $\lambda_n = n!$ is a simple pole of the resolvent, but a double eigenvalue. We then let B be the compact operator defined by

$$B\phi_n^+ = (n!)^{-1}\phi_n^+, \quad B\phi_n^- = \{(n-1)!\}^{-1}\phi_n^+.$$

It may be noted that if we realize $\mathfrak{X}$ as a space of L_2 functions, taking

$$\phi_n^+ = \cos 2\pi nx, \quad \phi_n^- = \sin 2\pi nx,$$

say, then B is an operator defined as an integral transform with an analytic kernel. At any rate, this peturbation breaks up the double eigenvalue $n!$ into two single eigenvalues $n!$ and $n! + (n!)^{-1}$, with the corresponding eigenfunctions $n\phi_n^+ - \phi_n^-$ and ϕ_n^+. A brief calculation shows that the corresponding projections $E(n!)$ are defined by

$$E(n!)\phi_j^{\pm} = 0 \text{ for } n \neq j,$$

$$E(n!)\phi_n^{+} = 0,$$

$$E(n!)\phi_n^{-} = \phi_n^{-} - n\phi_n^{+}.$$

Thus, the spectral measures of the points in the spectrum of $T + B$ are not uniformly bounded, so that $T + B$ is surely not spectral.

This example also indicates that the spectral property of $T + B$ fails because we do not group the two projections arising out of the double eigenvalues of T together in forming our spectral sums. We shall see later that this is very typical behavior.

In view of the importance for our proof of the property described in Lemma 3, we shall give an example which shows it to fail if we allow regular operators with an infinity of double poles of the resolvent. This is:

EXAMPLE 2. We introduce an orthonormal basis for Hilbert space $\mathfrak{X}$ consisting of two infinite sequences of vectors ϕ_n^{+}, ϕ_n^{-}, as in Example 1. We let T be the smallest closed operator satisfying

$$T\phi_n^{+} = n^2\phi_n^{+} + n\phi_n^{-}, \quad T\phi_n^{-} = n^2\phi_n^{-}.$$

Then $\sigma(T)$ is the set of points n^2, and $(\lambda - T)^{-1}$ is defined by

$$(\lambda - T)^{-1}\phi_n^{+} = (n^2 - \lambda)^{-1}(\phi_n^{+} - n(n^2 - \lambda)^{-1}\phi_n^{-})$$

$$(\lambda - T)^{-1}\phi_n^{-} = (n^2 - \lambda)^{-1}\phi_n^{-}.$$

Hence T is regular. If we put $k_n = n^2 - n^{1/2}$, then

$$d(k_n, \sigma(T)) = n^{1/2}$$

for all large n, while

$$(k_n - T)^{-1}\phi_n^{+} = n^{-1/2}\phi_n^{+} - \phi_n^{-}$$

has norm at least 1.

5. Basic properties of ordinary differential operators. We wish ultimately to apply our abstract theory to the study of linear differential operators. We shall take our formal differential operators to have the form

$$\tau = \sum_{i=0}^{n} a_i(x)\left(\frac{d}{dx}\right)^i, \tag{3}$$

where

$$a_n(x) \equiv 1, \quad a_{n-1}(x) \equiv 0,$$

and where the coefficient function $a_i(x)$ belongs to the class $C^\infty\,[0,1]$. The restriction on the coefficients a_n and a_{n-1} is not as severe as might at first appear, since any operator τ of the form (3) in which $a_n(x) \neq 0$ and $a_n(x)$ is real can be reduced to one of the restricted form we have chosen by an elementary transformation.

In connection with the study of the n-th order differential operator τ, it is convenient to introduce the Banach space $A^n = A^n[0, 1]$ consisting of those functions f in C^{n-1} such that $f^{(n-1)}(x)$ is absolutely continuous and such that $f^{(n)} \in L_2[0, 1]$. We introduce the norm in A^n by the definition

$$|f| = \left\{\int_0^1 |f^{(n)}(x)|^2\,dx\right\}^{1/2} + \max_{0 \leq x \leq 1} \; \max_{0 \leq i \leq n-1} |f^{(i)}(x)|.$$

A fundamental formula in the study of τ is then the Green's formula, which we can obtain readily by partial integration:

$$\int_0^1 \tau f(x)\,\overline{g(x)}\,dx - \int_0^1 f(x)\,\overline{\tau^* g(x)}\,dx = F_1(f, g) - F_0(f, g). \tag{4}$$

Here, f and g are arbitrary elements of $A^n[0, 1]$, τ is the formal differential operator

$$\tau = \sum_{i=0}^{n} a_i(x)\left(\frac{d}{dx}\right)^i,$$

and τ^* is the formal differential operator

$$\tau^* = \sum_{i=0}^{n} b_i(x)\left(\frac{d}{dx}\right)^i,$$

where

$$b_i(x) = \sum_{j=i}^{n} (-1)^j \binom{j}{i}\left(\frac{d}{dx}\right)^{j-i} \overline{a_j(x)}\,.$$

The operator τ^* is called the formal, or Lagrange, adjoint of τ. The bilinear forms $F_1(f, g)$ and $F_0(f, g)$ are given by the formulas

$$F_1(f,g) = \sum_{i,j=0}^{n-1} \alpha_{ij} f^{(i)}(1)\overline{g^{(j)}(1)},$$

$$F_0(f,g) = \sum_{i,j=0}^{n-1} \beta_{ij} f^{(i)}(0)\overline{g^{(j)}(0)},$$

where the coefficients α_{ij} and β_{ij} are calculated readily from the functions $a_i(x)$. We can see, in particular, that

$$\beta_{ij} = \alpha_{ij} = 0 \text{ for } i + j \geq n - 1,$$

$$\beta_{n-1-k,k} = -\alpha_{n-1-k,k} = (-1)^k.$$

Thus, the matrices β_{ij} and α_{ij} are nonsingular subdiagonal matrices, and hence define nonsingular bilinear forms.

If a formal differential operator τ is given, we set up a corresponding unbounded operator T_0 in the Hilbert space $L_2[0,1]$ as follows:

(a) $\mathfrak{D}(T_0)$ is the set of all C^n functions f defined in $[0,1]$ and vanishing outside some compact subset of the interior of $[0,1]$.

(b) If $f \in \mathfrak{D}(T_0)$, $T_0 f$ is defined simply as τf.

Our principal analytic problem at this point is to determine the adjoint of T_0. The solution is contained in the following:

LEMMA 5. *The adjoint T_0^* of the operator T_0 is the operator T_1 defined as follows*:

(a) *Its domain is $A^{(n)}$.*

(b) *If $f \in A^{(n)}$, $T_0^* f = \tau^* f$.*

Proof. It follows immediately from Green's formula that $T_1 \subseteq T_0^*$. To prove the opposite inclusion, we proceed by stages.

(a) Consider first an element $z \in L_2$ such that $T_0^* z = 0$. That is, $(z, T_0 y) = 0$ for every $T_0 y$ in the range of T_0. We shall show that $z \in C^n$. Let Σ be the n-dimensional space of solutions of $\tau^*\sigma = 0$. We shall show that if $f \in L_2$ is orthogonal to Σ, then $(f, z) = 0$. Since Σ is finite dimensional and hence closed, we shall be able to conclude that $z \in \Sigma$, which will give us the desired result $z \in C^n$.[5] We begin by proving the somewhat weaker statement contained in:

SUBLEMMA 5. *If*

(a) f *is orthogonal to* Σ,

(b) $f \in C^n$,

(c) $f(x) \equiv 0$ *outside some compact subset of* $(0,1)$,

then f *is orthogonal to* z.

Proof. We know by the standard theory of ordinary differential equations that the equation $\tau\hat{f} = f$ has a unique solution $\hat{f} \in C^n$ which satisfies the boundary conditions

$$0 = \hat{f}(0) = \hat{f}'(0) = \cdots = \hat{f}^{(n-1)}(0).$$

If we can only verify that $\hat{f}(x) \equiv 0$ outside some closed subinterval of $(0,1)$, we will know that $\hat{f} \in \mathfrak{D}(T_0)$, so that $f = T_0\hat{f}$, and therefore $(f, z) = 0$. Now $\hat{f}$ is, in some interval $[0, \epsilon]$, the unique solution of the equation $\tau\hat{f} = 0$ satisfying the boundary conditions

$$0 = \hat{f}(0) = \hat{f}'(0) = \cdots = \hat{f}^{(n-1)}(0) = 0.$$

Hence $\hat{f}(x) \equiv 0$ in $[0, \epsilon]$. We could apply the same argument to an interval $[1-\epsilon, 1]$, if only we knew that

$$0 = \hat{f}(1) = \hat{f}'(1) = \cdots = \hat{f}^{(n-1)}(1),$$

and it is this which we propose to verify. This we can do as follows: let $\sigma \in \Sigma$. Then we have, from Green's formula,

$$0 = \int_0^1 \tau\hat{f}(x)\sigma(x)\,dx - \int_0^1 \hat{f}(x)\overline{\tau^*\sigma(x)}\,dx$$

$$= F_1(\hat{f}, \sigma) - F_0(\hat{f}, \sigma) = F_1(\hat{f}, \sigma).$$

That is, $F_1(f, \sigma) = 0$ for every $\sigma \in \Sigma$. Since there exists a $\sigma \in \Sigma$ with any pre-assigned values

$$\sigma(1), \sigma'(1), \cdots, \sigma^{n-1}(1),$$

[5] It may be noted that the method of proof of this lemma is actually that adapted to proving the following result:

THEOREM. *Let a distribution* δ *satisfy an ordinary linear differential equation with* C^∞ *coefficients. Then* δ *is itself a* C^∞ *function.*

In connection with this proof, see [9, Theorem 1.1], where the same result is proved by a different method.

it follows that

$$\hat{f}(1) = \hat{f}'(1) = \cdots = \hat{f}^{(n-1)}(1) = 0,$$

by the nonsingularity of the form $F_1(\hat{f}, \sigma)$. This concludes the proof of the sublemma.

Now we must show that hypotheses (b) and (c) of the sublemma can be dropped without invalidating the conclusion. Indeed, let f be a function which is orthogonal to Σ. Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be an orthonormal basis for Σ. Then, by approximating σ_i sufficiently closely by a C^n function ϕ_i which vanishes outside a compact subinterval of $(0, 1)$, we can ensure that the matrix $(\phi_i, \sigma_j) = m_{ij}$ is nonsingular. Now, let f be approximated by a sequence f_k of C^n functions which vanish outside a closed subinterval of $(0, 1)$. Then, if $\hat{m}_{ij}$ is the inverse matrix of m_{ij},

$$\hat{f}_k = f_k - \sum_{j=1}^{n} \sum_{l=1}^{n} (f_k, \sigma_j)\hat{m}_{jl}\phi_l$$

is a sequence of C^n functions orthogonal to Σ which vanish outside a compact subinterval of $(0, 1)$, and such that $\lim_{k \to \infty} f_k = f$. Since, by the sublemma, $(\hat{f}_k, z) = 0$, we are able to conclude that $(f, z) = 0$.

To complete the proof of Lemma 5 it still remains to consider the case $T_0^* z = g$, where $g \neq 0$ and $g \in L_2$, and to show that $z \in A^n$. We know by the standard theory of ordinary differential equations that there exists a solution $z_1 \in A^n$ of the equation $\tau^* z_1 = g$. Now, as remarked at the beginning of the proof of Lemma 5, $z_1 \in \mathfrak{D}(T_0^*)$. Hence

$$T_0^*(z - z_1) = 0.$$

By what we have already proved, $z - z_1 \in C^n$ and

$$\tau^*(z - z_1) = 0.$$

Hence it follows that $z \in A^n$ and that

$$\tau^* z = \tau^* z_1 = g = T_0^* z.$$

Thus the proof of Lemma 5 is complete.

Lemma 5 has as a consequence an interesting topological property of our formal differential operators, expressed in:

LEMMA 6. *Suppose that f_m is a sequence of elements of A^n, and that f_m and τf_m converge (weakly) in the topology of $L_2[0,1]$. Then f_m converges (weakly) in the topology of $A^n[0,1]$ (and conversely).*

Proof. Let us introduce a norm in $\mathfrak{D}(T_0^*)$ in two ways:

$$|f|_1 = \left\{\int_0^1 |f(x)|^2 dx\right\}^{\frac{1}{2}} + \left\{\int_0^1 |\tau^* f(x)|^2 dx\right\}^{\frac{1}{2}},$$

$$|f|_2 = |f|_1 + \max_{0 \le x \le 1} \max_{0 \le i \le n-1} |f^{(i)}(x)|.$$

Then, since T_0^* is closed, $\mathfrak{D}(T_0^*)$ is complete in the first norm. On the other hand, it follows from this that $\mathfrak{D}(T_0^*)$ is complete in the second norm. Since $|f|_2 \le |f|_1$, it follows from a well-known principle in the theory of Banach spaces [7, Theorem 11.7] that $|f|_1$ and $|f|_2$ are equivalent. On the other hand, it is evident on inspection that $|f|_2$ and the norm introduced for A^n determine the same topology. Hence it follows that $|f|_1$ determines the same topology in A^n as the norm of A^n, and this proves our lemma.

On the basis of these two lemmas we can proceed systematically to set up the exact operator theory of differential operators. We first make:

DEFINITION 2. Let τ be a formal differential operator of order n, and let

$$(5) \qquad A_j(f) = \sum_{i=0}^{n-1} A_{ji} f^{(i)}(0) + \sum_{i=0}^{n-1} \hat{A}_{ji} f^{(i)}(1) = 0, \qquad (j = 1 \cdots k)$$

be a set k linear boundary conditions. Then we define an operator T in $L_2[0,1]$ by putting:

$$\text{(a)}\ \mathfrak{D}(T) = \left\{ f \in A^n \,\middle|\, \sum_{i=0}^{n-1} A_{ji} f^{(i)}(0) + \sum_{i=0}^{n-1} \hat{A}_{ij} f^{(i)}(1) = 0,\ j = 1, \cdots, k \right\}.$$

(b) If $f \in \mathfrak{D}(T)$, $Tf = \tau f$.

Then T is said to be the differential operator determined by the formal operator τ and the boundary conditions [3]. Any such operator is called a *differential operator.*

LEMMA 7. *Any differential operator T is a closed operator in Hilbert space with a dense domain. Moreover, the range of T is closed.*

Proof. Let $f_n \longrightarrow f$, $Tf_n \longrightarrow g$. Then, by Lemma 6, we have $f \in A^n$, $f_n \longrightarrow f$ in the topology of A^n. It is then evident that f satisfies the boundary conditions which define T, so that $f \in \mathfrak{D}(T)$. Moreover, if T is defined by the formal operator τ, we have $\tau f_n \longrightarrow \tau f$ in the topology of L_2, so that $Tf = \tau f = g$; thus T is closed.

Let T_1 be the differential operator defined by the formal operator τ and the boundary conditions

$$f(0) = f'(0) = \cdots = f^{(n-1)}(0) = f(1) = f'(1) = \cdots = f^{(n-1)}(1) = 0.$$

Then T is an extension of T_1. Now, it is clear that the differential operator T defined by the boundary conditions $A_j(f) = 0$ will remain the same if we drop from our list of conditions all A_j which are linear combinations of A_k with $k < j$. Hence, without loss of generality, we can suppose that the vectors

$$[A_{j0} \cdots A_{j(n-1)} \hat{A}_{j0} \cdots \hat{A}_{j(n-1)}]$$

form a linearly independent set. Thus, we can find a finite set of functions $\phi_1, \phi_2, \cdots, \phi_k \in \mathfrak{D}(T)$ such that

$$A_j(\phi_i) = \delta_{ji}.$$

It follows that

$$\mathfrak{D}(T) = \mathfrak{D}(T_1) + S,$$

where S is the finite dimensional space generated by the vectors ϕ_i ($i = 1, 2, \cdots, k$). Hence, if $\mathfrak{R}(T)$ denotes the range of T, we have

$$\mathfrak{R}(T) = \mathfrak{R}(T_1) + \hat{S},$$

where $\hat{S}$ is a finite dimensional space. Hence, we have only to show that $\mathfrak{R}(T_1)$ is closed. Now, suppose that $\mathfrak{R}(T_1)$ is not closed. Then there exists an element g and a sequence $f_n \in \mathfrak{D}(T_1)$ such that $T_1 f_n \longrightarrow g$, but $g \notin \mathfrak{R}(T_1)$. It then follows from the closure of the operator T_1 that f_n does not converge. Hence there exists $\epsilon > 0$ and a sequence m_i, n_i of indices approaching infinity such that

$$|f_{m_i} - f_{n_i}| > \epsilon.$$

Putting

$$f_{m_i} - f_{n_i} = g_i,$$

we have $|g_i| > \epsilon$, $T_1 g_i \longrightarrow 0$. If we then put

$$\hat{g}_i = \hat{g}_i / |g_i|,$$

we have $|\hat{g}_i| = 1$, $T_1 g_i \longrightarrow 0$. A subsequence of $\hat{g}_i$ converges weakly: we can suppose without loss of generality that this subsequence is the sequence $\hat{g}_i$ itself. It then follows by Lemma 6 that $\hat{g}_i$ converges weakly in the topology of A^n, and hence in the topology of C^0. Therefore $\hat{g}_i(x)$ is a uniformly bounded sequence which converges at each x $(0 \le x \le 1)$; this implies that $\hat{g}_i$ converges in the topology of $L_2[0,1]$. From the closure of T_1 we find, putting

$$\hat{g} = \lim_{i \to \infty} \hat{g}_i,$$

that $|\hat{g}| = 1$, $T_1 \hat{g} = 0$. But then $\hat{g}$ is a nonzero function in C^n which satisfies the equation $\tau g = 0$ and the boundary conditions

$$\hat{g}(0) = \hat{g}'(0) = \cdots = \hat{g}^{(n-1)}(0) = 0;$$

this contradiction proves Lemma 7.

If we examine the part of the foregoing proof which concerns the operator T_1, we see that we have actually shown:

COROLLARY. *Let T be a differential operator with an inverse T^{-1}. Then T^{-1} is a continuous mapping from the range $\mathfrak{R}(T)$ of T to L_2.*

We strengthen this conclusion in:

LEMMA 8. *Let T be a differential operator with an inverse T^{-1}. Then T^{-1} is a continuous mapping from the range $\mathfrak{R}(T)$ of T into A^n, and a compact mapping from $\mathfrak{R}(T)$ into $L_2[0,1]$.*

Proof. We know that if Tf_n converges, f_n converges. It follows by Lemma 6 that f_n converges in the topology of A^n, proving the first part of the lemma. Now suppose that Tf_n converges weakly: since T^{-1} is continuous, f_n converges weakly. It follows by Lemma 6 that f_n converges weakly in the topology of A^n, and hence in the topology of C^0; so $f_n(x)$ is a uniformly bounded sequence of functions converging at each $x \in [0,1]$. Then it follows that f_n converges in the topology of L_2. Since T^{-1} thus transforms weakly convergent sequences into strongly convergent sequences, T^{-1} is compact.

LEMMA 9. *Let T be the differential operator defined by the formal operator τ of order n and by the boundary conditions* (5). *Then T^* is the differential* *operator T_1 defined by the formal operator τ^* and by a set of boundary conditions*

$$B_i(f) = \sum_{j=0}^{n-1} B_{ij} f^{(j)}(0) + \sum_{j=0}^{n-1} \hat{B}_{ij} f^{(j)}(1) = 0 \qquad (i = 1, 2, \cdots, k')$$

obtained from the conditions (5) *as follows:*

Let $S_i\,[\sigma_i^0 \cdots \sigma_i^{2n-1}]$ $(i = 1 \cdots k')$ be a basis for the set of solutions of the equations

$$A_i(\sigma) = \sum_{j=0}^{n-1} A_{ij}\sigma^j + \sum_{j=0}^{n-1} \hat{A}_{ij}\sigma^{n+j} \qquad (i = 1 \cdots k)$$

derived from equations (5), *and let*

$$F_1(f, g) - F_0(f, g) = \sum_{i,j=0}^{n-1} \{ \alpha_{ij} f^{(i)}(1)\overline{g^{(j)}(1)} - \beta_{ij} f^{(i)}(0)\overline{g^{(j)}(0)} \}$$

be the bilinear functional arising in Green's formula (4). *Then*:

$$B_{ij} = \sum_{l=0}^{n-1} \overline{\alpha}_{lj}\,\overline{\sigma}_i^{\,l} \quad \text{and} \quad \hat{B}_{ij} = - \sum_{l=0}^{n-1} \overline{\beta}_{lj}\overline{\sigma}_i^{\,n+l}.$$

Proof. It follows immediately from Green's formula that $T_1 \subseteq T^*$. To prove the converse, let ϕ_i be a C^n function such that

$$\phi_i^{(j)}(0) = \sigma_i^j, \quad \phi_i^{(j)}(1) = \sigma_i^{j+n-1}.$$

Then $A_m(\phi_i) = 0$ $(m = 1 \cdots k)$, so that $\phi_i \in \mathfrak{D}(T)$. If $f \in \mathfrak{D}(T^*)$, it follows that

$$0 = (T\phi_i, f) - (\phi_i, T^* f) = F_1(\phi_i, f) - F_0(\phi_i, f)$$

$$= \sum_{j=0}^{n-1} B_{ij} f^{(j)}(0) + \sum_{j=0}^{n-1} \hat{B}_{ij} f^{(j)}(1),$$

so that $f \in \mathfrak{D}(T_1)$. From this it follows immediately that $T_1 = T^*$.

LEMMA 10. *Let T be a differential operator, and suppose that for some* *complex* λ *both* $T-\lambda$ *and* $T^*-\bar{\lambda}$ *have an inverse. Then* $T(T^*)$ *is a regular operator,* T *and* T^* *have spectra related by* $\sigma(T)=\bar{\sigma}(T^*)$, *and determine spectral measures* E_1 *and* E_2 *related by* $E_1(\lambda)=E_2^*(\bar{\lambda})$.

In this case, we call T a regular differential operator.

Proof. By Lemma 7 and its corollary, the range of $T-\lambda$ is closed and $(T-\lambda)^{-1}$ is continuous. To show that $(T-\lambda)^{-1}$ is everywhere defined, that is, that

$$\mathfrak{R}(T-\lambda)=H,$$

we have then only to show that no nonzero $z\in H$ is orthogonal to $(T-\lambda)\mathfrak{D}(T)$. However, any such z would satisfy $(T^*-\bar{\lambda})z=0$, and we have ruled out this possibility in our hypothesis. This, together with Lemma 8, proves the first part of our lemma. The remaining parts follow, via the remark after Lemma 1, from the corresponding results for bounded operators, all of which are well known (Cf. [7, Lemma V.4].)

For application to the spectral theory of differential operators we shall need the criterion contained in:

LEMMA 11. *Let* T *be a regular operator in a Banach space* $\mathfrak{X}$ *and let* $\lambda_0\in\sigma(T)$. *Let* $f_1^*, f_2^*, \cdots, f_n^*$ *be a basis for the solutions of* $(T^*-\lambda_0)f=0$,[6] *and let* Σ *be the space of solutions of* $(T-\lambda_0)\sigma=0$. *Then* λ_0 *is a multiple pole of the resolvent* $(T-\lambda)^{-1}$ *if and only if some nonzero* $\sigma\in\Sigma$ *satisfies* $f_i^*(\sigma)=0$ $(i=1,2,\cdots,n)$.

Proof. We can readily see, by Lemma 1 (c), that λ_0 is a multiple pole of the resolvent if and only if there exists a solution g of the equation $(T-\lambda_0)^2g=0$ which is not a solution of $(T-\lambda_0)g=0$; that is, if and only if some nonzero $\sigma\in\Sigma$ is in the range of $(T-\lambda_0)$. Now, if $\sigma=(T-\lambda_0)g$, then

$$f_i^*(\sigma)=f_i^*((T-\lambda_0)g)=(T^*-\lambda_0)f_i^*(g)=0.$$

Conversely, if $f_i^*(\sigma)=0$, then it follows that σ is in the closure of the range of $T-\lambda_0$, and our lemma will be proved once we show that $T-\lambda_0$ has a closed

[6] The general theory of adjoint unbounded operators in a Banach space is discussed more fully in Lemmas 18 and 19 below. It is well to remark, however, that we are faced with the usual confusion as to adjoints in Hilbert space, where, contrary to our practice in other Banach spaces, we make use of the *Hermitian*, rather than the pure Banach-space, adjoint. This has the effect of introducing complex conjugates in many of the Hilbert-space formulas where the corresponding Banach-space formulas do not have complex conjugates. This should not cause any essential difficulty to the reader.

range. This, however, is easy to show since

$$(T-\lambda_0)\mathfrak{D}(T) = (T-\lambda_0)E(\lambda_0)\mathfrak{D}(T) + (T-\lambda_0)(I-E(\lambda_0))\mathfrak{D}(T)$$

$$= (T-\lambda_0)E(\lambda_0)\mathfrak{D}(T) + (I-E(\lambda_0))\mathfrak{X}.$$

The first space on the right is finite dimensional and the second is closed, so that $(T-\lambda_0)\mathfrak{D}(T)$ is closed.

LEMMA 12. *Let E be a projection of a B-space $\mathfrak{X}$ onto a finite dimensional range, and let $E^*: \mathfrak{X}^* \longrightarrow \mathfrak{X}^*$ be its adjoint. Then, if $\phi_1, \phi_2, \cdots, \phi_n$ is a basis for $E\mathfrak{X}$ we can find a unique basis $\psi_1^*, \psi_2^*, \cdots, \psi_n^*$ of $E\mathfrak{X}^*$ such that $\psi_i^*(\phi_j) = \delta_{ij}$; and then*

$$Ef = \sum_{i=1}^{n} \phi_i \psi_i^*(f) \quad \textit{for any} \quad f \in \mathfrak{X}.$$

Proof. Any element Ef can be written uniquely as

$$Ef = \sum_{i=1}^{n} \phi_i \alpha_i(f),$$

where the $\alpha_i(f)$ are linear functionals. If $f_m \longrightarrow f$ and $\alpha_i(f_m) \longrightarrow \alpha_i$, it is clear that $\alpha_i = \alpha_i(f)$. Hence, by the closed graph theorem of Banach spaces [7, Theorem 11.8] the uniquely determined linear functionals α_i are continuous. Hence $\alpha_i(f) = \psi_i^*(f)$ for some $\psi_i^* \in \mathfrak{X}^*$.

From

$$Ef = \sum_{i=1}^{n} \phi_i \psi_i^*(f)$$

it follows readily that

$$E^*\psi^* = \sum_{i=1}^{n} \psi_i^* \psi^*(\phi_i),$$

so that $\psi_1^*, \psi_2^*, \cdots, \psi_n^*$ span $E^*\mathfrak{X}^*$. To see that $\psi_1^*, \psi_2^*, \cdots, \psi_n^*$ are linearly independent, let $\sum_{i=1}^{n} \alpha_i \psi_i^* = 0$; then

$$\alpha_j = \left(\sum_{i=1}^{n} \alpha_i \psi_i^* \right) \phi_j = 0,$$

so that Lemma 12 is completely proved.

As the final lemma of this section, we state a useful elementary principle in the theory of spectral differential operators.

LEMMA 13. *Let T be a spectral differential operator, and let λ_i be an enumeration of the points in $\sigma(T)$. Then, if $f \in \mathfrak{D}(T)$, the "expansion"*

$$\sum_{i=1}^{\infty} E(\lambda_i)f$$

converges unconditionally in the topology of A^n.

Proof. The series $\sum_{i=1}^{\infty}(\lambda_i)f$ certainly converges unconditionally in the topology of L_2. On the other hand, so does the series

$$T\left(\sum_{i=1}^{\infty} E(\lambda_i)f\right) = \sum_{i=1}^{\infty} E(\lambda_i)(Tf).$$

Hence, by Lemma 6, the original series converges unconditionally in the topology of A^n.

6. Application. The second order differential operator. In this section we wish to apply the theory developed up to now to various second order differential operators arising out of the formal differential operator

$$\tau = -\left(\frac{d}{dx}\right)^2 + q(x).$$

Our peturbation theorem, Theorem 1, reduces the study of this operator to the much simpler operator $-(d/dx)^2$. What we need about the latter is summarized, however, in:

LEMMA 14. *The unbounded operator T defined by the formal differential operator $\tau = -(d/dx)^2$ and the boundary conditions*

(6) $\quad f(0) - k_0 f'(0) = 0, \quad f(1) - k_1 f'(1) = 0, \qquad k_0, k_1$ *arbitrary,*

is a spectral operator satisfying all the hypotheses of case (b) of Theorem 1.

REMARK. We can also admit the boundary conditions determined by $k_0 = \infty$ and/or $k_1 = \infty$; that is, the conditions $f'(0) = 0$ and $f'(1) = 0$, respectively.

Proof. Since it is easy to treat all special cases in which k_0 or k_1 is zero or infinity by a separate argument much like the argument given below, we shall assume for simplicity that we have none of these special cases to deal with. If we put $\lambda = s^2$, the general solution of the equation

$$-f''(x) - \lambda f(x) = 0$$

is $\sin s(x + \alpha)$, where α is an arbitrary constant. This satisfies the boundary condition at zero if

$$\tan s\alpha = k_0 s,$$

and satisfies the boundary conditions at one if

$$\tan s(1 + \alpha) = k_1 s.$$

Thus, $T - \lambda$ can only fail to have an inverse if $\lambda = s^2$, where s is a root of the equation

$$\tan s = \frac{(k_1 - k_0)s}{1 + k_0 k_1 s^2} = \frac{cs}{1 + ds^2}, \qquad d \neq 0. \tag{7}$$

It is readily seen by making use of Lemma 9 that T^* is the differential operator defined by τ^* and by the adjoint boundary conditions

$$f(0) - \bar{k}_0 f'(0) = 0; \; f(1) - \bar{k}_1 f'(1) = 0.$$

Thus the adjoint operator $T^* - \bar{\lambda}$ can only fail to have an inverse if $T - \lambda$ fails to have an inverse; that is, if and only if s satisfies (7). Since not every s satisfies (7), it follows immediately from Lemma 10 that T is regular.

Our next task is to locate the zeros of (7) more exactly. Since $\tan s$ is periodic of period π and has only the zero $s = 0$ in its period-strip, it follows readily that (7) has a countable sequence $z_k, z_{k+1}, \cdots$ of zeros which can be numbered in such a way that

$$z_n = n\pi + O(1).$$

From this preliminary estimate we readily obtain the estimate

$$\tan z_n \sim \frac{cn\pi}{1 + d(n\pi)^2} \sim c(dn\pi)^{-1}$$

Hence it follows that

$$z_n = n\pi + c(dn\pi)^{-1} + O(n^{-2}).$$

We thus obtain an enumeration λ_n $(n = k, k + 1, \cdots)$[7] of the eigenvalues of T such that

$$\lambda_n = (n\pi)^2 + 2cd^{-1} + O(n^{-1}).$$

Hence, if d_n is the distance from λ_n to the remainder of the spectrum,

$$d_n \sim \pi^2(2n + 1),$$

so that

$$\sum_{n=k}^{\infty} d_n^{-2} < \infty .$$

It is evident from the form of the boundary conditions defining our operator that each λ_n can correspond to at most one function ϕ_n (up to a scalar multiple) which satisfies

$$(T - \lambda_n)\phi_n = 0.$$

Thus, if $E(\lambda_n)$ is to be anything but a projection onto a one-dimensional range, λ_n must be a multiple pole of the resolvent. By Lemma 11, the condition for this is $(\phi_n, \psi_n) = 0$, where ψ_n is the (unique) solution of

$$(T^* - \overline{\lambda}_n)\psi_n = 0.$$

Since, however, T^* is defined by the complex-conjugate boundary conditions of those that define T, it is clear that

$$\psi_n(x) = \overline{\phi_n(x)}.$$

Hence, λ_n can only be a multiple pole of the resolvent of T if

$$\int_0^1 (\phi_n(x))^2 dx = 0.$$

Now, we have

$$\phi_n(x) = \sin z_n(x + \alpha_n) = \sin(z_n x + \beta_n),$$

where β_n must be determined so as to satisfy

$$k_0^{-1} z_n^{-1} \sin \beta_n = \cos \beta_n .$$

It follows readily that

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[7] Note: k need not be equal to one.

$$\beta_n = \pi/2 - (n\pi k_0)^{-1} + O(n^{-2}),$$

so that

$$\phi_n(x) = \cos(z_n x + \delta_n), \ \delta_n = (n\pi k_0)^{-1} + O(n^{-2}).$$

It follows that

$$\int_0^1 (\phi_n(x))^2 dx \sim \int_0^1 \cos^2 n\pi x \, dx = \frac{1}{2},$$

so that only a finite set of λ_n can be multiple poles of the resolvent of T. For those λ_n which are simple poles of the resolvent of T, the projection $E(\lambda_n)$ is, by Lemma 12, the operator determined by the integral kernel

$$\hat{\phi}_n(x)\hat{\phi}_n(y) = E_n(x, y),$$

where $\hat{\phi}_n$ is a scalar multiple of ϕ_n, the scalar being chosen so as to make

$$\int_0^1 (\hat{\phi}_n(x))^2 dx = 1.$$

We have $\hat{\phi}_n = c_n\phi_n$, and a simple computation reveals that

$$c_n = 2^{-\frac{1}{2}} + O(n^{-2});$$

hence it follows that

$$E_n(x, y) = \frac{1}{2}\cos n\pi x \cos n\pi y - \frac{(cd^{-1}x + k_0)}{\sqrt{2}\, n\pi} \sin n\pi x \cos n\pi y$$
$$- \frac{(cd^{-1}y + k_0)}{\sqrt{2}\, n\pi} \sin n\pi y \cos n\pi x + O(n^{-2}),$$

which gives a decomposition of E_n into four terms

$$(8) \qquad E_n = \hat{E}_n + A_n + B_n + \Delta_n.$$

It is now trivial to find a uniform bound for

$$\Big|\sum_{n \in J} E_n\Big|,$$

J an arbitrary finite set of integers, by making use of the decomposition (8).

We have

$$\left|\sum_{n\in J}\hat{E}_n\right| \leq 1,$$

since the $\hat{E}_n$ are a family of orthogonal projections. We have

$$\left|\sum_{n\in J}\Delta_n\right| \leq M,$$

since

$$|\Delta_n| = O(n^{-2}) \quad\text{and}\quad \sum_{n=1}^{\infty} n^{-2} < \infty.$$

The operators A_n and B_n have the form

$$A_n = \hat{E}_n\hat{A}_n \quad\text{and}\quad B_n = \hat{B}_n\hat{E}_n,$$

where

$$|\hat{A}_n| = O(n^{-1}) \quad\text{and}\quad |\hat{B}_n| = O(n^{-1}),$$

a situation studied above in the proof of part (b) of Theorem 1, where the argument given proves not only the uniform boundedness of $\sum_{n\in J} A_n$, but also, with suitable slight modifications, the law

$$\lim_{n\to\infty}\left|\sum_{m=n}^{\infty} A_m\right| = 0.$$

All that remains to complete the proof of our lemma is a proof that

$$\sum_{i=k}^{\infty} E(\lambda_i) = I.$$

By Lemma 15 below,

$$E_\infty = I - \sum_{i=k}^{\infty} E(\lambda_i)$$

either projects onto an infinite dimensional space or is zero. But,

$$\lim_{m\to\infty}\left|\left(I - \sum_{n=m}^{\infty} E(\lambda_n)\right) - \left(I - \sum_{n=m}^{\infty}\hat{E}_n\right)\right| = 0.$$

Hence, by Lemma 4,

$$I - \sum_{n=m}^{\infty} E(\lambda_n)$$

has a finite dimensional range for all sufficiently large m, and hence, a fortiori, E_∞ has a finite dimensional range.

THEOREM 2. *Let T be the unbounded differential operator defined by the formal differential operator $\tau = -(d/dx)^2$ and the boundary conditions*

$$(9) \qquad f(0) - k_0 f'(0) = 0 \quad f(1) - k_1 f'(1) = 0,$$

where k_0 and k_1 are arbitrary, possibly infinite, complex numbers. Then if B is an arbitrary bounded operator, $T + B$ is a spectral operator.

Proof. This follows from Lemma 14 and Theorem 1.

COROLLARY 1. *Let T be the unbounded differential operator defined by the formal differential operator*

$$\tau = -\left(\frac{d}{dx}\right)^2 + q(x)$$

and by the boundary conditions (9), where $q(x) \in C^\infty$.[8] *Then T is a spectral operator.*

This corollary is the "convergence in mean" form of the theorem of Birkhoff-Hilb. As far as pointwise convergence is concerned, we can state:

COROLLARY 2. *Let T be as in Corollary 1, and let $f \in \mathfrak{D}(T)$. Then if λ_i is an enumeration of $\sigma(T)$, the series*

$$\sum_{i=1}^{\infty} E(\lambda_i) f$$

converges unconditionally in the topology of A^2.[9]

Proof. This follows immediately from Corollary 1 and Lemma 13.

[8] This much is what we have proved explicitly. But, with a little more "analytic care," we would see that it is sufficient that $q(x)$ be measurable and bounded.

[9] We shall see (Corollary 2 of Theorem 3) that this series converges to f.

It may be noted, moreover, that Theorem 1 and Lemma 14 yield a much wider class of spectral operators than the analytic method of Birkhoff-Hilb. For instance, the differential-difference operator

$$\tau f(x) = \left(\frac{d}{dx}\right)^2 f(x) + q(x) f(x+\alpha)$$

(in which $x+\alpha$ is understood to be taken modulo 1, and $q(x)$ is bounded and measurable), with appropriate boundary conditions, is immediately seen to be spectral, as is the integro-differential operator

$$\tau f(x) = \left(\frac{d}{dx}\right)^2 f(x) + \int_0^1 K(x,y) f(y)\, dy,$$

provided only that the integral kernel K defines a bounded operator.

7. Theorems on the spectral measure of infinity. Suppose that T is an unbounded regular spectral operator in a Banach space $\mathfrak{X}$, and that $\{\lambda_i\}$ is its spectrum. Let $E(\lambda_i)$ be the associated spectral measure. Then we put

$$E(\infty) = I - \sum_{i=1}^{\infty} E(\lambda_i).$$

It is clear that $E(\infty) f = f$ if and only if

$$E(\lambda_i) f = 0, \qquad \text{for } 1 \leq i < \infty.$$

This leads us to the following more general:

DEFINITION 3. If T is an unbounded regular operator in the Banach space $\mathfrak{X}$, with spectrum $\{\lambda_i\}$ and spectral measure $E(\lambda_i)$, we put

$$S_\infty(T) = \{f \mid E(\lambda_i) f = 0, \quad 1 \leq i < \infty\}.$$

LEMMA 15. *The space $S_\infty(T)$ either is infinite dimensional or consists only of zero.*

Proof. We can suppose without loss of generality that $0 \notin \sigma(T)$, and put $U = T^{-1}$. It then follows by the remark following Lemma 1 that

$$\sigma(U) = \{\lambda_i^{-1}\} \cup \{0\},$$

and that the spectral measure $\hat{E}$ of U is defined by

$$\hat{E}(\lambda_i^{-1}) = E(\lambda_i).$$

Hence, if $f \in S_\infty = S_\infty(T)$, we have

$$\hat{E}(\lambda_i^{-1})Uf = U\hat{E}(\lambda_i^{-1})f = 0,$$

so that $US_\infty \subseteq S_\infty$. Moreover, by [**15**, Theorem 8.2c], $(U-\lambda)^{-1}f$ is regular at every point λ_i^{-1} if $f \in S_\infty$; thus if $f \in S_\infty$, $(U-\lambda)^{-1}$ has no singularity other than the origin. Hence U, regarded as an operator in S_∞, is quasi-nilpotent. If S_∞ were finite dimensional, it would follow that for some finite k, $U^k S_\infty = 0$. Since U has the inverse T, this would imply that S_∞ contained no nonzero vector.

LEMMA 16. *The space $S_\infty(T)$ is the set of all $f \in \mathfrak{X}$ for which $(T-\lambda)^{-1}f$ is an entire function of λ.*

Proof. If $(T-\lambda)^{-1}f$ is entire, then if we let C be a small circle around λ_i we find that

$$0 = \frac{1}{2\pi i}\int_C (T-\lambda)^{-1} f d\lambda = -E(\lambda_i)f$$

Conversely, if $E(\lambda_i)f = 0$, it follows from [**15**, Theorem 8.2c] that $(T-\lambda)^{-1}f$ is regular at λ_i. Since this holds for every $\lambda_i \in \sigma(T)$, it follows that $(T-\lambda)^{-1}f$ is entire.

LEMMA 17. *Let T be a regular spectral operator in a Banach space $\mathfrak{X}$. Suppose that all but a finite number of the poles μ_i of the resolvent function $(T-\lambda)^{-1}$ are simple, and that $S_\infty(T) = 0$. Let*

$$d_i = \operatorname{dist}(\mu_i, \sigma(T)),$$

and let B be bounded.

(a) *If $d_i \longrightarrow \infty$, $T+B$ is regular.*

(b) *If $\underline{\lim}_{i\to\infty} d_i > 0$, there exists an $\epsilon > 0$ such that $T+B$ is regular whenever $|B| < \epsilon$.*

(c) *If $\underline{\lim}_{i\to\infty} d_i > 0$ and B is compact, $T+B$ is regular.*

Proof. This lemma is needed to make the statement of Theorem 3 below plausible and possible. The proof results incidentally from the proof of Theorem

3, so that it is not necessary to give the details here.

THEOREM 3. *Let T be a regular spectral operator in the Banach space $\mathfrak{X}$. Suppose that all but a finite number of the points in $\sigma(T)$ are simple poles of the resolvent function $(T-\lambda)^{-1}$ and that $S_\infty(T)=0$. Let U_i be a sequence of bounded domains with $\bigcup_{i=1}^\infty U_i$ the entire plane, and put V_i = boundary (U_i); let*

$$V_i \cap \sigma(T) = \phi \quad \text{and} \quad d_i = \operatorname{dist}(V_i, \sigma(T));$$

and let B be a bounded operator.

(a) *If $d_i \longrightarrow \infty$, $S_\infty(T+B)=0$.*

(b) *If $\underline{\lim}_{\,i\to\infty}\, d_i > 0$, there exists an $\epsilon > 0$ such that $S_\infty(T+B)=0$ whenever $|B| \le \epsilon$.*

(c) *If $\underline{\lim}_{\,i\to\infty}\, d_i > 0$, and B is compact, $S_\infty(T+B)=0$.*[10]

Proof. We first show that if $\mu_1, \mu_2, \cdots, \mu_N$ is a finite set of points in the plane, we can find a domain U containing all of them such that V = boundary (U) has a minimum distance from $\sigma(T)$ greater than $d = 1/2\ \underline{\lim}\ d_i$ (or, in case (a), greater than an arbitrarily prescribed d) and such that the minimum distance from V to μ_i is greater than a constant D which may be as large as we please. This is done as follows: we take j_0 so large that

$$d_i > \frac{1}{2} \underline{\lim}_{k\to\infty} d_k \qquad \text{if } i \ge j_0,$$

and let K be a prescribed very large closed circular domain. Put

$$K' = K \cup \bigcup_{i=1}^{j_0} \overline{U_i},$$

and let $U_1, U_2, \cdots, U_M$ be a covering of K'. Then we have only to take

$$U = \bigcup_{i=1}^{M} U_i.$$

[10] It would be interesting to know that in case (b) of Theorem 3 we can dispense with the restriction $|B| < \epsilon$, but I do not know whether or not this is possible.

Now, let $f \in S_\infty(T+B)$, and let

$$f(\lambda) = (T + B - \lambda)^{-1} f.$$

We shall show that the entire function $f(\lambda)$ is uniformly bounded, so that $f(\lambda)$ is constant, $f(\lambda) = g$, and hence $f = (T + B - \lambda)g$ for all λ. From this it is evident that $g = 0$, so that $f = 0$. To demonstrate the uniform boundedness we proceed as follows: Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the set of all multiple poles of the resolvent, and let Λ be an arbitrary point in the complex plane. Take, in the first part of this proof,

$$\mu_1, \mu_2, \cdots, \mu_N = \Lambda, \lambda_1, \cdots, \lambda_n.$$

Then, by Lemma 3, there exists an absolute constant c such that $|R_\lambda| < cd^{-1}$ for $\lambda \in V$, where $R_\lambda = (T - \lambda)^{-1}$. If we put

$$\overline{R}_\lambda = (T + B - \lambda)^{-1},$$

we have (cf. formula (1) in the proof of Theorem 1)

$$\overline{R}_\lambda = (I + R_\lambda B)^{-1} R_\lambda.$$

Hence, if

$$|B| \leq c^{-1} d(1 - \delta)$$

with $\delta > 0$, $\overline{R}_\lambda$ exists for $\lambda \in V$, and

$$|\overline{R}_\lambda| < \delta^{-1} cd^{-1}.$$

But then

$$|\overline{R}_\lambda f| \leq \delta^{-1} cd^{-1} |f|$$

for $\lambda \in V$, so that, by the maximum modulus principle,

$$|\overline{R}_\lambda f| \leq \delta^{-1} cd^{-1} |f|$$

everywhere in U. Hence we have

$$|f(\Lambda)| = |\overline{R}_\Lambda f| \leq \delta^{-1} cd^{-1} |f|;$$

that is, $f(\lambda)$ is uniformly bounded. This proves Theorem 3 in cases (a) and (b).

To handle case (c), we observe that since $\sum_{i=1}^{N} E(\lambda_i)$ converges strongly to I, $\sum_{i=1}^{N} E(\lambda_i)f$ converges to f uniformly as f ranges over any compact subset of $\mathfrak{X}$. Since we now assume that B is compact, it follows that $\sum_{i=1}^{N} E(\lambda_i)B$ converges to B in the uniform topology of operators. We choose N_0 so large that

$$\left| B - \sum_{i=1}^{N_0} E(\lambda_i)B \right| \leq c^{-1}d(1-\delta).$$

Then, if we put

$$C = B - \sum_{i=1}^{N_0} E(\lambda_i)B,$$

we have

$$\overline{R}_\lambda = \left(I + R_\lambda C + \sum_{i=1}^{N_0} R_\lambda E(\lambda_i)B \right)^{-1} R_\lambda.$$

However, if d_1 is the minimum distance from λ to any of the points λ_i, it follows by the discussion of the functional calculus of T preceding Lemma 3 that there exists an absolute constant c_1 such that

$$|R_\lambda E(\lambda_i)| \leq c_1 d_1^{-1}$$

for $1 \leq i \leq N_0$ and for d_1 sufficiently large. We now determine the domain U of the first paragraph of this proof by putting

$$\mu_1, \mu_2, \cdots, \mu_N = \Lambda, \lambda_1, \cdots, \lambda_{N_1},$$

where $N_1 \geq N_0$ is so large that the set $\lambda_1, \lambda_2, \cdots, \lambda_{N_1}$ includes all the multiple poles of the resolvent, and where

$$D = 2|B|N_0 c_1 \delta^{-1}.$$

It then follows, as in the proof of parts (a) and (b) of Theorem 3, that $\overline{R}_\lambda$ exists for $\lambda \in V$, and that

$$|\overline{R}_\lambda| < z\delta^{-1} c d^{-1};$$

from this point on we can argue just as in cases (a) and (b).

Thus all cases of Theorem 3 are proved.

COROLLARY 1. *Under the hypotheses of Theorem* 1, $T + B$ *is a spectral operator such that* $S_\infty(T + B) = 0$.

Proof. We choose the domains U_i of Theorem 3 as follows: If i is even, $i = 2n$, we take U_i to be the interior of a circle of radius d_i about the point λ_i, where d_i is the distance from λ_i to the rest of $\sigma(T)$. If i is odd, $i = 2n + 1$, we take U_i to be the set of all points z with $|z| < n$ but $|z - \lambda_i| > d_i/4$ for all i.

COROLLARY 2. *If* T *is the differential operator of Theorem* 2, *and* B *is bounded, then every function* $f \in L_2[0, 1]$ *can be expanded in a series of eigenfunctions (including, possibly, a finite number of solutions of equations of the type*

$$(T + B - \lambda)^k f = 0)$$

of $T + B$ *which converges unconditionally in the topology of* L_2. *Any function of class* A^2 *which satisfies the appropriate boundary conditions can be expanded in a series of eigenfunctions converging unconditionally in the topology of* A^2.

Theorem 3 also applies to a class of operators which are not necessarily spectral. To discuss this class of operators, we shall first extend the elementary theory of the adjoint from closed operators in Hilbert space to closed operators in an arbitrary reflexive Banach space. If $\mathfrak{X}$ is a reflexive Banach space, so is the direct sum $\mathfrak{X} \oplus \mathfrak{X}$ (in any suitable norm), and we have evidently

$$(\mathfrak{X} \oplus \mathfrak{X})^* = \mathfrak{X}^* \oplus \mathfrak{X}^*.$$

The space $\mathfrak{X} \oplus \mathfrak{X}$ admits the evident automorphisms

$$A_1 : (x, y) \longrightarrow (y, x),$$

$$A_2 : (x, y) \longrightarrow (-y, x).$$

We have

$$A_1^2 = -A_2^2 = I, \quad A_1 A_2 = -A_2 A_1.$$

If M is a closed manifold in a Banach space Y, its annihilator $M^\perp$ is the closed subspace of Y^* defined by

$$M^\perp = \{y^* \in Y^* \mid y^*(M) = 0\}.$$

If Y is reflexive, we have evidently $M^{\perp\perp} = M$. If T is a linear transformation in

$\mathfrak{X}$ (Note: we continue to suppose that $\mathfrak{D}(T)$ is dense in $\mathfrak{X}$.), its graph $\Gamma(T)$ is the subset of $\mathfrak{X} \oplus \mathfrak{X}$ defined by

$$\Gamma(T) = \{(x, Tx) \mid x \in D(T)\}.$$

Clearly, $\Gamma(T)$ is closed if and only if T is closed. We have evidently

$$\Gamma(T^{-1}) = A_1\Gamma(T),$$

whenever T^{-1} is defined (or, equivalently, whenever $A_1\Gamma(T)$ is the graph of a single-valued operator). We define the closed linear operator T^* in $\mathfrak{X}^*$ by putting

$$\Gamma(T^*) = [A_2\,\Gamma(T)]^{\perp}.$$

The operator T^* is single valued, since $(0, y^*) \in \Gamma(T^*)$ is equivalent to $y^*(x) = 0$ for all $x \in D(T)$; and since $D(T)$ is dense in $\mathfrak{X}$, this gives $y = 0$. It may also be remarked that if T is bounded, this definition of T^* agrees with the usual one.

LEMMA 18. (a) *$D(T^*)$ is dense.*

(b) *$T^{**} = T$.*

(c) *T and T^* have both bounded inverses if either does, and $(T^{-1})^* = (T^*)^{-1}$.*

(d) *If B is a bounded operator, $(T + B)^* = T^* + B^*$.*

Proof. The proofs are exactly like those in the Hilbert-space case. If $D(T^*)$ is not dense, we can find an $x \in \mathfrak{X}$ such that

$$xD(T^*) = 0,$$

while $x \neq 0$. Then

$$A_2(0, x) = (-x, 0) \in \Gamma(T^*) = A_2\Gamma(T),$$

so that

$$(0, x) \in A_2^2\,\Gamma(T) = \Gamma(T),$$

and hence $x = T(0) = 0$, a contradiction. This proves (a).

To prove (b), we observe that

$$\Gamma(T^{**}) = (A_2\Gamma(T^*))^{\perp} = A_2(\Gamma(T^*))^{\perp} = A_2(A_2\,\Gamma(T)) = -\,\Gamma(T).$$

To prove (c), we observe that

$$\Gamma((T^*)^{-1}) = A_1\Gamma(T^*) = A_1(A_2\Gamma(T))^{\perp} = (A_2(A_1\Gamma(T)))^{\perp} = \Gamma((T^{-1})^*).$$

Thus $(T^*)^{-1} = (T^{-1})^*$ even if either or both of the transformations are unbounded, multi-valued, or not everywhere defined, so that (c) follows as a special case.

To prove (d) we note that it is evident that

$$\Gamma(T^* + B^*) \subseteq \Gamma((T + B)^*).$$

On the other hand, if $x^* \in D((T + B)^*)$, so that

$$x^*((T + B)y) = (T + B)^* x^*(y)$$

for every $y \in D(T)$, we have clearly

$$x^*(Ty) = \{(T + B)^* x^* - B^* x^*\}(y)$$

for $y \in D(T)$; thus $x^* \in D(T^*)$, and

$$T^* x^* + B^* x^* = (T + B)^* x^*.$$

LEMMA 19. (a) *If one of T and T^* is regular, both are.*

(b) *We have $\sigma(T) = \sigma(T^*)$.*

(c) *If T and T^* are regular, their spectral measures E and $\hat{E}$ are related by $\hat{E}(\lambda) = E^*(\lambda)$.*

(d) *If T and T^* are regular and one is spectral, so is the other.*

Proof. By Lemma 18 (c) and (d), we have

$$((T - \lambda)^{-1})^* = ((T - \lambda)^*)^{-1} = (T^* - \lambda)^{-1}$$

with both sides of this equation existing as bounded operators for exactly the same λ. This proves (b) and (a), since for bounded operators U and U^* are either both compact or both not compact.

To prove (c), we note that $E(\lambda)$ may be characterized as

$$E(\lambda) = -\frac{1}{2\pi i}\int_C (T - \lambda)^{-1} d\lambda,$$

where C is a sufficiently small circle about λ. But then

$$E^*(\lambda) = -\frac{1}{2\pi i}\int_C (T^* - \lambda)^{-1}\,d\lambda = \hat{E}(\lambda)$$

is evident. However, since (d) follows immediately from (c), Lemma 19 is entirely proved.

Suppose that T is a regular operator in $\mathfrak{X}$. Then by $\mathrm{sp}(T)$, the *spectral span* of T, we denote the smallest closed manifold containing all the manifolds $E(\lambda)\mathfrak{X}$. Thus, $x \in \mathrm{sp}(T)$ if and only if x can be approximated by linear combinations of solutions f of equations

$$(T-\lambda)^k f = 0,$$

that is, by generalized eigenvectors of T. Thus, if T is known to be a regular spectral operator,

$$\mathrm{sp}(T) = \left(\sum_{i=1}^{\infty} E(\lambda_i)\right)\mathfrak{X}.$$

For nonspectral regular operators in a reflexive space, however, we may state:

LEMMA 20. *If T is a regular operator in the (reflexive) Banach space $\mathfrak{X}$, then $sp(T) = S_\infty(T^*)^\perp$.*

REMARK. For spectral operators, the conditions

$$\mathrm{sp}(T) = \mathfrak{X} \text{ and } S_\infty(T) = 0$$

are clearly equivalent; but for nonspectral operators the condition for $\mathrm{sp}(T) = \mathfrak{X}$ given by the lemma is $S_\infty(T^*) = 0$ and not $S_\infty(T) = 0$. Indeed, H. Hamburger [**10**, pp. 74-79] has constructed an example of a compact operator U in Hilbert space $\mathfrak{X}$ whose generalized eigenvectors span $\mathfrak{X}$, and which is such that an infinite dimensional closed subspace $\mathfrak{X}_0$ of $\mathfrak{X}$ exists such that $U\mathfrak{X}_0 \subseteq \mathfrak{X}_0$, and U is quasi-nilpotent in $\mathfrak{X}_0$. If we put $T = (U^*)^{-1}$, we have $\mathrm{sp}(T) \neq \mathfrak{X}$, while $S_\infty(T) = 0$.

Proof of Lemma 20. It is clear that if $\lambda \in \sigma(T)$ and we have

$$E(\lambda)f = f, \text{ while } E(\mu)^* g^* = 0 \text{ for every } \mu \in \sigma(T) = \sigma(T^*),$$

then

$$g^*(f) = g^*(E(\lambda)f) = E(\lambda)^* g^*(f) = 0.$$

Thus, it is clear that $\mathrm{sp}(T) \subseteq S_\infty(T^*)^\perp$. Conversely, if $f \notin \mathrm{sp}(T)$, there exists a functional $g^* \in \mathfrak{X}^*$ such that

$$g^*(f) = 1, \quad g^*(\mathrm{sp}(T)) = 0.$$

Since $g^*(E(\lambda)f') = 0$ for any $f' \in \mathfrak{X}$ and any $\lambda \in \sigma(T)$, it follows that

$$E(\lambda)^* g^* = 0 \quad \text{for every} \quad \lambda \in \sigma(T) = \sigma(T^*).$$

Thus $g^* \in S_\infty(T^*)$; and since $g^*(f) = 1$, it follows that $f \notin S_\infty(T^*)$.

Lemma 20 and Theorem 3 together give us a fairly general insight into the range of situations in which a "spectral density" property $\mathrm{sp}(T) = \mathfrak{X}$ is to be expected of an operator T. However, in applying these results it is convenient to be able to deal, wherever possible, with solutions of the equation $(T - \lambda)f = 0$, rather than with solutions of the equation $(T - \lambda)^k f = 0$. The next lemma describes a simple case in which this is possible.

LEMMA 21. *Let T be a regular spectral operator in the Banach space $\mathfrak{X}$. Suppose that all but a finite number of the countable set $\{\lambda_n\}$ of points in $\sigma(T)$ are simple poles of the resolvent function and correspond to one-dimensional eigenspaces. Let d_n be the minimum distance from λ_n to the other points in $\sigma(T)$. Then all but a finite number of points in $\sigma(T+B)$ are simple poles corresponding to one-dimensional eigenspaces if*

(a) *d_i approaches infinity, and B is bounded; or*

(b) *$\underline{\lim}_{i \to \infty} d_i > 0$, and $|B|$ is less than some positive constant $\epsilon(T)$; or*

(c) *$\underline{\lim}_{i \to \infty} d_i > 0$, and B is compact.*

Proof. The proof in each of these three cases is very much like the proof in the corresponding case of Theorem 3. We shall show that there exists an N such that $\mu \in \sigma(T+B)$ and $|\mu| \geq N$ imply that μ is a simple pole of the resolvent $\overline{R}_\lambda$ of $T + B$ and corresponds to a one-dimensional eigenspace. Indeed, if $\lambda \in \sigma(T+B)$, there exists an $f \in \mathfrak{X}$ such that $|f| = 1$ and such that

$$(T + B - \lambda)f = 0, \text{ so that } (T - \lambda)f = -Bf.$$

From this last equation it is evident that if $(T - \lambda)^{-1} = R_\lambda$ exists, it must have a norm which is at least $|B|^{-1}$. By Lemma 3, there exists an absolute constant $c = c(T)$ such that

$$|(T - \lambda)^{-1}| \leq cd^{-1}$$

if λ is not within a distance d of any point in $\sigma(T)$, and if N is so great that every multiple pole λ_0 of R_λ satisfies $|\lambda_0| < N$. It follows that every point μ of $\sigma(T+B)$ with $|\mu| \leq N$ is within a distance $c^{-1}|B|$ of a point $\lambda_n \in \sigma(T)$. Moreover, if we suppose that $c^{-1}|B| < d_n/2$ (which covers cases a and b), then we see as in the proof of Theorems 1 and 3 that the resolvent

$$\overline{R}_\lambda = (T + B - \lambda)^{-1}$$

exists everywhere on the circle C_n with center λ_n and radius $d_n/2$, at least if N is chosen to be sufficiently large (or, in case b, for $|B|$ sufficiently small). We have, as usual,

$$\overline{R}_\lambda = (I + R_\lambda B)^{-1} R_\lambda;$$

and, for N sufficiently large (or $|B|$ sufficiently small), this leads, as in the proof of Theorems 1 and 3, to an estimate

$$|E(\lambda_n) - E_n| < \frac{1}{2} \; |E(\lambda_n)|^{-1}.$$

In this formula, E_n is the sum of all the projections $\overline{E}(\mu)$ for μ interior to C_n, where $\overline{E}$ is the spectral measure corresponding to $T+B$. If follows by Lemma 4 that E_n is a projection onto a one-dimensional subspace, so that there is exactly one point $\mu_n \in \sigma(T)$ interior to C_n, and $\overline{E}(\mu_n)$ is a one-dimensional projection. Since we have already shown that every $\mu \in \sigma(T)$ with $|\mu| \leq N$ must belong to the interior of some C_n, our Lemma is proved in cases (a) and (b).

It is not hard to see that the same argument will work in case (c) as soon as we are able to show that $|R_{\lambda_n'} B| \longrightarrow 0$ if λ_n' is a sequence with $|\lambda_n'| \longrightarrow \infty$, and with

$$\operatorname{dist}(\lambda_n', \sigma(T)) > \epsilon > 0.$$

However, since it is evident from the functional calculus that $R_{\lambda_n'}$ converges strongly to zero as $n \longrightarrow \infty$, and since B is compact, it follows that $|R_{\lambda_n'} B| \longrightarrow 0$ as $n \longrightarrow \infty$. In this way we are able to dispose successfully of case (c), so that Lemma 21 is proved in entirety.

REMARK. It is not hard to see that a proof like that of Lemma 21 will establish the existence of certain cases in which the hypothesis that the resolvent R_λ of T has only simple poles corresponding to one-dimensional eigenspaces will yield the corresponding property for $T+B$, so that we can be sure that not even one pole of the resolvent $\overline{R}_\lambda$ of $T+B$ is multiple. In general, the situation is this: Multiple poles of $\bar{R}_\lambda$ can only arise out of multiple poles of R_λ, or out of simple poles of R_λ which are multiple eigenvalues, or, finally, out of the "fusion" of several poles of T under the influence of the perturbation B. If we rule out the first two causes, and demand that B be too small to move any pole of R_λ far enough to cause two poles of R_λ to meet, we can be sure that $\bar{R}_\lambda$ has only simple poles. On the other hand, it is clear that if R_λ has multiple poles or multiple eigenvalues, no demand that B be small can be strong enough to ensure that $\bar{R}_\lambda$ has no multiple poles.[11]

8. Applications to differential equations. Theorem 1 is usually inapplicable in the theory of partial and singular ordinary differential operators because the very simple behavior of the eigenvalues required in the hypotheses of Theorem 1 ordinarily fails. However, even in these cases, Theorem 3 can often be applied to yield interesting results. Let us begin by considering the ordinary singular differential operator

$$\tau = -\left(\frac{d}{dx}\right)^2 + q(x)$$

on the half-open interval $I = [0, \infty)$, and make the assumption that $q'(x) > 0$, $q(x) \longrightarrow \infty$. Then, as is well known (cf. [**16**, p. 19]), any boundary condition

$$f(0) + kf'(0) = 0 \qquad (0 \leq k \leq \infty)$$

determines a self-adjoint operator T as follows:

(a) $\mathfrak{D}(T)$ is the set of all functions f which belong to $A^2[0, N]$ for every $N > 0$, such that $\tau f \in L_2$, and such that $f(0) + kf'(0) = 0$.

(b) $Tf = \tau f$ for $f \in \mathfrak{D}(T)$.

Moreover (cf. [**16**, p. 113 and p. 134]), the operator T is without continuous spectrum, and has only a finite number $N(\lambda)$ of eigenvalues (counted with appropriate multiplicities) below any fixed λ. This number is given asymptotically as $\lambda \longrightarrow +\infty$ by the formula

$$N(\lambda) = \int_0^{\mu(\lambda)} (\lambda - q(x))^{\frac{1}{2}},$$

where $\mu(\lambda)$ is the uniquely determined solution of $q(\mu(\lambda)) = \lambda$. This formula makes it easy for us to evaluate

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[11] For a detailed discussion of this type of question, c.f. [**18**].

$$c = c(\tau) = \underline{\lim}_{\lambda \to \infty} \lambda^{-1} N(\lambda),$$

and by use of Theorem 3 we are able to state:

THEOREM 4. (a) *If the singular differential operator τ is such that $c(\tau) = +\infty$, and T is the self-adjoint operator in Hilbert space $\mathfrak{X}$ associated above with τ, then* $\mathrm{sp}(T + B) = \mathfrak{X}$ *for every bounded operator B.*

(b) *If instead of $c(\tau) = +\infty$ we have $c(\tau) > 0$, then* $\mathrm{sp}(T + B) = \mathfrak{X}$ *for all bounded operators B with $|B| < \epsilon = \epsilon(\tau)$, and for every compact operator B.*

REMARK. It is easy to see that $\epsilon(\tau) = 1/2\,c(\tau)$ is an acceptable determination.

Proof. The proof results immediately from Lemma 20 and Theorem 3, the only point in question being the method by which we are to choose the domains U_i of Theorem 3. However, it is clearly possible to choose arbitrarily large real λ_i such that the distance from λ_i to $\sigma(T)$ is not less than $c(\tau)/2$. If we put

$$U_i = \{x + iy \mid x < \lambda_i\},$$

we complete our proof.

The same argument evidently applies to any self-adjoint operator T which is is without continuous spectrum, and for which we have

$$c(T) = \underline{\lim}_{\lambda \to \infty} \lambda^{-1} N(\lambda) > 0,$$

where $N(\lambda)$ is the number of eigenvalues μ (counted with multiplicities and supposed finite) such that $-\lambda \leq \mu \leq \lambda$. This observation applies to an extensive class of elliptic partial differential operators. Thus, for instance, Hilbert-Courant [**12**, Chap. 6, Theorem 17] gives the value

$$c(T) = (4\pi)^{-1} \iint_G p^{-1}(x, y)\, dxdy$$

for the partial differential operator T defined in terms of the formal operator

$$\tau = -\frac{\partial}{\partial x} p(x, y) \frac{\partial}{\partial x} - \frac{\partial}{\partial y} p(x, y) \frac{\partial}{\partial y} + q(x, y)$$

and in terms of any one of a wide family of boundary conditions. Here, G is a bounded domain whose boundary is of measure zero, and T is an unbounded self-adjoint operator in the Hilbert space $\mathfrak{X} = L_2(G)$. The functions $p(x,y)$ and $q(x,y)$ are required to be real and infinitely differentiable in a neighborhood of the closure of G, while we assume that $p(x,y) > 0$ everywhere on the closure of G. This means, however, that the corresponding partial differential operator $T + B$, defined in terms of the formal operator[12]

$$\tau' = -\frac{\partial}{\partial x} p(x,y) \frac{\partial}{\partial x} - \frac{\partial}{\partial y} p(x,y) \frac{\partial}{\partial y} + q(x,y) + iq'(x,y),$$

has the property $\mathrm{sp}(T+B) = \mathfrak{X}$, provided only that $|q'(x,y)| < \epsilon$ for some sufficiently small $\epsilon > 0$.

Many other instances are known in which a self-adjoint formal elliptic operator τ has nonzero constant $c(\tau)$. For instance, Gårding [8] shows that if the domain $G \subseteq E^n$ is bounded, and τ is real, formally self-adjoint, or order m, and has constant coefficients, then we have an asymptotic expression of the form

$$N(\lambda)\lambda^{-n/m} \sim d(\tau).$$

This allows us to apply Theorem 3, case (a) whenever $n < m$, and cases (b) and (c) of Theorem 3 whenever $n = m$.

To apply Theorem 3 when $n > m$, we must proceed in a slightly different way. Let us suppose that T is an unbounded self-adjoint operator without continuous spectrum such that $N(\lambda)$ is finite and

$$\varliminf_{\lambda \to \infty} N(\lambda)\lambda^{-\epsilon} > 0,$$

where $\epsilon > 0$. Then the operator T^k satisfies

$$\lim_{\lambda \to \infty} N_1(\lambda)\lambda^{-1} = \infty$$

for some sufficiently large k ($N_1(\lambda)$ is the number of eigenvalues μ_1 of T^k such

[12] To define exactly the functional domains and boundary conditions involved in the theory of partial differential equations would involve us in a very extensive analytic discussion, which has, after all, nothing to do with our problem, since we can take the same domain for $T + B$ as was required to make T self-adjoint (or, more generally, spectral). This difficulty leads to a slight vagueness in the formulations of the rest of this section, but not to any real lack of rigor in the results.

that $-\lambda \leq \mu_1 \leq \lambda$; that is, the number of eigenvalues μ of T such that $-\lambda^{1/k} \leq \mu \leq \lambda^{1/k}$). Now, if B is a bounded operator such that every product

$$T^{i_1} B^{j_1} T^{i_2} \ldots B^{j_n},$$

with at least one j_i nonzero, is a bounded operator, it follows readily that $(T + B)^k$ satisfies all the hypotheses of Theorem 3. It then follows that

$$\mathrm{sp}((T + B)^k) = \mathfrak{X}.$$

However, from [**15**, Theorem 9.4] it follows readily that

$$\mathrm{sp}(S) = \mathfrak{X} \text{ and } \mathrm{sp}(S^k) = \mathfrak{X}$$

are equivalent restrictions on a regular operator S. That is, we can conclude that $\mathrm{sp}(T + B) = \mathfrak{X}$.

To give a concrete example of a case in which this argument applies, we have only to use the result of Gårding, and consider the formal operator $\tau + K$, in which τ is self-adjoint elliptic partial differential operator with constant coefficients in a bounded domain G, and K is an integral operator

$$Kf(x) = \int_G K(x, y) f(y)\, dy$$

in which the kernel is a C^∞ function of both its arguments, defined when each argument is in a neighborhood of the closure of G. We are able to conclude that the appropriate unbounded operators T defined in terms of such formal operators also have the "spectral spanning" property $\mathrm{sp}(T) = \mathfrak{X}$.

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