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SEMI-ORTHOGONALITY IN RICKART RINGS

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This note initiates a study of the semi-orthogonality relation on the lattice of principal left ideals generated by idempotents of a Rickart ring. It will be seen that two left ideals in a von Neumann algebra are semi-orthogonal if and only if their unique generating projections are non-asymptotic. Connections between semi-orthogonality, dual modularity, von Neumann regularity, and algebraic equivalence will be established; those Rickart rings with a superabundance of semi-orthogonal left ideals will be characterized.

A *regular ring* is a ring A with identity in which each element $a \in A$ is *regular* in the sense that $aba = a$ for some element $b \in A$. A *Rickart ring* is a ring A with identity in which the left (and right) annihilator of each element is a principal left (right) ideal generated by an idempotent. Regular rings and Baer rings, as defined by Kaplansky [4], are special cases of Rickart rings: in particular, then, a von Neumann algebra is a Rickart ring. Rickart rings are called Baer rings in [2]. Throughout this note, A will denote a Rickart ring. $L(M)$ and $R(M)$ will denote respectively the left and right annihilators of a subset M of A. The letters e, f, g, h and k will denote idempotents and the letters E, F, G, H and K will denote the left ideals they generate.

Ordered by set inclusion, the set $L(A)$ of principal left ideals generated by idempotents forms a lattice. If E and F form a modular pair in $L(A)$, we shall write $(E, F)M$; if E and F form a dual modular pair in $L(A)$, we shall write $(E, F)M^*$. Following S. Maeda [6], we shall say that two left ideals E and F in $L(A)$ are *semi-orthogonal*, $E \sharp F$, if they are generated by orthogonal idempotents. Maeda shows that the semi-orthogonality relation $\sharp$ on $L(A)$ has these properties: (1) If $E \sharp E$, then $E = (0)$; (2) If $E \sharp F$, then $F \sharp E$; (3) If $E_1 \leqq E$ and $E \sharp F$, then $E_1 \sharp F$; (4) If $E \sharp F$ and $E \vee F \sharp G$, then $E \sharp F \vee G$; (5) If $E \leqq F$, then there is a left ideal G in $L(A)$ such that $E \vee G = F$ and $E \sharp G$.

The results herein form a portion of the author's dissertation, submitted to the Graduate School of the University of Massachusetts and directed by Professor D. J. Foulis.

2. Semi-orthogonal left ideals.

In this section, we give geometric meaning to Maeda's canonical semi-orthogonality relation in $L(A)$.

THEOREM 1. *Let $E = Ae$ and $F = Af$. Then the following conditions are equivalent*:

(1) $E \sharp F$.

(2) $E \cap F = (0)$ *and* $e(1 - f)$ *is regular in* A.

(3) $E \oplus F = E \vee F$ *in* $L(A)$.

Proof. The proofs of (1) implies (2) and of (3) implies (1) are routine. To see that (2) implies (3), we suppose that $e(1 - f)xe(1 - f) = e(1 - f)$ for some $x \in A$. Put $g = (1 - f)xe(1 - f)$. Then $fg = 0 = gf$ and $eg = e(1 - f)xe(1 - f) = e(1 - f) = e - ef$. Then $g^2 = (1 - f)xe(1 - f)g = (1 - f)xeg = (1 - f)xe(1 - f) = g$ and $(f + g)^2 = f + fg + gf + g = f + g$.

We claim that $E \oplus F = A(f + g)$. But $f = (f + g) - g(f + g) \in A(f + g)$ and $e = ef + eg = e(f + g) \in A(f + g)$. Thus $E \oplus F \leqq A(f + g)$. Conversely, $f + g = f + (1 - f)xe(1 - f) = (1 - f)xe + (1 - xe + fxe)f \in E \oplus F$. Hence $E \oplus F = A(f + g) \in L(A)$.

We can find perspicacious geometric and topological interpretations for each of these equivalent conditions in the ring of bounded operators on a Hilbert space or, more generally, in any von Neumann algebra. In such a ring, any left annihilator is a principal left ideal generated by a unique projection (= self-adjoint idempotent). Let e and f denote the unique generating projections of E and F respectively: we shall identify these projections with their ranges.

If $e \wedge f = 0$, e and f are said to be *asymptotic* if $\sup |\langle \alpha, \beta \rangle| = 1$, where $||\alpha|| = 1 = ||\beta||$, $\alpha \in e$, $\beta \in f$; otherwise e and f are said to be *non-asymptotic*. It is known [5, p. 166 and pp. 172–174] that these conditions are equivalent: (1) e and f form a non-asymptotic pair; (2) The projection map of the subspace $e \oplus f$ onto e is continuous; (3) The vector sum of e and f is a closed subspace; (4) $(e, f)M^*$ in the projection lattice of the ring of all bounded operators on the underlying Hilbert space. The relation of semi-orthogonality to non-asymptoticity is provocative; for, by modifying results of Jacob Feldman [1, pp. 12–14], it is easy to verify that $E \sharp F$ if and only if e and f form a non-asymptotic pair.

Our next result, though appearing an immediate consequence of Theorem 1 (2), seems to require a measure of prestidigitatorial skill with idempotents.

COROLLARY 1. ef *is regular if and only if* $(1 - f)(1 - e)$ *is regular*.

Proof. We prefer to demonstrate the obviously equivalent statement: If $e(1 - f)$ is regular, then so is $f(1 - e)$. To this end, choose an idempotent h with $Ah = Ae \cap Af$. Put $e_1 = e + h - eh$ and $f_1 = f + h - fh$. Then e_1 and f_1 are idempotent generators for Ae and Af respectively and $h = he_1 = e_1h = hf_1 = f_1h$. By direct computation, we have $e_1(1 - f_1) = e(1 - f)(1 - h)$ and $f_1(1 - e_1) = f(1 - e)(1 - h)$. Since $e(1 - f)$ is regular, $e(1 - f)xe(1 - f) = e(1 - f)$ for some $x \in A$. Then, an easy computation shows $e_1(1 - f_1)[(1 - f)x]e_1(1 - f) = e_1(1 - f_1)$; thus $e_1(1 - f_1)$ is regular.

Put $e_0 = e_1(1 - h)$ and $f_0 = f_1(1 - h)$. Then $e_0(1 - f_0) = e_1(1 - f_1)$ is regular. Moreover, if $z \in Ae_0 \cap Af_0 \leqq Ae_1 \cap Af_1 = Ah$, then $z = zh$ $(ze_0)h = ze_1(1-h)h = 0$; so $Ae_0 \cap Af_0 = (0)$. Then by Theorem 1 (2), we have $Ae_0 \# Af_0$.

Consequently, $f(1 - e)(1 - h) = f_1(1 - e_1) = f_0(1 - e_0)$ is regular. Then $f(1 - e)(1 - h)yf(1 - e)(1 - h) = f(1 - e)(1 - h)$ for some element $y \in A$. But this means that $f(1 - e)(1 - h)yf(1 - e) - f(1 - e) = f(1 - e)(1 - h)yf(1 - e)h - f(1 - e)h$ is an element of $A(1 - e) \cap Ah = A(1 - e) \cap Ae \cap Af = (0)$. Thus $f(1 - e)[(1 - h)y]f(1 - e) = f(1 - e)(1 - h)yf(1 - e) = f(1 - e)$, showing that $f(1 - e)$ is regular in A.

COROLLARY 2. *If $E \# F$, then $(E, F)M$ and $(E, F)M^*$ in $L(A)$.*

Proof. A proof that E and F form a modular pair is given by Maeda [6, Lm. 1]. Now suppose that $Ae \# Af$ with $Af \leqq Ag \leqq Ae \oplus Af$. Then $g = xe + yf$ for some elements x and y in A. Then $xe = g - yf \in Ae \cap Ag$ and we have $g = xe + yf \in (Ae \cap Ag) \oplus Af$. Thus $Ag \leqq (Ae \cap Ag) \oplus Af$. Since the opposite inclusion is evident, $Ag = (Ae \cap Ag) \oplus Af$. Hence $(Ae, Af)M^*$.

3. **Equivalence of left ideals.** Two left ideals E and F in $L(A)$ are *semi-orthogonally perspective* via G, $G: E \sim F$, if $E \oplus G = E \vee F = G \oplus F$ with $E \# G$ and $G \# F$. The importance of this relation is exemplified in the following result:

THEOREM 1. *If $G: E \sim F$, then the mapping $E_0 \to \varphi(E_0) = (E_0 \oplus G) \cap F$ is a lattice isomorphism of the principal lattice ideal generated by E in $L(A)$ onto the principal lattice ideal generated by F in $L(A)$. Under this mapping, moreover, semi-orthogonal left ideals contained in E correspond with semi-orthogonal left ideals contained in F.*

Proof. The proof is entirely lattice theoretic. Define a mapping ψ by $F_0 \to (G \oplus F_0) \cap E$ for each $F_0 \leqq F$; clearly both φ and ψ are isotone maps. By Corollary 2.2, we have $(F, G)M^*$ and $(G, E)M$. With these modularity relations, it is easy to compute $(\psi \circ \varphi)(E_0) = E_0$ for all $E_0 \leqq E$. Similarly $(\varphi \circ \varphi)(F_0) = F_0$ for all $F_0 \leqq F$. Thus φ is a lattice isomorphism with ψ its inverse mapping.

Now suppose $E_1, E_2 \leqq E$ with $E_1 \# E_2$. Since $E \# G$, $E_1 \oplus E_2 \# G$ also. Then $E_1 \oplus G \# E_2$ and we may compute $\varphi(E_1) \oplus G = [(E_1 \oplus G) \cap F] \oplus G = (E_1 \oplus G) \cap (F \oplus G) = (E_1 \oplus G) \cap (E \oplus G) = E_1 \oplus G \# E_2$, since $(F, G)M^*$. Thus $\varphi(E_1) \# E_2 \oplus G$, so that $\varphi(E_1) \# \varphi(E_2)$. Conversely, if $F_1, F_2 \leqq F$ with $F_1 \# F_2$, a similar argument shows $\psi(F_1) \# \psi(F_2)$.

LEMMA 1. [7, Th. 2]. *Let $eA = aA$ and $Af = Aa$. Then there exists a unique element $a^+ \in A$ such that*

(1) $aa^+ = e$.

(2) $fa^+ = a^+$.

Moreover,

(3) $a^+a = f$.

(4) $Ae = Aa^+$.

(5) $fA = a^+A$.

(6) $a = aa^+a$.

(7) $a^+ = a^+aa^+$.

Two idempotents e and f are *algebraically equivalent* via a and $b(a, b\colon e \sim f)$ if $e = ab$, $f = ba$, $a \in eAf$ and $b \in fAe$. This is easily seen to be an equivalence relation. The idempotents e and f are algebraically equivalent if and only if Ae and Af are isomorphic A-modules; moreover, in that case, the mapping $x \to bxa$ is a ring isomorphism of eAe onto fAf [4, pp. 21-23].

Notice that by Lemma 1, if $eA = aA$ and $Af = Aa$, then e and f are algebraically equivalent via a, a^+. This observation enables us to relate algebraic equivalence in A to semi-orthogonal perspectivity in $L(A)$.

THEOREM 2. *If $Ae \sim Af$, then $e \sim f$.*

Proof. Suppose $Ag\colon Ae \sim Af$. Put $a = e(1 - g)$ and $b = f(1 - g)$; then a and b are regular by Theorem 2.1 (2). An easy computation shows $eA = RL(e) = RL(e(1 - g)) = RL(a) = aA$ and similarly $fA = bA$. Moreover, $Ae \oplus Ag = Ag \oplus Af$ implies $R(a) = R(b)$; thus $Aa = LR(a) = LR(b) = Ab$. Choose an idempotent h with $Ah = Aa = Ab$. Then by our observation above, $e \sim h$ and $h \sim f$. Hence $e \sim f$.

For semi-orthogonal left ideals, the converse of Theorem 2 is also valid. We prove this as a first consequence of Lemma 2. With $Ae \# Af$, this fundamental lemma establishes a bijection of eAf onto, what might be termed, the set of relative semi-orthocomplements of Af in $Ae \oplus Af$.

LEMMA 2. *Let $E = Ae$ and $F = Af$ with $E \# F$.*

(1) *If $G \oplus F = E \oplus F$ with $G \in L(A)$, then $G = A(e - a)$ for some*

unique $a \in eAf$.

(2) *If* $a \in eAf$, *then there exists a left ideal* $G \in L(A)$ *such that*

(i) $G = A(e - a)$.

(ii) $G \oplus F = E \oplus F$.

(iii) $E \vee G = E \oplus LR(a)$.

(iv) $E \cap G = E \cap L(a)$.

Proof. To prove (1), let g be an idempotent generator for G. Choose w and x in A such that $e = wg + xf$. Then $e = ewg + exf$. Put $a = exf$. Then $e - a = ewg \in G$; so $A(e - a) \leqq G$. Conversely, $g = ye + zf = y(e - a) + ya + zf = yewg + ya + zf$ for some $y, z \in A$. But $g - yewg = ya + zf \in G \cap F = (0)$, so that $g = yewg = y(e - a)$. Hence $G = Ag \leqq A(e - a)$.

If also $b \in F = Af$ with $e - b \in G$, then $a - b = (e - b) - (e - a) \in G \cap F = (0)$; so $a = b$. This establishes the uniqueness of a.

To prove (2), let e_0 and f_0 denote orthogonal idempotent generators for E and F respectively. Put $g = e_0 - e_0a$ and $G = Ag$. Since $ae_0 = afe_0 = aff_0e_0 = 0$, we find that $g = g^2$. Thus $G \in L(A)$. Now $g = e_0(e - a)$ and $e - a = e(e_0 - e_0a) = eg$ implies $G = Ag = A(e - a)$, proving (i). The remaining parts of (2) are straightforward computations.

THEOREM 2. *Let* $Ae \# Af$. *Then* $Ae \sim Af$ *if and only if* $e \sim f$.

Proof. Suppose $a, b\colon e \sim f$. Put $G = A(e - a)$ and $H = A(f - b)$. Then by Lemma 2 (2), $G \oplus Af = Ae \oplus Af = Ae \oplus H$. But $e - a = ab - a = a(b - f) = -a(f - b)$ and $f - b = ba - b = b(a - e) = -b(e - a)$, showing that $G = A(e - a) = A(f - b) = H$. Thus $Ae \oplus G = Ae \oplus Af = G \oplus Af$.

4. Regularity. In this section, we characterize those Rickart rings A in which $E \cap F = (0)$ implies $E \# F$ for all E and F in $L(A)$. It will be convenient in the two lemmas and in Theorem 1 to adopt some notation. Let a and b denote regular elements with $Ae = Aa$ and $fA = bA$. Choose a^+ and b^+ by Lemma 3.1 so that $a^+a = e$ and $bb^+ = f$; choose idempotent generators g and h of $LR(ab)$ and $RL(ab)$ respectively. In the context of Rickart $*$-semigroups, Theorem 1 is due to D. J. Foulis [2].

LEMMA 1. *If* eb *or* af *is regular, then so is* ab.

Proof. Suppose eb is regular. Choose an idempotent generator k for Aeb and choose $(eb)^+$ so that $(eb)^+eb = k$. Put $x = (eb)^+a^+h$. Then $xab = (eb)^+a^+hab = (eb)^+a^+ab = (eb)^+eb = k$. Then $abxab = abk = (ae)bk = a(eb)k = a(eb) = (ae)b = ab$, showing that ab is regular. The argument for af is similar.

LEMMA 2. *If ab is regular, so are eb and af.*

Proof. Choose $(ab)^+$ so that $ab(ab)^+ = h$. Let k denote an idempotent generator of $LR(ef)$ and put $x = kb(ab)^+$. Then $afx = afkb(ab)^+ = (ae)fkb(ab)^+ = a(ef)kb(ab)^+ = a(ef)b(ab)^+ = (ae)fb(ab)^+ = afb(ab)^+ = ab(ab)^+ = h$. Hence $afxaf = haf = habb^+ = abb^+ = af$, showing that af is regular. Similarly eb is regular.

THEOREM 1. *ab is regular if and only if ef is regular.*

Proof. If ab is regular, then so is eb by Lemma 2. Since eb is regular, so is ef by Lemma 2 again, applied with $a = e$.

Conversely, if ef is regular, then so is eb by Lemma 1, applied with $a = e$. Then since eb is regular, so is ab by Lemma 1 again.

THEOREM 2. *These conditions are equivalent:*

(1) *ef is regular for every idempotent e and f.*

(2) *If a and b are regular, then so is ab.*

(3) *If $E \cap F = (0)$, then $E \sharp F$.*

Moreover, if A is a matrix ring, we may add

(4) *A is a regular ring.*

Proof. The equivalence of (1) and (2) is a consequence of Theorem 1. That (1) implies (3) is a consequence of Theorem 2.1 (2). Using the notation of the proof of Corollary 2.1, we may show that (3) implies (1); with $E = Ae$ and $F = Af$, we have $Ae_0 \cap Af_0 = (0)$ as before. Then by (3), $Ae_0 \sharp Af_0$. Consequently, $e_1(1 - f_1) = e_0(1 - f_0)$ is regular by Theorem 2.1, and hence $e(1 - f)$ is regular. Thus (3) implies $e(1 - f)$ is regular for every idempotent e and f, and this is evidently equivalent to (1).

Let us now suppose that A is a Rickart matrix ring of order $\geqq 2$. If A is a regular ring, then $E \cap F = (0)$ implies $E \sharp F$ for all E and F in $L(A)$ by Theorem 2.1. Conversely, if this condition holds for all E and F in $L(A)$, we show that A is a regular ring. To this end, let e_{ij}, $1 \leqq i, j \leqq n$, be a family of matrix units for A. We shall show that $e_{11}Ae_{11}$ and hence A, which is isomorphic to the $n \times n$ matrix ring over $e_{11}Ae_{11}$, is a regular ring.

Let $e_{11}xe_{11}$ denote an arbitrary element in $e_{11}Ae_{11}$; put $a = e_{11}xe_{12}$ and choose idempotent generators e and f for $RL(a)$ and $LR(a)$ respectively. Since $R(f) = R(a)$, $ae_{ii} = 0$ for $i \neq 2$ implies $fe_{ii} = 0$ for $i \neq 2$; since $L(e) = L(a)$, $e_{22}a = 0$ implies $e_{22}e = 0$. Thus $fe = f(\Sigma e_{ii})e = (\Sigma fe_{ii})e = (fe_{22})e = f(e_{22}e) = 0$, showing that $Ae \cap Af = (0)$. Moreover $f(1 - e) = f$ is regular. Hence $Ae \sharp Af$.

Now let e_0 and f_0 denote orthogonal idempotents generating Ae and Af respectively. Put $g = e_0 - e_0a$. Then, as in the proof of Lemma 3.2, $a = e(1 - g)$ and $Ag = A(e - a)$. Thus $Ae \cap Ag = Ae \cap L(a) = Ae \cap L(e) = (0)$. Then by hypothesis, $Ae \# Ag$. But this means that $a = e(1 - g)$ is regular in A. Choose an element b in A with $aba = a$.

Then

$$(e_{11}xe_{12})b(e_{11}xe_{12}) = aba = a = e_{11}xe_{12}$$

or equivalently

$$(e_{11}xe_{12})b(e_{11}xe_{11}) = e_{11}xe_{11}.$$

Thus

$$(e_{11}xe_{11})(e_{12}be_{11})(e_{11}xe_{11}) = e_{11}xe_{11},$$

showing that $e_{11}xe_{11}$ is a regular element of $e_{11}Ae_{11}$.

Hence $e_{11}Ae_{11}$ is a regular ring.

Recall that two left ideals in a von Neumann algebra A are semi-orthogonal if and only if their unique generating projections are non-asymptotic. Therefore, a von Neumann matrix algebra with no asymptotic pairs of projections must be regular and hence finite dimensional [8, pp. 85–87]. The definitive result in the general case is due to D. M. Topping [9]. Topping shows that in a von Neumann algebra these conditions are equivalent: (1) A has no asymptotic pairs of projections; (2) A contains no infinite orthogonal sequence of non-abelian projections; (3) A is the direct sum of an abelian subalgebra and a finite dimensional subalgebra. As a consequence of this result, a type II_1 von Neumann algebra may contain asymptotic pairs of projections, although its projection lattice is necessarily modular. Thus semi-orthogonality and dual modularity are in general distinct concepts. Using Foulis' characterization of dual modularity in terms of range-closedness, this same example shows that the product of two projections in a von Neumann algebra may have a closed range without being $*$-regular.

A simple proof, in the spirit of this paper, of (1) implies (2) in Baer $*$-rings would be worthwhile; for this would show that a complete $*$-regular ring can contain no infinite orthogonal sequence of non-abelian projections and hence no infinite orthogonal sequence of equivalent projections. A complete $*$-regular ring must, therefore, be of finite type. This is a difficult step in Irving Kaplansky's proof [3] that an orthocomplemented complete modular lattice is a continuous geometry.

REFERENCES

1. J. Feldman, *Isomorphisms of rings of operators*, Thesis, University of Chicago, 1954.

2. D. J. Foulis, *Relative inverses in Baer $*$-semigroups*, Mich. Math. J., **10** (1963), 65-84.
3. I. Kaplansky, *Any orthocomplemented complete modular lattice is a continuous geometry*, Ann. of Math., **61** (1955), 524-541.
4. ———, *Rings of Operators*, New York: W. A. Benjamin, Inc., 1968.
5. G. W. Mackey, *On infinite dimensional linear spaces*, Trans. Amer. Math. Soc., **57** (1945), 155-207.
6. S. Maeda, *On relatively semi-orthocomplemented lattices*, J. Sci. Hiroshima Univ., **24A** (1960), 155-161.
7. D. D. Miller and A. H. Clifford, *Regular* D-*classes in semigroups*, Trans. Amer. Math. Soc., **82** (1956), 270-280.
8. J. von Neumann, *Continuous Geometry*, Princeton: Princeton Univ. Press, 1960.
9. D. M. Topping, *Asymptoticity and semimodularity in projection lattices*, Pacific. J. Math., **20** (1967), 317-325.

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