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ON SPLITTING IN HEREDITARY TORSION THEORIES

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Let $(\mathscr{T}, \mathscr{F})$ denote a hereditary torsion theory for the category of modules over a ring R. In this paper the splitting of projective modules is studied, and it is shown that this is not equivalent to the splitting of quasi-projective modules. In addition, situations arising from the class of torsion modules $\mathscr{T}$ (or the class of torsionfree modules $\mathscr{F}$) being contained in the injective or in the projective modules are considered, and several conditions sufficient for an especially strong form of splitting are given. Finally when $\mathscr{T}$ is closed under injective envelopes the following is shown: every module splits if R is an artinian generalized uniserial ring, and projective modules split if R is a QF-2 ring.

The term "ring" will mean an associative ring with unity 1, and all modules are assumed to be unitary left modules. We denote the category of all modules over a ring R by ${}_R\mathscr{M}$. Dickson [6] defined a torsion theory for ${}_R\mathscr{M}$ to be a pair $(\mathscr{T}, \mathscr{F})$ of classes of modules satisfying the following:

(a) $\mathscr{T} \cap \mathscr{F} = 0$;

(b) $\mathscr{T}$ is closed under homomorphic images and $\mathscr{F}$ is closed under submodules;

(c) For each module M there exists a (unique) submodule $M_t \in \mathscr{T}$ such that $M/M_t \in \mathscr{F}$.

A torsion theory $(\mathscr{T}, \mathscr{F})$ is said to be *hereditary* if $\mathscr{T}$ is closed under submodules, and *stable* if $\mathscr{T}$ is closed under injective envelopes. We remark that from (b) above it is clear that $\text{Hom}(T, F) = 0$ for all $T \in \mathscr{T}$ and all $F \in \mathscr{F}$; also Dickson has shown that $\mathscr{T}$ is closed under submodules if and onlf if $\mathscr{F}$ is closed under injective envelopes. In this paper we shall always be concerned with hereditary torsion theories.

If $\mathscr{T}$ is a hereditary torsion class, then Gabriel [8] has shown that $\mathscr{T}$ is uniquely associated with an (topologizing and) idempotent *filter*

$F(\mathscr{T}) = \{L \subseteqq R \mid L$ is a left ideal of R and $R/L \in \mathscr{T}\}$. Moreover, $\mathscr{T}$ is a torsionfree class for some torsion class $\mathscr{C}$ if and only if $F(\mathscr{T})$ contains a unique minimal left ideal (see [9]); in this case Jans has called $\mathscr{T}$ a *torsion-torsionfree* (TTF) *class*, and we shall call $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{T})$ the torsion theories *associated* with $\mathscr{T}$. If R is a right perfect ring, Alin [1] has shown that every hereditary torsion class for ${}_R\mathscr{M}$ is a TTF class.

If $(\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory for ${}_R\mathscr{M}$ and if $M \in {}_R\mathscr{M}$, we say that M *splits* provided $M = M_t \oplus M'$; we shall call $(\mathscr{T}, \mathscr{F})$ *splitting* if every module in ${}_R\mathscr{M}$ splits. We say that $(\mathscr{T}, \mathscr{F})$ is *centrally splitting* provided $\mathscr{T}$ is a TTF class with associated torsion theories $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{T})$, and $M = M_t \oplus M_c$ (i.e., M is the direct sum of its two torsion submodules) for every $M \in {}_R\mathscr{M}$. Centrally splitting is clearly a strong form of splitting; the interested reader may see [5] for more information on this topic.

1. Splitting in projective modules. In this section we shall study the dual for projective modules to the following result of Armendariz [3] on the splitting of injective modules. We denote the injective envelope of a module M by $E(M)$.

THEOREM A (Armendariz). *If $(\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory, then the following are equivalent:*

(1) *$\mathscr{T}$ is stable;*
(2) *Every injective module splits;*
(3) *Every quasi-injective module splits;*
(4) *$E(M_t) = E(M)_t$ for every $M \in {}_R\mathscr{M}$.*

If N is a submodule of the module M, we call N *invariant* in M provided that $f(N) \subseteq N$ for every endomorphism f of M. We call N *small* in M provided that if K is a submodule of M and if $K + N = M$, then $K = M$. We shall say that a class $\mathscr{C}$ of modules is *closed under projective covers* provided that whenever $M \in \mathscr{C}$ has a projective cover $P(M)$, then $P(M) \in \mathscr{C}$.

THEOREM 1.1. *Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_R\mathscr{M}$ such that every torsionfree module has a projective cover. Then the following are equivalent:*

(1) *$\mathscr{F}$ is closed under projective covers;*
(2) *Every projective module splits.*

Proof. (1) $\rightarrow$ (2): Let Q be a projective module, and let $\pi: P(Q/Q_t) \rightarrow Q/Q_t$ be the projective cover of Q/Q_t. Let n be the natural epimorphism of Q onto Q/Q_t. By [4, Lemma 2.3] there exists a monomorphism $h: P(Q/Q_t) \rightarrow Q$ such that $nh = \pi$ and such that $Q = Imh + Q'$, where $Q' \subseteq \operatorname{Ker} n = Q_t$. But Imh is torsionfree; so that $Imh \cap Q_t = 0$ and $Q = Imh \oplus Q_t$.

(2) $\rightarrow$ (1): Choose $M \in \mathscr{F}$, and let $\pi: P(M) \rightarrow M$ be the projective cover of M. Then $P(M)_t \subseteq \operatorname{Ker} \pi$, so that $P(M)_t$ is small in $P(M)$. But $P(M)$ splits by hypothesis; thus $P(M) \in \mathscr{F}$.

EXAMPLE 1.2. The splitting of projective modules does not imply the splitting of quasi-projective modules in left artinian generalized uniserial rings.

Let K be a field, and let R be the ring of 4×4 upper triangular matrices over K. Let

$$I = \left\{ \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{vmatrix} \,\middle|\, a_{ij} \in K \right\};$$

then I is an idempotent, two-sided ideal of R. Thus by a result of Jans [9], $\mathscr{T} = \{M \in {}_R\mathscr{M} \mid IM = 0\}$ is a TTF class. Further $R \in \mathscr{F}$, so that every free module is torsionfree. Hence every projective module is torsionfree, and thus splits. Now let e_{ij} denote the matrix with 1 in the i^{th} row and j^{th} column and 0 elsewhere, and let $J = Re_{14}$. Then J is a two-sided ideal of R, and hence $M = Re_{44}/Je_{44} = Re_{44}/J$ is quasi-projective and indecomposable. But $M \notin \mathscr{T}$, and $Re_{24}/J \subseteqq M_t$. Thus M_t is a nontrivial submodule of M.

We next turn our attention to the quasi-projective cover; this was introduced in [12], and there it was shown that a sufficient (but not necessary) condition for the quasi-projective cover of a module M to exist is that the projective cover of M exist.

PROPOSITION 1.3. *Let M be a quasi-projective module which has a projective cover. If N is an invariant submodule of M, then the module M/N is quasi-projective.*

Proof. Let $\pi: P(M) \to M$ be the projective cover of M, and choose an endomorphism f of $P(M)$. By [12, Proposition 2.2], f induces an endomorphism g of M such that $g\pi = \pi f$. Let $K = \pi^{-1}(N)$; then $\pi f(K) = g\pi(K) = g(N) \subseteqq N$, and hence $f(K) \subset \pi^{-1}(N) = K$. We have shown that K is invariant in $P(M)$; thus by [12, Proposition 2.1] we have $P(M)/K \cong M/N$ is quasi-projective.

THEOREM 1.4. *Let M be a module with a projective cover, let $\pi': QP(M) \to M$ denote the quasi-projective cover of M, and let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_R\mathscr{M}$. If $M \in \mathscr{F}$, then $QP(M) \in \mathscr{F}$.*

Proof. Let $\pi: P(M) \to M$ denote the projective cover of M; by [12, Propositions 2.6, 2.1 and 2.2] we have that $QP(M) \cong P(M)/X$, where X is the unique maximal invariant submodule of $P(M)$ contained in Ker π. Let n denote the natural epimorphism of $P(M)$ onto

$QP(M)$. Since $\operatorname{Ker} n \subseteqq \operatorname{Ker} \pi$, we have that $\operatorname{Ker} n$ is small in $P(M)$, and thus $n: P(M) \to QP(M)$ is the projective cover of $QP(M)$. Further $QP(M)_t \subseteqq \operatorname{Ker} \pi'$ since $M \in \mathscr{F}$, and also $QP(M)_t$ is invariant in $QP(M)$. Hence $QP(M)/QP(M)_t$ is quasi-projective by Proposition 1.3; thus $QP(M)_t = 0$ by condition (3) of the definition of the quasi-projective cover in [12].

2. Classes of projective and injective modules. Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_R\mathscr{M}$. In this section we investigate the following condition:

$\mathscr{T}$ is stable and all torsionfree modules are injective. This has been studied previously in [3] (also see [2] for the special case that $\mathscr{T}$ is the Goldie torsion class), where it was shown to imply that $(\mathscr{T}, \mathscr{F})$ is splitting. In Theorem 2.2 we shall give a statement equivalent to this condition, and, in addition, we shall show that it implies the much stronger result: $(\mathscr{T}, \mathscr{F})$ is centrally splitting. Finally we shall obtain a dual to Theorem 2.2.

LEMMA 2.1. *Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_R\mathscr{M}$, let $R = R_t \oplus K$, and let $\mathscr{F}$ be closed under homomorphic images. Then $R = R_t \dotplus K$(ring direct sum), $\mathscr{T}$ is a* TTF *class, and $(\mathscr{T}, \mathscr{F})$ is centrally splitting.*

Proof. Since right multiplication by an element of R is a left R-homomorphism on K, and since $\mathscr{F}$ is closed under homomorphic images, K is a two-sided ideal of R and $R = R_t \dotplus K$.

By [5, Theorem 1] it now suffices to see that $\mathscr{T}$ is a TTF class. Choose $L \in F(\mathscr{T})$; then $K \cap L \in F(\mathscr{T})$, and hence $R/K \cap L \in \mathscr{T}$. Thus $K/K \cap L \in \mathscr{T}$. But $K \to K/K \cap L \to 0$ is exact and $K \in \mathscr{F}$; thus $K/K \cap L \in \mathscr{T} \cap \mathscr{F} = 0$ and $K = K \cap L \subseteqq L$. We have shown that K is the unique minimal ideal in $F(\mathscr{T})$; thus $\mathscr{T}$ is a TTF class.

THEOREM 2.2. *If $(\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory for ${}_R\mathscr{M}$, then the following are equivalent:*

(1) *$\mathscr{T}$ is stable, and all torsionfree modules are injective;*

(2) *$\mathscr{F}$ is closed under homomorphic images, and all torsionfree modules are projective;*

(3) *$\mathscr{F}$ is clossed under homomorphic images, $R = R_t \dotplus K$ (ring direct sum), and K is a semi-simple ring with minimum condition.*

In addition, whenever (1), (2), *and* (3) *are true, then $\mathscr{T}$ is a* TTF *class and $(\mathscr{T}, \mathscr{F})$ is centrally spilitting.*

Proof. (1) $\to$ (3) follows from [3, Theorem 3.2].

(3) $\to$ (2): If M is a torsionfree module, then $R_t M = 0$ and hence

M is a projective K-module. But now M is a direct summand of a free K-module, and hence M is a direct summand of a free R-module. Thus M is projective as an R-module.

$(2) \to (1)$: Choose $M \in \mathscr{T}$, and let n be the natural epimorphism of $E(M)$ onto $E(M)/E(M)_t$. Since this torsionfree module is projective, there exists a monomorphism f from $E(M)/E(M)_t$ into $E(M)$ such that $E(M) = \operatorname{Ker} n \oplus \operatorname{Im} f$. But $M \subseteq \operatorname{Ker} n$ and M is large in $E(M)$; hence $\operatorname{Im} f = 0$ and $E(M) = E(M)_t \in \mathscr{T}$. Thus $\mathscr{T}$ is stable. Now choose $M \in \mathscr{F}$; then $E(M) \in \mathscr{F}$ and so the module $E(M)/M$ is torsionfree, and hence projective. Thus $E(M) \cong M \oplus E(M)/M$. This proves that M is injective.

The final statement follows from Lemma 2.1.

THEOREM 2.3. *Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_R\mathscr{M}$ for which cyclic torsionfree modules have projective covers; the following are equivalent:*

(1) *$\mathscr{F}$ is closed under projective covers, and every torsion module is projective;*

(2) *$\mathscr{F}$ is closed under homomorphic images, and every torsion module is injective;*

(3) *$\mathscr{F}$ is closed under homomorphic images, $R = R_t \dotplus K$ (ring direct sum), and R_t is a semi-simple ring with minimum condition.*

In addition, whenever (1), (2), *and* (3) *are true, then $\mathscr{T}$ is a* TTF *class and $(\mathscr{T}, \mathscr{F})$ is centrally splitting.*

Proof. $(1) \to (3)$: Choose $N \in \mathscr{F}$, and let L be a homomorphic image of N. Since L_t is projective, there exists a monomorphism f from L_t to N. But $\operatorname{Hom}(L_t, N) = 0$; thus $L_t = 0$ and $L \in \mathscr{F}$. Thus $\mathscr{F}$ is closed under homomorphic images.

Since R/R_t is a cyclic module, it has a projective cover $\pi: P(R/R_t) \to R/R_t$, and $P(R/R_t) \in \mathscr{F}$ by hypothesis. If n denotes the natural epimorphism from R onto R/R_t, then there exists a homomorphism $f: P(R/R_t) \to R$ such that $R = \operatorname{Im} f + \operatorname{Ker} n = \operatorname{Im} f + R_t$. But $\operatorname{Im} f \in \mathscr{F}$, so that $R_t \cap \operatorname{Im} f = 0$ and $R = R_t \oplus \operatorname{Im} f$. Thus $R = R_t \dotplus K$ — and we also get the final statement of the theorem — by Lemma 2.1.

Finally, it is easy to see that R_t is a completely reducible ring since every torsion module is projective; this is equivalent to saying that R_t is a semi-simple ring with minimum condition.

$(3) \to (2)$: If $M \in \mathscr{T}$, then $KM = 0$ since $\mathscr{F}$ is closed under homomorphic images. Hence M is an injective R_t-module, and, by Baer's Lemma, it is easy to see that M is an injective R-module.

$(2) \to (1)$: Let $M \in \mathscr{F}$ have a projective cover $\pi: P(M) \to M$; then

$P(M)_t$ is injective and $P(M) = P(M)_t \oplus P'$. Further, $\mathrm{Hom}(P(M)_t, M) = 0$ and thus $P(M)_t \subseteq \mathrm{Ker}\,\pi$. Hence $P(M)_t$ is small in $P(M)$, and $P(M) = P' \in \mathscr{F}$. Thus $\mathscr{F}$ is closed under projective covers.

Since R_t is injective, we have $R = R_t \oplus K$. Thus, by Lemma 2.1, we have that $R = R_t \dotplus K$. Since $\mathscr{F}$ is closed under homomorphic images, one can easily see that $M \in \mathscr{T}$ if and only if $KM = 0$. But if every R_t-module is injective, then every R_t-module is projective. Thus every torsion R-module is projective.

3. Stable torsion theories. In [5] the following result is given; its proof depends strongly upon the dualities present in quasi-Frobenius rings.

THEOREM B. *Let R be a quasi-Frobenius ring and let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_R\mathscr{M}$. The following are equivalent:*

(1) *$\mathscr{T}$ is stable;*

(2) *$(\mathscr{T}, \mathscr{F})$ is splitting;*

(3) *$(\mathscr{T}, \mathscr{F})$ is centrally splitting.*

It is easily seen that the implications $(3) \rightarrow (2) \rightarrow (1)$ are always true, regardless of the type of ring involved. We are motivated to examine the remaining implications in types of left artinian rings more general that the quasi-Frobenius ones, especially since Fuller [7] has shown that QF-3 rings possess dualities somewhat similar to those in quasi-Frobenius rings.

THEOREM 3.1. *Let R be a left artinian generalized uniserial ring, and let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_R\mathscr{M}$. Then $\mathscr{T}$ is stable if and only if $(\mathscr{T}, \mathscr{F})$ is splitting.*

Proof. We need only consider the case where $\mathscr{T}$ is stable. Now every module M is a direct sum of indecomposable cyclic submodules, and each of these submodules is a homomorphic image of a left ideal Re where e is a primitive idempotent of R [10]. But each such Re has a lattice of submodules which is a finite chain, and thus every homomorphic image of an Re has a lattice of submodules which is a finite chain.

If L is an indecomposable cyclic submodule of M, then by the preceding its socle, denoted $\mathrm{soc}(L)$, is simple. Thus either $\mathrm{soc}(L) \in \mathscr{T}$ or $\mathrm{soc}(L) \in \mathscr{F}$. But $\mathrm{soc}(L)$ is large in L, so that L is contained in the injective envelope of $\mathrm{soc}(L)$. By hypothesis either $E(\mathrm{soc}(L)) \in \mathscr{T}$ or $E(\mathrm{soc}(L)) \in \mathscr{F}$; thus either $L \in \mathscr{T}$ or $L \in \mathscr{F}$. Hence M splits.

EXAMPLE 3.2. Splitting does not imply centrally splitting in left artinian generalized uniserial rings.

Let K be a field, and let R be the ring of two by two upper triangular matrices over K. Let

$$I = \left\{ \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} \mid a, b \in K \right\};$$

then I is an idempotent, two-sided ideal of R. Thus by Jans [9],

$$\mathscr{T} = \{M \in {}_R\mathscr{M} \mid IM = 0\}$$

is a TTF class with associated torsion theories $(\mathscr{T}, \mathscr{F})$ and $(\mathscr{C}, \mathscr{T})$, where

$$\begin{aligned} \mathscr{F} &= \{N \in {}_R\mathscr{M} \mid \operatorname{Hom}(T, N) = 0 \quad \text{for all } T \in \mathscr{T}\} \text{ and} \\ \mathscr{C} &= \{L \in {}_R\mathscr{M} \mid \operatorname{Hom}(L, T) = 0 \quad \text{for all } T \in \mathscr{T}\} \\ &= \{L \in {}_R\mathscr{M} \mid IL = L\}. \end{aligned}$$

Clearly $(\mathscr{C}, \mathscr{T})$ does not split, since $R_c = I$ is not a direct summand of R. Hence $\mathscr{T}$ is not centrally splitting.

Note that $F(\mathscr{T}) = \{I, R\}$; thus for $M \in {}_R\mathscr{M}$, $M_t = \{x \in M \mid (0: x) \in F(\mathscr{T})\} = \{x \in M \mid I \subseteqq (0: x)\}$, where $(0: x) = \{r \in R \mid rx = 0\}$. Since I is the only large proper left ideal of R, we see that M_t is the singular submodule $Z(M)$ of M. Also $Z(R) = 0$, so that $\mathscr{T}$ is the Goldie — and $E(R)$ — torsion class (see [1] and [9] for an explanation of these). It is well-known that the Goldie torsion class is stable; thus $(\mathscr{T}, \mathscr{F})$ splits by Theorem 3.1.

As an aside, we note that the class $\mathscr{C}$ above is hereditary but is not stable. Also we remark that Teply [11, Propositions 4.5 and 4.7] gives several necessary and sufficient conditions for splitting to imply centrally splitting.

PROPOSITION 3.3. *Let R be a QF-2 ring, and let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_R\mathscr{M}$. If $\mathscr{T}$ is stable, then every projective module splits.*

Proof. If e is a primitive idempotent in R, then $\operatorname{soc}(Re)$ is both a simple module and is large in Re. Hence Re is contained in the injective envelope of $\operatorname{soc}(Re)$, and thus either $Re \in \mathscr{T}$ or $Re \in \mathscr{F}$. But any projective module P over a left artinian ring R is isomorphic to a direct sum of modules Re_α, where each e_α is a primitive idempotent of R. Thus every projective module splits.

If $\mathscr{T}$ is a stable hereditary torsion class for a QF-2 ring, then, by Theorem A and Proposition 3.3, every quasi-injective and every projective module splits. It seems reasonable to conjecture that every module will split, and in fact we have been unable to find examples to the contrary.

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