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RATIONAL EXTENSIONS OF MODULES

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It is shown, that a module B is a rational extension of a submodule A if and only if B/A is a torsion module with respect to the largest torsion theory for which B is torsion-free. The rational completion of a module can thus be viewed as a module of quotients. The behavior of rationally complete modules under the formation of direct sums and products is studied. It is also shown, that a module is rationally complete provided it contains a copy of every nonprojective simple module.

In the second part of the paper, rational extensions of modules over a left perfect ring are studied. Necessary and sufficient conditions are given for a semi-simple module to be rationally complete. This characterization depends only on the idempotents of the ring. If R is left and right perfect and if every simple right module is rationally complete, then every module is rationally complete.

1. Filters and rational extensions. We first recall a number of definitions and results concerning filters of ideals and torsion theories. Our main reference is Lambek [15], whose terminology we follow. The reader may wish to consult some related papers, e.g., [6, 11, 12, 16, 17].

All rings have a unit element, all modules are unital and, unless otherwise stated, "module" means "right R-module".

A set $\mathfrak{F}$ of right ideals of R is called a *filter*, if the following conditions are satisfied:

(1) Every right ideal containing a member of $\mathfrak{F}$ belongs to $\mathfrak{F}$.

(2) $\mathfrak{F}$ is closed under finite intersections.

(3) If $I \in \mathfrak{F}$ and $r \in R$, then $r^{-1}I = \{x \in R \mid rx \in I\} \in \mathfrak{F}$. The filter $\mathfrak{F}$ is called *idempotent*, if.

(4) If $I \in \mathfrak{F}$ and J is a right ideal, such that $a^{-1}J \in \mathfrak{F}$ for all $a \in I$, then $J \in \mathfrak{F}$.

The set of filters is partially ordered by inclusion. The filter $\mathfrak{F}_0$ consisting of R alone is the smallest, the filter $\mathfrak{F}_\infty$ consisting of all right ideals is the largest filter. Both are idempotent.

Let $\mathfrak{F}$ be an idempotent filter and M a module. We define the *$\mathfrak{F}$-torsion submodule* $\mathfrak{F}(M)$ to be the set of all $m \in M$, whose annihilator is in $\mathfrak{F}$. M is said to be *$\mathfrak{F}$-torsion* if $\mathfrak{F}(M) = M$ and *$\mathfrak{F}$-torsion-free* if $\mathfrak{F}(M) = 0$. The module $M/\mathfrak{F}(M)$ is always $\mathfrak{F}$-torsion-free.

A module is called *$\mathfrak{F}$-divisible* if every homomorphism from I to M, where $I \in \mathfrak{F}$ can be extended to a homomorphism from R to M

or, equivalently, if $E(M)/M$ is $\mathfrak{F}$-torsion-free. Here and throughout the paper, $E(M)$ denotes the injective hull of M.

If M is $\mathfrak{F}$-torsion-free, then there is a unique (up to isomorphism) module $Q_{\mathfrak{F}}(M)$ (hereafter denoted by Q) containing M such that (i) $M \subseteqq Q$ is essential, (ii) Q/M is $\mathfrak{F}$-torsion and (iii) Q is $\mathfrak{F}$-divisible. Q is $\mathfrak{F}$-torsion-free and it is given by the formula $Q/M = \mathfrak{F}(E(M)/M)$ or, explicitly, $Q = \{x \in E(M) \mid x^{-1}M \in \mathfrak{F}\}$, where $x^{-1}M = \{r \mid xr \in M\}$. For an arbitrary M, we define $Q_{\mathfrak{F}}(M)$ to be $Q_{\mathfrak{F}}(M/\mathfrak{F}(M))$. It can be shown, that $Q_{\mathfrak{F}}(M) = \lim_{\rightarrow} \mathrm{Hom}_R(I, M/\mathfrak{F}(M))(I \in \mathfrak{F})$. Q is called the *module of quotients* of M with respect to $\mathfrak{F}$.

Given any M, the set $\mathfrak{F}_M = \{I \mid \mathrm{Hom}_R(R/I, E(M)) = 0\}$ is an idempotent filter and it is the largest among all idempotent filters $\mathfrak{G}$ such that M is $\mathfrak{G}$-torsion-free.

PROPOSITION 1.1.

(a) *$I \in \mathfrak{F}_M$ if and only if $xI \neq 0$ for every $x \in E(M)$, $x \neq 0$.*

(b) *$I \in \mathfrak{F}_M$ if and only if for all $m \in M, m \neq 0$, and all $r \in R$ there is an $s \in R$ such that $ms \neq 0$, $rs \in I$.*

Proof. (a) There is a nonzero map $R/I \rightarrow E(M)$ if and only if there is an $x \in E(M), x \neq 0$, such that $xI = 0$. (b) is proved in [15, Prop. 0.2].

A module P is $\mathfrak{F}_M$-torsion if and only if $\mathrm{Hom}_R(P, E(M)) = 0$. As an example, consider $\mathfrak{F}_R = \mathfrak{F}_{E(R)}$. This is the filter of dense right ideals [14, p. 96].

LEMMA 1.2.

(a) *$L \subseteqq M$ implies $\mathfrak{F}_L \supseteqq \mathfrak{F}_M$.*

(b) *If $L \subseteqq M$ is an essential extension, then $\mathfrak{F}_L = \mathfrak{F}_M$.*

(c) *If $\{M_\alpha\}$ is an arbitrary family of modules and if $S = \sum_\alpha M_\alpha$, $P = \Pi_\alpha M_\alpha$, then $\mathfrak{F}_S = \mathfrak{F}_P = \bigcap_\alpha \mathfrak{F}_{M_\alpha}$.*

The proof is straightforward by (1.1, b).

In [9], Findlay and Lambek define a relation between three modules. The write $A \leqq B(M)$ if $A \subseteqq B$ and if for every homomorphism $\phi: C \rightarrow M$, where $A \subseteqq C \subseteqq B, \phi(A) = 0$ implies $\phi = 0$. Equivalently, $A \leqq B(M)$ if and only if $\mathrm{Hom}_R(B/A, E(M)) = 0$, i.e., if B/A is $\mathfrak{F}_M$-torsion.

$A \subseteqq B$ is called a *rational extension* if $A \leqq B(B)$. In view of the remarks above and [9, Prop. 2.2], we have.

LEMMA 1.3. *The following statements are equivalent:*

(a) *$A \subseteqq B$ is a rational extension.*

(b) *$A \subseteqq B$ is an essential extension and B/A is $\mathfrak{F}_A$-torsion.*

(c) *$A \subseteqq B$ and B/A is $\mathfrak{F}_B$-torsion.*

A module is called *rationally complete* if it has no proper rational extensions. In particular, an injective module is rationally complete. Findlay and Lambek [9] have shown, that every module has a maximal rational extension $\bar{M}$ which is rationally complete. $\bar{M}$ is unique up to isomorphism and it is called the *rational completion* of M.

PROPOSITION 1.4.
(a) *$\bar{M}$ is the module of quotients $Q_{\mathfrak{F}_M}(M)$,*
(b) $\bar{M} = \{x \in E(M) \mid x^{-1}M \in \mathfrak{F}_M\}$.

Proof. By (1.3) $\bar{M}$ satisfies properties (i) and (ii) of the module of quotients. Also by (1.3) $\bar{M} \subseteqq E(M)$. Let now $T/\bar{M}$ be the $\mathfrak{F}_M$-torsion-submodule of $E(M)/\bar{M}$, then T is a rational extension of $\bar{M}$ by (1.3), since $\mathfrak{F}_M = \mathfrak{F}_T$. But $\bar{M}$ has no proper rational extensions, hence $T = \bar{M}$ and $E(M)/\bar{M}$ is $\mathfrak{F}_M$-torsion-free, thus satisfying condition (iii). (b) is just the explicit description.

COROLLARY 1.5. *A module M is rationally complete if and only if M is $\mathfrak{F}_M$-divisible.*

PROPOSITION 1.6. (Brown [4]). *The direct product of rationally complete modules is rationally complete.*

Proof. Let $P = \Pi_\alpha M_\alpha$ and suppose each M_α is rationally complete. By (1.2, c) $\mathfrak{F}_P \subseteqq \mathfrak{F}_{M_\alpha}$ for each α. Since each M_α is $\mathfrak{F}_P$-divisible, P is also $\mathfrak{F}_P$-divisible.

Following Goldman, [12], a filter $\mathfrak{F}$ is called *Noetherian*, if it has the following property: If $I_1 \subseteqq I_2 \subseteqq \cdots$ is a (countable) ascending chain of right ideals whose union is in $\mathfrak{F}$, then some I_n is in $\mathfrak{F}$. This condition is satisfied, if every right ideal in $\mathfrak{F}$ contains a finitely generated right ideal also in $\mathfrak{F}$.

PROPOSITION 1.7. *The direct sum of any family of rationally complete modules is rationally complete if and only if every idempotent filter is Noetherian.*

Proof. Let $\{M_\alpha\}$ be a family of rationally complete modules and S be their sum. Then $\mathfrak{F}_S = \cap_\alpha \mathfrak{F}_{M_\alpha}$ by (1.2) and each M_α is $\mathfrak{F}_S$-torsion-free and $\mathfrak{F}_S$-divisible. Since $\mathfrak{F}_S$ is Noetherian by assumption, S is $\mathfrak{F}_S$-divisible by [12, Thm. 4.4], hence rationally complete by (1.5).

Conversely, suppose there is a filter $\mathfrak{F}$ which is not Noetherian. Then there is a right ideal $I \in \mathfrak{F}$ such that I is the union of a chain $I_1 \subseteqq I_2 \subseteqq \cdots$, where no I_i is in $\mathfrak{F}$.

Let M be an injective module such that $\mathfrak{F} = \mathfrak{F}_M$. Such a module exists by [12, Thm. 5.3]. M is rationally complete, but we claim, that the sum S of countably many copies of M is not rationally complete. By assumption. $I_i \notin \mathfrak{F}_M$ for all i, hence by (1.1, a), there exist nonzero elements $x_i \in M$ such that $x_i I_i = 0$. If $d \in I$, then $d \in I_k$ for some k, hence $x_i d = 0$ for $i \geqq k$ and $\phi(d) = (\cdots, x_i d, \cdots)$ defines a homomorphism $I \rightarrow S$. Suppose that S is rationally complete, then ϕ extends to a map $\phi' : R \rightarrow S$ and $\phi'(1) = (y_1, y_2, \cdots, y_n, 0, 0, \cdots)$. Thus $x_i I = 0$ for all $i > n$, contrary to the assumption.

The proof shows a little bit more: If every injective (or rationally complete) module M is countably Σ-rationally complete (i.e., a countable sum of copies of M is rationally complete), then the sum of any family of rationally complete modules is rationally complete. This is an analogue of the situation for injective modules [8, p. 205].

It is not true in general that a direct summand of a rationally complete module is rationally complete (see (1.9) below). However, we have

PROPOSITION 1.8. *Let $\{M_\alpha\}$ be a family of modules such that $\mathfrak{F}_{M_\alpha} = \mathfrak{F}$ for all α. If $S = \sum_\alpha M_\alpha$ or $P = \Pi_\alpha M_\alpha$ is rationally complete, then M_α is rationally complete for every α.*

Proof. Let $\phi : I \rightarrow M_\alpha$ be a homomorphism; where $I \in \mathfrak{F}_{M_\alpha}$. Because $\mathfrak{F}_{M_\alpha} = \mathfrak{F}_P$ by (1.2) ϕ extends to $\phi' : R \rightarrow P$ (after some identifications) and $\pi_\alpha \phi' : R \rightarrow M_\alpha$ is the desired extension of ϕ, where π_α is the projection $P \rightarrow M_\alpha$. The proof for S is similar.

PROPOSITION 1.9. *Let $\{S_\alpha\}$ be a set of nonisomorphic simple modules, representing all nonprojective simple modules. Then every module containing the module $T = \sum_\alpha S_\alpha$ is rationally complete.*

Proof. First note that a simple module $S \cong R/A$ (A a maximal right ideal) is projective if and only of it is isomorphic to a direct summand of R or if and only if A is a direct summand of R.

Suppose now $T \subseteq M$. Suppose $x \in \bar{M} \subseteq E(M)$, then $I = x^{-1}M \in \mathfrak{F}_M$. Since $E(M)$ is an essential extention of M, I is a large [14, p. 70] right ideal of R. Suppose $I \neq R$, then I is contained in a maximal right ideal A, which is also large. Furthermore, $A \in \mathfrak{F}_M$. The simple module $S = R/A$ is not projective by what has been said above, hence there is a nonzero homomorphism $R/A \rightarrow T \rightarrow M \rightarrow E(M)$, which contradicts the fact that $A \in \mathfrak{F}_M$. Hence $I = R$ and $x \in M$, i.e., $M = \bar{M}$.

A similar result was proved in [4] with $T = \sum R/L$, L running through all large right ideals.

An immediate consequence is that the class of rationally complete modules is closed under submodules (or factor modules) if and only if every module is rationally complete. A characterization of the rings with the latter property is given in [5]. As Brown [4] has remarked, (1.9) also implies that every rationally complete module is injective if and only if R is completely reducible (i.e., semisimple Artinian). We also have

COROLLARY 1.10. *Every rationally complete module is projective if and only if R is completely reducible.*

Proof. Choose one simple module from each isomorphism class and let U be their sum. Then U is rationally complete by (1.9), hence every simple module is projective. Therefore R has no proper large ideal and R is completely reducible by [14, p. 61]. The converse is obvious.

COROLLARY 1.11. *Every cogenerator is rationally complete.*

Proof. A cogenerator contains a copy of every simple module.

PROPOSITION 1.12. *Suppose every simple module is isomorphic to a minimal right ideal. Then every faithful module is rationally complete.*

Proof. If M is faithful, then $R \subseteqq \Pi M$ for some index set. Since $T \subseteqq R$ by assumption, ΠM is rationally complete and hence so is M by (1.8).

One might ask whether any module S with the property that $S \oplus M$ is rationally complete for all M, has to contain the module T defined above. This is not so in general but the following discussion shows that this situation arises for the ring $\mathbf{Z}$ of integers.

EXAMPLE 1.13. Let us first describe the idempotent filters of $\mathbf{Z}$. Given any subset $\mathscr{P}$ of the set of all primes, the ideals generated by products of powers of primes from $\mathscr{P}$ form an idempotent filter. Conversely, any idempotent filter (except $\mathfrak{F}_\infty$) is of this form. This follows from unique factorization and the fact that in a commutative Noetherian ring a filter $\mathfrak{F}$ is idempotent if and only if $I, J \in \mathfrak{F}$ implies $IJ \in \mathfrak{F}$ [17, 1.22].

If M is any nonzero Abelian group, we let $\mathscr{P}$ be the set of primes, such that $pm = 0$ ($p \in \mathscr{P}$, $m \in M$) implies $m = 0$. (Thus M is $\mathscr{P}$-torsion-free in the sense of [13]). $\mathfrak{F}_M$ is then the filter "generated" by $\mathscr{P}$. By (1.5) M is rationally complete if and only if M is divisible by all primes in $\mathscr{P}$ ($\mathscr{P}$-divisible [13]). In particular, any torsion group is rationally complete and a torsion-free group is rationally complete if and only if it is divisible.

The group T is the sum of all $Z/(p)$. We claim that T is the smallest group such that $T \oplus M$ is rationally complete for all M. Indeed, suppose U does not contain $Z/(q)$. Then U is q-torsion-free and so is $U \oplus Z$. But $U \oplus Z$ is not q-divisible, hence not rationally complete.

2. Rational extensions of modules over left perfect rings.

From now on R shall denote a left perfect ring [3] with Jacobson radical N. We shall use the following properties of R: Every right R-module M has nonzero socle Soc M and R satisfies the minimum condition on principal right ideals. Furthermore, the following facts, which are well-known for Artinian rings [2] are also true for perfect rings: The unit element of R can be written as a sum of orthogonal primitive idempotents. If e is any primitive idempotent, then eR has a unique maximal submodule eN, hence eR/eN is simple. Every simple module S is of this form: $S \cong eR/eN$ if and only if $Se \neq 0$. Two primitive idempotents e and f are called isomorphic if $eR \cong fR$, or equivalently if $eR/eN \cong fR/fN$. Furthermore e and f are isomorphic if and only if there exist $u, v \in R$ such that $e = ufv$ and $f = veu$ [14, p. 63], this shows that the concept is left-right symmetric.

From now on, we consider a fixed representation of the unit element as a sum of orthogonal primitive idempotents.

$$(*) \qquad 1 = e_{11} + \cdots + e_{1k_1} + \cdots + e_{n1} + \cdots + e_{nk_n} ,$$

where e_{ij} is isomorphic to e_{rs} if and only if $i = r$. We also set $e_i = e_{i1}$.

Once (*) is fixed there is, for any simple module S, a unique primitive idempotent e_i such that $Se_i \neq 0$. We shall say, that e_i corresponds to S. If M is any module, we let e be the sum of the e_i corresponding to the simple submodules of M. Again we say, that e corresponds to M.

LEMMA 2.1. *Let e correspond to M and let $m \in M$. Then $mRe = 0$ implies $m = 0$.*

Proof. If $m \neq 0$, then there is an $r \in R$ such that $0 \neq mr \in$ Soc M. Thus xrR is a sum of simple submodules of M and $xrRe \neq 0$.

LEMMA 2.2. *Let S be a simple module with corresponding idempotent e_i and let J be a two-sided ideal. Then $SJ = S$ if and only if $e_i \in J$.*

Proof. The "if" part is clear. To prove the "only if" part suppose $SJ \neq 0$. Now $J = fR + P$ where $f^2 = f$ and $P \subseteqq N$. (See e.g., [2, Thm. 2.6 B] for a proof in the Artinian case). f can be written as a sum of primitive idempotents $f_j \in J$ and since $SP = 0$, it follows that $Sf_k \neq 0$ for some k. Therefore f_k is isomorphic to e_i and since J is two-sided $e_i \in J$.

PROPOSITION 2.3. *Let M be any module over the left perfect ring R and let e correspond to M. Then $\mathfrak{F}_M$ consists of all right ideals containing ReR.*

Proof. *Suppose $ReR \subseteqq I$.* Since e corresponds also to $E(M)$, (2.1) implies that $xI \neq 0$ for any nonzero $x \in E(M)$. Thus $I \in \mathfrak{F}_M$ by (1.1).

Conversely, let $I \in \mathfrak{F}_M$. We claim, that the two-sided ideal $J = \{r \mid Rr \subseteqq I\}$ is also in $\mathfrak{F}_M$. To prove this, it will be sufficient to show, that $K = \{r \mid xJ = 0 \Rightarrow xr = 0 \text{ for all } x \in E(M)\}$ equals R. Suppose $K \neq R$, then we can choose an $a \notin K$, such that aR is minimal in the set $\{cR \mid c \notin K\}$ of principal right ideals. Then $a \notin J$, hence there exists a b such that $ba \notin I$. If $as \notin K$ for some $s \in S$, then $asR = aR$ by minimality. Thus $bas \in I$ implies $as \in K$ (for if not, $ba \in I$ by the preceding remark) and we conclude that $(ba)^{-1}I \subseteqq a^{-1}K = L$ and $L \in \mathfrak{F}_M$. Since $aL \subseteqq K$, $xJ = 0$ implies $xaL = 0$ and hence $xa = 0$ by (1.1, a), contradicting the assumption that $a \notin K$.

Since $J \in \mathfrak{F}_M, SJ \neq 0$ for every simple submodule S of M. Lemma (2.2) implies then $e \in J$ and it follows that $ReR \subseteqq J \subseteqq I$.

COROLLARY 2.4. *Every idempotent filter of the left perfect ring R is of the form $\{I \mid ReR \subseteqq I\}$ for some idempotent e.*

Proof. By [12, Thm. 5.3], every idempotent filter is of the form $\mathfrak{F}_M$.

This implies that the product of $\mathfrak{F}$-torsion modules is $\mathfrak{F}$-torsion for any $\mathfrak{F}$. See [1], [7].

We also point out, that ReR, where $e = e_{i_1} + \cdots + e_{i_k}$ depends only on the isomorphism classes of the e_{i_j}. Furthermore ReR contains every idempotent isomorphic to one of the e_{i_j}. Thus, the left perfect ring R possesses 2^n idempotent filters of left ideals, where n is the number of isomorphism classes of simple modules.

We shall use (2.3) to describe the rational completion $\bar{M}$ of a module M over a left perfect ring R. Note that if $\mathfrak{F} = \{I \mid ReR \subseteqq I\}$, then the $\mathfrak{F}$-torsion-submodule $\mathfrak{F}(M)$ of M is given by

$$\mathfrak{F}(M) = \{m \mid mRe = 0\} .$$

PROPOSITION 2.5. *Let R be left perfect and let M be a module* *with corresponding idempotent e.*

(a) $\bar{M} = \{x \in E(M) \mid xRe \subseteqq M\}$

(b) *If M is semi-simple, then* $\bar{M} = \{x \in E(M) \mid xReN = 0\}$.

Proof. (a) follows from (1.4, b) and (2.3). To prove (b), note that $xRe \subseteqq M$ if and only if $xReN = 0$.

LEMMA 2.6. *Let M be any module with corresponding idempotent e. Let I, J be right ideals of R. Then*

(a) $\{x \in E(M) \mid xI = 0\} = \{x \in E(M) \mid xIe = 0\}$

(b) *Suppose that* $I \subseteqq J$ *and that*

$$\{x \in E(M) \mid xI = 0\} = \{x \in E(M) \mid xJ = 0\} .$$

Then $Ie = Je$.

Proof. (a) This is the first part of Lemma 1.1, (a) in [10]. The proof given there still works in the present, slightly more general case.

(b) Suppose $Ie \neq Je$. Then $(J/I)e \neq 0$, hence there is a $y \in J/I$ such that $ye_i \neq 0$ for some primitive idempotent e_i corresponding to a simple submodule of M. The module ye_iR is a nonzero homomorphic image of e_iR and therefore maps onto e_iR/e_iN. It follows, that there are right ideals A, B with $I \subseteqq B \subseteqq A \subseteqq J$ such that

$$A/B \cong e_iR/e_iN \subseteqq E(M) .$$

Thus, there is a nonzero homomorphism $A \to A/B \to E(M)$, which extends to a nonzero $\phi : R \to E(M)$ given by $\phi(1) = x \neq 0$. Now $B \subseteqq \operatorname{Ker} \phi$, hence $xI = 0$, but $\phi(A) \neq 0$, hence $xJ \neq 0$. This contradicts the assumption.

PROPOSITION 2.7. *Let R be left perfect and let M be a semi-simple module with corresponding idempotent e. Then*

(a) $\bar{M} = E(M)$ *if and only if* $eNe = 0$,

(b) $\bar{M} = M$ (i.e., M is rationally complete) *if and only if* $ReNe = Ne$,

(c) $M = E(M)$ (i.e., M is injective) *if and only if* $Ne = 0$,

(d) *M is projective if and only if* $eN = 0$.

Proof. (a) If $\bar{M} = E(M)$ we have by (2.5)

$$E(M) = \{x \in E(M) \mid x \cdot 0 = 0\} = \bar{M} = \{x \in E(M) \mid xReN = 0\} .$$

Applying (2.6, b) to $I = 0$, $J = ReN$, we get that $ReNe = 0$, which is equivalent to $eNe = 0$. Conversely, if $ReNe = 0$, then (2.6, a) applied to $I = ReN$, yields $\bar{M} = E(M)$.

(b) Suppose $\bar{M} = M$. Since M is the socle of $E(M)$,

$$M = \{x \in E(M) \mid xN = 0\} .$$

Applying (2.6, b) again, this time to $I = ReN, J = N$, we obtain $ReNe = Ne$. The converse is proved as above.

(c) This follows from (a) and (b).

(d) A simple module e_iR/e_iN is projective if and only if $e_iN = 0$.

COROLLARY 2.8. *If the rational completion of every semisimple module is injective, then R is completely reducible.*

Proof. Let M be a semi-simple module with corresponding idempotent $e = e_1 + \cdots + e_n$. Then $ReR = R$ and $eNe = 0$ implies $N = 0$.

The ring of 2×2 triangular matrices over a field is not completely reducible, yet the rational completion of every simple module is injective.

Another immediate consequence of (2.7) is the following.

COROLLARY 2.9. *A projective semi-simple module is rationally complete if and only if it is injective.*

A ring is called *primary* if it is a simple Artinian ring modulo its radical.

PROPOSITION 2.10. *Let R be left and right perfect. Then the following are equivalent:*

(a) *Every right R-module is rationally complete.*

(b) *Every simple right R-module is rationally complete.*

(c) *R is a finite product of primary left and right perfect rings.*

Proof. (a) $\Rightarrow$ (b) is trivial. (c) $\Rightarrow$ (a) is a part of [5, Main Theorem]. It remains to prove (b) $\Rightarrow$ (c).

Using the decomposition (*), we set $f_i = e_{i1} + \cdots + e_{ik_i}$ and, as before, $e_i = e_{i1}$ for $i = 1, \cdots, n$. By assumption and (2.7), we have that $Re_iNe_i = Ne_i$ for all i. This implies $Rf_iNf_i = Nf_i$ for all i, because $e_i = ue_{ij}v$ and $e_{ij} = ve_iu$ for some $u, v \in R$, whence $Re_iN = Rf_iN$.

Let now e and f be any two different idempotents from the set $\{f_i\}$. Since eRf annihilates every simple R-module, we have that $eRf \subseteqq N$. From this and $RfNf = Nf$ it follows that

$$ReRfR = (ReRfR)Nf .$$

However, since R is right perfect, a generalized version of Nakayama's

Lemma holds: $MN = M$ implies $M = 0$ for every right module M (see [3, p.473]). Thus $eRf = 0$ and we conclude that all idempotents f_i are central. This implies (c).

REMARKS 2.11.

(a) Since (2.10, c) is left-right symmetric, Proposition (2.10) is also true for left modules instead of right modules.

(b) There exist rings such that every simple module but not every module is rationally complete; e.g., the ring $\boldsymbol{Z}$ of integers (see (1.13)). As a matter of fact, Brown [4] has shown that every simple module over a commutative ring is rationally complete.

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