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SOME RESTRICTED PARTITION FUNCTIONS: CONGRUENCES MODULO 11

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Ramanujan's congruences for the unrestricted partition function $p(n)$ with 5, 7 and 11 as moduli can be shown to be equivalent to precisely similar congruences for some restricted partition functions of the type

$$ {}^{t}_{r}p(n)\,, \tag{1} $$

where to determine the value of (1) we count all the unrestricted partitions of n excepting those which contain any number of the forms tn or $tn \pm r$ as a part. The purpose of the present paper is to deal with congruences modulo 11.

In [5] the author has established a number of congruences modulo 3 for (1) with certain selected values of t and r. Functions of the type (1) are not new in number theory literature; for example, in the combinatorial interpretation of the famous Rogers-Ramanujan identities one finds

$$ {}^{5}_{1}p(n)\,, \qquad {}^{5}_{2}p(n)\,. $$

2. **The final results.** The restricted partition function (1) with $t = 363$ and $r = 121$ has a somewhat simpler interpretation. It is easily seen that this function counts the unrestricted partitions of n excepting those which contain 121 or any multiple thereof as a part. We use the simpler notation

$$ {}^{121}p(n) = {}^{121}_{0}p(n) = {}^{363}_{121}p(n)\,, \tag{2} $$

in the theorems to emphasize this interpretation.

The phrase 'for almost all values of n' appearing in Theorem 1 means that the number of integers $n \leqq N$ for which any specified congruence does not hold is $o(N)$. We assume ${}^{t}_{r}p(m)$ to be 1 when $m = 0$, and 0 when $m < 0$.

THEOREM 1. *For almost all values of n the following congruences with respect to the modulus 11 hold.*

$$(1)\qquad {}^{121}p(n) \equiv 0\ ,$$

$$(2)\qquad {}^{363}_{176}p(n) \equiv -{}^{363}_{55}p(n-22)\ ,$$

$$(3)\qquad {}^{363}_{154}p(n) \equiv -{}^{363}_{88}p(n-11)\ ,$$

$$(4)\qquad {}^{363}_{143}p(n) \equiv -{}^{363}_{22}p(n-33)\ ,$$

$$(5)\qquad {}^{363}_{77}p(n) \equiv \ \ {}^{363}_{44}p(n-11)\ ,$$

$$(6)\qquad {}^{363}_{110}p(n) \equiv \ \ {}^{363}_{11}p(n-33)\ .$$

THEOREM 2. *For all values of* $n \geqq 0$

$$ {}^{121}p(11n+6) \equiv 0 \qquad (\text{mod } 11)\ ,$$

and more generally with $0 \leqq \lambda \leqq 16$

$$ {}^{363}_{11\lambda}p(11n+6) \equiv 0 \qquad (\text{mod } 11)\ .$$

THEOREM 3. *The following congruences modulo* 11 *are true for all values of* $n \geqq 0$.

$$(1)\qquad -{}^{363}_{176}p(11n+10) - {}^{363}_{55}p(11n-12) \equiv 5\ {}^{121}p(11n+5)\ ,$$

$$(2)\qquad {}^{363}_{154}p(11n+7) + {}^{363}_{88}p(11n-4) \equiv 5\ {}^{121}p(11n+3)\ ,$$

$$(3)\qquad {}^{363}_{143}p(11n+4) + {}^{363}_{22}p(11n-29) \equiv 5\ {}^{121}p(11n+1)\ ,$$

$$(4)\qquad {}^{363}_{77}p(11n-2) - {}^{363}_{44}p(11n-13) \equiv 5\ {}^{121}p(11n+8)\ ,$$

$$(5)\qquad -{}^{363}_{110}p(11n\quad) + {}^{363}_{11}p(11n-33) \equiv 5\ {}^{121}p(11n+2)\ ,$$

$$(6)\qquad -{}^{363}_{176}p(11n+9) - {}^{363}_{55}p(11n-13) \equiv 5\ {}^{121}p(11n+4)\ ,$$

$$(7)\qquad {}^{363}_{154}p(11n+4) + {}^{363}_{88}p(11n-7) \equiv 5\ {}^{121}p(11n\quad)\ ,$$

$$(8)\qquad {}^{363}_{143}p(11n+10) + {}^{363}_{22}p(11n-23) \equiv 5\ {}^{121}p(11n+7)\ ,$$

$$(9)\qquad {}^{363}_{77}p(11n\quad) - {}^{363}_{44}p(11n-11) \equiv 5\ {}^{121}p(11n+10)\ ,$$

$$(10)\qquad -{}^{363}_{110}p(11n+7) + {}^{363}_{11}p(11n-26) \equiv 5\ {}^{121}p(11n+9)\ .$$

THEOREM 4. *The following congruences with respect to the modulus* 11 *hold for all values of* $n \geqq 0$.

$$(1)\qquad {}^{363}_{176}p(121n + 121) - {}^{363}_{176}p(11n + 11) \equiv -{}^{363}_{55}p(121n + 99) + {}^{363}_{55}p(11n - 11)\ ,$$

$$(2)\qquad {}^{363}_{154}p(121n + 120) - {}^{363}_{154}p(11n + 10) \equiv -{}^{363}_{88}p(121n + 109) + {}^{363}_{88}p(11n - 1\ )\ ,$$

$$(3)\qquad {}^{363}_{143}p(121n + 119) - {}^{363}_{143}p(11n + 9) \equiv -{}^{363}_{22}p(121n + 86) + {}^{363}_{22}p(11n - 24)\ ,$$

$$(4)\qquad {}^{363}_{77}p(121n + 106) - {}^{363}_{77}p(11n - 4) \equiv {}^{363}_{44}p(121n + 95) - {}^{363}_{44}p(11n - 15)\ ,$$

$$(5)\qquad {}^{363}_{110}p(121n + 114) - {}^{363}_{110}p(11n + 4) \equiv {}^{363}_{11}p(121n + 81) - {}^{363}_{11}p(11n - 29)\ .$$

3. **Notations and conventions.** Ramanujan [8] defined

$$(3)\qquad \Phi_{r,s}(x) = \sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty} \alpha^r\beta^s x^{\alpha\beta} = \sum_{n=1}^{\infty} n^r\sigma_{s-r}(n)x^n\ ,$$

where $\sigma_k(n)$ is, as usual, the sum of the kth powers of the divisors of n. The author has found it convenient to simplify the notation to $\Phi_{r,s}$, [3], and even to just (r, s), [4], so that

$$(4)\qquad (r, s) = \sum_{n=1}^{\infty} n^r\sigma_{s-r}(n)x^n\ .$$

The meanings of $f(x)$, u_r, v and $\sum_v [\mp V(v)]$ as given below are the same as in the previous paper [5], but those of U_i and $P_i(v)$ are different:

$$(5)\qquad f(x) = \prod_1^{\infty}(1 - x^n) = \sum_{-\infty}^{+\infty}(-1)^m x^{m(3m+1)/2} = \sum_{n=0}^{\infty} a_n x^n\ ,$$

$$(6)\qquad [f(x)]^{-1} = \left[\prod_1^{\infty}(1 - x^n)\right]^{-1} = \sum_{n=0}^{\infty} p(n)x^n\ .$$

$$(7)\qquad u_r = \sum_{n=0}^{\infty} n^r a_n x^n \sum_{n=0}^{\infty} p(n)x^n\ .$$

$\sum_v$ denotes summation over the pentagonal numbers v, where

$$(8) \qquad v = \frac{1}{2} m(3m+1), \qquad m = 0, \pm 1, \pm 2, \cdots ;$$

$$\sum_v [\mp V(v)]$$

implies that the sign to be prefixed is negative or positive according as v is of the form $(2m+1)(3m+2)$ or $m(6m+1)$. (The first form $(2m+1)(3m+2)$ is equivalent to $(2m+1)(3m+1)$ as given in the previous paper [5]; m ranges over all integers positive, zero or negative.) It is obvious that

$$(9) \qquad u_r = \sum_v (\mp v^r x^v)/f(x) .$$

We define the U_i's by

$$(10) \qquad \begin{cases} U_0 = 2u_5 - 5u_4 + 2u_3 + 5u_2 - 5u_1 + u_0 , \\ U_1 = 2u_5 - 3u_4 - u_3 + 4u_2 - u_1 , \\ U_2 = 2u_5 - u_4 + 5u_2 + 5u_1 , \\ U_4 = 2u_5 + 3u_4 + 3u_3 - 5u_2 - 3u_1 , \\ U_5 = u_5 - 3u_4 - 3u_3 + 4u_2 + u_1 , \\ U_7 = 2u_5 - 2u_4 - u_3 - 2u_2 + 3u_1 . \end{cases}$$

We also need polynomials $P_i(v)$ in v which like U_i are defined only for $i = 0, 1, 2, 4, 5$ and 7 and which are obtained by replacing U_i by $P_i(v)$ and u_r by v^r in the above relations (10).

4. **Some lemmas.** For the pentagonal numbers v which fall only in the residue classes $i = 0, 1, 2, 4, 5$ and 7 modulo 11 the following lemma can be verified.

LEMMA 1. *If v is a pentagonal number, then*

$$\begin{aligned} P_i(v) &\equiv 1 \pmod{11}, \ \textit{if } v \equiv i \pmod{11} \\ &\equiv 0 \pmod{11}, \ \textit{if } v \not\equiv i \pmod{11} . \end{aligned}$$

Applying relation (9) to (10) we obtain

$$(11) \qquad U_i = \sum_v [\mp P_i(v) x^v]/f(x) ;$$

and then the use of Lemma 1 leads to Lemma 2.

LEMMA 2. $$U_i \equiv \sum_{v \equiv i} (\mp x^v)/f(x) \pmod{11} ,$$

the summation being extended over all pentagonal numbers $v \equiv i$ (mod 11).

The following lemma can be verified without difficulty by writing $11m + j$ with $j = 0, -4; -1, -3; 1, -5; 3, 4; -2;$ and $2, 5$ respectively in place of m in the expression $\frac{1}{2}m(3m + 1)$ for the pentagonal numbers, and in $(-1)^m$ its associated sign. It is also to be remembered (when j is negative, say, $-j'$) that $\frac{1}{2}(11m - j')(33m - 3j' + 1)$ and $\frac{1}{2}(11m + j')(33m + 3j' - 1)$ represent the same set of numbers.

LEMMA 3. *With respect to the modulus* 11 *the pentagonal numbers* v *fall in the six residue classes* $i = 0, 1, 2, 4, 5$ *and* 7; *and the solutions of*

$$v \equiv i \pmod{11}$$

and the corresponding associated signs are as follows.

i	solutions (1st set):	sign	solutions (2nd set):	sign
0	$\frac{1}{2}(363m^2 + 11m)$	$(-1)^m$	$\frac{1}{2}(363m^2 + 253m) + 22,$	$(-1)^m$
1	$\frac{1}{2}(363m^2 + 55m) + 1,$	$(-1)^{m+1}$	$\frac{1}{2}(363m^2 + 187m) + 12,$	$(-1)^{m+1}$
2	$\frac{1}{2}(363m^2 + 77m) + 2,$	$(-1)^{m+1}$	$\frac{1}{2}(363m^2 + 319m) + 35,$	$(-1)^{m+1}$
4	$\frac{1}{2}(363m^2 + 209m) + 15,$	$(-1)^{m+1}$	$\frac{1}{2}(363m^2 + 275m) + 26,$	$(-1)^m$
5	$\frac{1}{2}(363m^2 + 121m) + 5,$	$(-1)^m$	———	——
7	$\frac{1}{2}(363m^2 + 143m) + 7,$	$(-1)^m$	$\frac{1}{2}(363m^2 + 341m) + 40,$	$(-1)^{m+1}$

The identities given in the next lemma are simple applications of a special case of a famous identity of Jacobi [2, p. 283] viz.,

$$\prod_{n=0}^{\infty} [(1 - x^{2kn+k-l})(1 - x^{2kn+k+l})(1 - x^{2kn+2k})] = \sum_{-\infty}^{+\infty} (-1)^m x^{km^2+lm} . \tag{12}$$

In establishing this lemma k and l are given values in conformity with the expressions quardratic in m given in Lemma 3.

LEMMA 4. *If* v *is a pentagonal number then, writing* $v \equiv i$ *simply for* $v \equiv i \pmod{11}$, *we have*

$$\sum_{v\equiv 0} (\mp x^v) = \prod_0^{\infty} [(1 - x^{363n+176})(1 - x^{363n+187})(1 - x^{363n+363})]$$
$$+ x^{22} \prod_0^{\infty} [(1 - x^{363n+55})(1 - x^{363n+308})(1 - x^{363n+363})] ,$$

$$\sum_{v\equiv 1} (\mp x^v) = -x \prod_0^{\infty} [(1 - x^{363n+154})(1 - x^{363n+209})(1 - x^{363n+363})]$$
$$- x^{12} \prod_0^{\infty} [(1 - x^{363n+88})(1 - x^{363n+275})(1 - x^{363n+363})] ,$$

$$\sum_{v\equiv 2}(\mp x^v) = -x^2\prod_0^\infty[(1-x^{363n+143})(1-x^{363n+220})(1-x^{363n+363})]$$
$$\qquad - x^{35}\prod_0^\infty[(1-x^{363n+22})(1-x^{363n+341})(1-x^{363n+363})]\ ,$$
$$\sum_{v\equiv 4}(\mp x^v) = -x^{15}\prod_0^\infty[(1-x^{363n+77})(1-x^{363n+286})(1-x^{363n+363})]$$
$$\qquad + x^{26}\prod_0^\infty[(1-x^{363n+4})(1-x^{363n+319})(1-x^{363n+363})]\ ,$$
$$\sum_{v\equiv 5}(\mp x^v) = x^5\prod_0^\infty(1-x^{121n+121})\ ,$$
$$\sum_{v\equiv 7}(\mp x^v) = x^7\prod_0^\infty[(1-x^{363n+110})(1-x^{363n+253})(1-x^{363n+363})]$$
$$\qquad - x^{40}\prod_0^\infty[(1-x^{363n+11})(1-x^{363n+352})(1-x^{363n+363})]\ .$$

The next lemma is derived from Lemma 2 after the substitution in it of the product expressions for $\sum_{v\equiv i}(\mp x^v)$ as given in Lemma 4. The following fact is to be used in addition.

$$\frac{\prod_{n=0}^\infty[(1-x^{363n+r})(1-x^{363n+363-r})(1-x^{363n+363})]}{f(x)} \tag{13}$$
$$= \frac{\prod_{n=0}^\infty[(1-x^{363n+r})(1-x^{363n+363-r})(1-x^{363n+363})]}{[(1-x)(1-x^2)(1-x^3)\cdots]} = \sum_{n=0}^\infty {}_{r}^{363}p(n)x^n\ .$$

LEMMA 5. *With respect to the modulus* 11

$$U_0 \equiv \sum_{n=0}^\infty {}_{176}^{363}p(n)x^n + \sum_{n=0}^\infty {}_{55}^{363}p(n-22)x^n\ ,$$
$$U_1 \equiv -\sum_{n=0}^\infty {}_{154}^{363}p(n-1)x^n - \sum_{n=0}^\infty {}_{88}^{363}p(n-12)x^n\ ,$$
$$U_2 \equiv -\sum_{n=0}^\infty {}_{143}^{363}p(n-2)x^n - \sum_{n=0}^\infty {}_{22}^{363}p(n-35)x^n\ ,$$
$$U_4 \equiv -\sum_{n=0}^\infty {}_{77}^{363}p(n-15)x^n + \sum_{n=0}^\infty {}_{44}^{363}p(n-26)x^n\ ,$$
$$U_5 \equiv \sum_{n=0}^\infty {}^{121}p(n-5)x^n\ ,$$
$$U_7 \equiv \sum_{n=0}^\infty {}_{110}^{363}p(n-7)x^n - \sum_{n=0}^\infty {}_{11}^{363}p(n-40)x^n\ .$$

We require a set of congruences which are directly derivable from the identities for $u_r = u_{r,0}$ given in [3], for $r = 1, 2, 3, 4$ and 5. These identities express u_r's as linear functions of $\Phi_{a,b}$'s. By suitable multiplications of both sides of these identities the fractional coefficients appearing in [3] may be made integral. Since we are concerned with congruences modulo 11 we have in the following lemma reduced these coefficients with respect to the modulus 11. For the sake of simplicity we have written (a, b) instead of $\Phi_{a,b}$.

LEMMA 6. *With respect to the modulus* 11 *for the congruences we have*

$$
\begin{aligned}
u_0 &= \quad 1 ; \\
u_1 &= -(0, 1) ; \\
u_2 &\equiv \quad (0, 1) + 4(1, 2) + 5(0, 3) ; \\
u_3 &\equiv \quad 2(0, 1) + 5(1, 2) + 5(2, 3) - 2(0, 3) - 3(1, 4) + 3(0, 5) ; \\
u_4 &\equiv -5(0, 1) - 5(1, 2) + (2, 3) + 4(3, 4) + 2(0, 3) - 5(1, 4) \\
&\quad - 2(2, 5) + 5(0, 5) - 4(1, 6) - 3(0, 7) ; \\
u_5 &\equiv -(0, 10) - 5(1, 2) - 4(2, 3) + (3, 4) - (4, 5) + 2(0, 3) \\
&\quad - 2(1, 4) + 5(2, 5) + 3(3, 6) + 2(0, 5) - (1, 6) + 3(2, 7) \\
&\quad + 2(0, 7) - 5(1, 8) .
\end{aligned}
$$

The next lemma is obtained by the substitution of the above values of u_r's in the expressions for U_i given in (10).

LEMMA 7. *With respect to the modulus* 11

$$
U_0 - 1 \equiv L_0
$$
$$
U_i \equiv L_i , \quad i = 1, 2, 4, 5, 7 ;
$$

where

$$
\begin{aligned}
L_i = {} & A_1(1, 8) + A_0(0, 7) \\
& + B_2(2, 7) + B_1(1, 6) + B_0(0, 5) \\
& + C_3(3, 6) + C_2(2, 5) + C_1(1, 4) + C_0(0, 3) \\
& + D_4(4, 5) + D_3(3, 4) + D_2(2, 3) + D_1(1, 2) + D_0(0, 1) ,
\end{aligned}
$$

the set of coefficients

$$
(A_1, A_0; B_2, B_1, B_0; C_3, C_2, C_1, C_0; D_4, D_3, D_2, D_1, D_0)
$$

being respectively

$$
\begin{array}{llllllllllllllll}
(& 1, & -3; & -5, & -4, & -4; & -5, & -2, & 4, & 4; & -2, & 4, & -3, & 1, & 4) ; \\
(& 1, & 2; & -5, & -1, & -3; & -5, & 5, & 3, & -2; & -2, & 1, & -5, & 5, & 5) ; \\
(& 1, & -4; & -5, & 2, & -1; & -5, & 1, & 1, & 5; & -2, & -2, & 2, & 4, & 3) ; \\
(& 1, & -5; & -5, & -3, & -5; & -5, & 4, & 5, & 1; & -2, & 3, & -1, & 3, & -2) ; \\
(& -5, & 0; & 3, & 0, & 0; & 3, & 0, & 0, & 0; & -1, & 0, & 0, & 0, & 0) ; \\
(& 1, & -1; & -5, & -5, & 2; & -5, & 3, & -2, & 3; & -2, & 5, & -4, & -2, & 1) ;
\end{array}
$$

for $i = 0, 1, 2, 4, 5$ *and* 7.

5. The basic theorem. By comparing the coefficients of the two expressions for U_i (mod 11) given in Lemmas 6 and 7 we obtain the following theorem from which our final conclusions are drawn.

THEOREM 0. *The following congruences are true for $n > 0$, the modulus being* 11.

$$\begin{aligned} &\tfrac{363}{176}p(n) + \tfrac{363}{55}p(n-22) \\ (1)\quad &\equiv (n-3)\sigma_7(n) - (5n^2+4n+4)\sigma_5(n) - (5n^3+2n^2-4n-4)\sigma_3(n) \\ &\qquad - (2n^4-4n^3+3n^2-n-4)\sigma(n)\,; \end{aligned}$$

$$\begin{aligned} &-\tfrac{363}{154}p(n-1) - \tfrac{363}{88}p(n-12) \\ (2)\quad &\equiv (n+2)\sigma_7(n) - (5n^2+n+3)\sigma_5(n) - (5n^3-5n^2-3n+2)\sigma_3(n) \\ &\qquad - (2n^4-n^3+5n^2-5n-5)\sigma(n)\,; \end{aligned}$$

$$\begin{aligned} &-\tfrac{363}{143}p(n-2) - \tfrac{363}{22}p(n-35) \\ (3)\quad &\equiv (n-4)\sigma_7(n) - (5n^2-2n+1)\sigma_5(n) - (5n^3-n^2-n-5)\sigma_3(n) \\ &\qquad - (2n^4+2n^3-2n^2-4n-3)\sigma(n)\,; \end{aligned}$$

$$\begin{aligned} &-\tfrac{363}{77}p(n-15) + \tfrac{363}{44}p(n-26) \\ (4)\quad &\equiv (n-5)\sigma_7(n) - (5n^2+3n+5)\sigma_5(n) - (5n^3-4n^2-5n-1)\sigma_3(n) \\ &\qquad - (2n^4-3n^3-n^2-3n+2)\sigma(n)\,; \end{aligned}$$

$$\begin{aligned} &121\,p(n-5) \\ (5)\quad &\equiv -5n\sigma_7(n) + 3n^2\sigma_5(n) + 3n^3\sigma_3(n) - n^4\sigma(n)\,; \end{aligned}$$

$$\begin{aligned} &\tfrac{363}{110}p(n-7) - \tfrac{363}{11}p(n-40) \\ (6)\quad &\equiv (n-1)\sigma_7(n) - (5n^2+5n-2)\sigma_5(n) - (5n^3-3n^2+2n-3)\sigma_3(n) \\ &\qquad - (2n^4-5n^3+4n^2+2n-1)\sigma(n)\,. \end{aligned}$$

6. Proofs of Theorems 1 and 2. In view of the well-known congruence [9, 1 p. 167]

$$\sigma_s(n) \equiv 0 \pmod{k} \tag{14}$$

for 'almost all' n for arbitrarily fixed k and odd s it is a straightforward matter to infer Theorem 1 from Theorem 0.

The first relation of Theorem 2 is also obtained immediately by writing $11n+11$ for n in the relation (5) of Theorem 0. The general result enunciated in Theorem 2 actually emanates from the first part, and the process of derivation has two stages. In the first stage Ramanujan's congruence modulo 11 for the partition function $p(n)$ is derived from the first relation, and then this derived relation is used in the second stage to establish the general proposition. It easily follows from (13) and (12) that ${}^{363}_{11\lambda}p(n)$ can be expressed in the (really finite) form,

$$ {}^{363}_{11\lambda}p(n) = p(n) + \sum_{n'=1}^{\infty} \varepsilon(n')p(n - 11n') , \tag{15} $$

where $\varepsilon(n') = 0$ or ± 1. For the special case corresponding to $\lambda = 11$ we have the fully specified expression,

$$ {}^{121}p(n) = \sum_{v} [\mp p(n - 121v)] . \tag{16} $$

Keeping in mind the first relation of Theorem 2, viz.,

$$ {}^{121}p(11n + 6) \equiv 0 \pmod{11} , \tag{17} $$

Ramanujan's congruence is seen to be valid by putting successively $n = 6, 17, 28, 39, \cdots$ in (16). Thus (17) implies Ramanujan's congruence. Conversely Ramanujan's congruence implies (17) as can be easily seen when n is replaced by $11n + 6$ in (16). Hence Ramanujan's congruence for the unrestricted partition function is equivalent to the congruence (17) for the restricted partition function. To derive the general proposition we merely write $11n + 6$ for n in (15) and make use of Ramanujan's congruence. It can be easily seen that this latter congruence is also equivalent to any particular case of the general preposition.

7. **Corollaries of the basic Theorem 0.** An interesting consequence of the congruences for the restricted partition functions so far established is that these enable us to deduce a certain congruence property of the divisor function $\sigma_k(n)$, viz., Lemma 8, which in its turn helps us to provide further congruences for the restricted partition functions. This lemma however, is also a particular case of a very general theorem established elsewhere [6].

LEMMA 8. *If n is not a multiple of* 11 *then*

$$ \sigma_7(n) \equiv \left(\frac{n}{11}\right) n^2 \sigma_3(n) \pmod{11} , $$

where $(n/11)$ is the Legendre symbol.

The congruence relation of the divisor functions given in Lemma 8 proves useful for the reduction of the basic congruences of the Theorem 0 into neater forms when attention is separately paid to the cases when n is a quadratic non-residue or a residue of 11. When n is a multiple of 11 this theorem reduces obviously to an elegant form.

COROLLARY 1. *If n is a quadratic non-residue of* 11, *then with respect to the modulus* 11,

$$(1)\qquad \begin{aligned}&{}^{363}_{176}p(n) + {}^{363}_{55}p(n-22)\\ &\equiv (5n^3+n^2+4n+4)\sigma_3(n) - (2n^4-4n^3+2n^2-n-4)\sigma(n)\ ,\end{aligned}$$

$$(2)\qquad \begin{aligned}&-{}^{363}_{154}p(n-1) - {}^{363}_{88}p(n-12)\\ &\equiv (5n^3+3n^2+3n-2)\sigma_3(n) - (2n^4-n^3+5n^2-5n-5)\sigma(n)\ ,\end{aligned}$$

$$(3)\qquad \begin{aligned}&-{}^{363}_{143}p(n-2) - {}^{363}_{22}p(n-35)\\ &\equiv (5n^3+5n^2+n+5)\sigma_3(n) - (2n^4+2n^3-2n^2-4n-3)\sigma(n)\ ,\end{aligned}$$

$$(4)\qquad \begin{aligned}&-{}^{363}_{77}p(n-15) + {}^{363}_{44}p(n-26)\\ &\equiv (5n^3-2n^2+5n+1)\sigma_3(n) - (2n^4-3n^3+n^2-3n+2)\sigma(n)\ ,\end{aligned}$$

$$(5)\qquad {}^{121}p(n-5) \equiv -3n^3\sigma_3(n) - n^4\sigma(n)\ ,$$

$$(6)\qquad \begin{aligned}&{}^{363}_{110}p(n-7) - {}^{363}_{11}p(n-40)\\ &\equiv (5n^3+4n^2-2n+3)\sigma_3(n) - (2n^4-5n^3+4n^2+2n-1)\sigma(n)\ .\end{aligned}$$

This corollary easily follows from Theorem 0 when use is made of Lemma 8 which enables replacement of the terms involving $\sigma_7(n)$ by terms involving $\sigma_3(n)$, and of the following relation (18) which makes redundant the terms involving $\sigma_5(n)$:

$$(18)\qquad \sigma_5(11n+i) \equiv 0 \pmod{11}$$

when i is a quadratic non-residue of 11. This congruence is a particular case of a more general relation [7, 4] which holds for any odd prime modulus.

When n is a quadratic residue of 11 there is no scope for using the relation (18) but Lemma 8 can still be used with some advantage, and the result is given in Corollary 2.

COROLLARY 2. *If n is a quadratic residue of 11 then with respect to the modulus 11,*

$$
\begin{aligned}
&{}^{363}_{176}p(n) + {}^{363}_{55}p(n-22) \\
(1)\qquad &\equiv -(5n^2+4n+4)\sigma_5(n) - (4n^3+5n^2-4n-4)\sigma_3(n) \\
&\quad - (2n^4-4n^3+3n^2-n-4)\sigma(n)\ ,
\end{aligned}
$$

$$
\begin{aligned}
&-{}^{363}_{154}p(n-1) - {}^{363}_{88}p(n-12) \\
(2)\qquad &\equiv -(5n^2+n+3)\sigma_5(n) - (4n^3+4n^2-3n+2)\sigma_3(n) \\
&\quad - (2n^4-n^3+5n^2-5n-5)\sigma(n)\ ,
\end{aligned}
$$

$$
\begin{aligned}
&-{}^{363}_{143}p(n-2) - {}^{363}_{22}p(n-35) \\
(3)\qquad &\equiv -(5n^2-2n+1)\sigma_5(n) - (4n^3+3n^2-n-5)\sigma_3(n) \\
&\quad - (2n^4+2n^3-2n^2-4n-3)\sigma(n)\ .
\end{aligned}
$$

$$
\begin{aligned}
&-{}^{363}_{77}p(n-15) + {}^{363}_{44}p(n-26) \\
(4)\qquad &\equiv -(5n^2+3n+5)\sigma_5(n) - (4n^3+n^2-5n-1)\sigma_3(n) \\
&\quad - (2n^4-3n^3+n^2-3n+2)\sigma(n)\ ,
\end{aligned}
$$

$$
\begin{aligned}
&{}^{121}p(n-5) \\
(5)\qquad &\equiv 3n^2\sigma_5(n) - 2n^3\sigma_3(n) - n^4\sigma(n)\ ,
\end{aligned}
$$

$$
\begin{aligned}
&{}^{363}_{110}p(n-7) - {}^{363}_{11}p(n-40) \\
(6)\qquad &\equiv -(5n^2+5n-2)\sigma_5(n) - (4n^3-2n^2+2n-3)\sigma_3(n) \\
&\quad - (2n^4-5n^3+4n^2+2n-1)\sigma(n)\ .
\end{aligned}
$$

Proof of Theorem 3. This theorem follows from the above corollaries. We shall show that the first set of five congruences of Theorem 3 is deducible from Corollary 1 whereas the last set is obtainable from the other corollary.

Eliminating $\sigma_3(n)$ between (5) and each of the remaining congruences of Corollary 1 we find that if n is a quadratic non-residue then

$$
\begin{aligned}
&3n^3\left[{}^{363}_{176}p(n) + {}^{363}_{55}p(n-22)\right] \\
(19)\qquad &\quad + (5n^3+n^2+4n+4)\,{}^{121}p(n-5) \\
&\equiv -n^3(2n^2+n-1)\sigma(n) \pmod{11}\ ,
\end{aligned}
$$

$$
\begin{aligned}
&-3n^3\left[{}^{363}_{154}p(n-1) + {}^{363}_{88}p(n-12)\right] \\
(20) \qquad &+ (5n^3 + 3n^2 + 3n - 2)\,{}^{121}p(n-5) \\
&\equiv n^3(4n^2 - 5n + 4)\sigma(n) \qquad (\text{mod } 11)\ ,
\end{aligned}
$$

$$
\begin{aligned}
&-3n^3\left[{}^{363}_{143}p(n-2) + {}^{363}_{22}p(n-35)\right] \\
(21) \qquad &+ (5n^3 + 5n^2 + n + 5)\,{}^{121}p(n-5) \\
&\equiv n^3(5n^2 - 4n - 2)\sigma(n) \qquad (\text{mod } 11)\ ,
\end{aligned}
$$

$$
\begin{aligned}
&-3n^3\left[{}^{363}_{77}p(n-15) - {}^{363}_{44}p(n-26)\right] \\
(22) \qquad &+ (5n^3 - 2n^2 + 5n + 1)\,{}^{121}p(n-5) \\
&\equiv n^3(3n^2 - 3n + 5)\sigma(n) \qquad (\text{mod } 11)\ ,
\end{aligned}
$$

$$
\begin{aligned}
&3n^3\left[{}^{363}_{110}p(n-7) - {}^{363}_{11}p(n-40)\right] \\
(23) \qquad &+ (5n^3 + 4n^2 - 2n + 3)\,{}^{121}p(n-5) \\
&\equiv n^3(n^2 + 2n + 3)\sigma(n) \qquad (\text{mod } 11)\ .
\end{aligned}
$$

By putting $11n + 10$, $11n + 8$, $11n + 6$, $11n + 13$ and $11n + 7$ in place of n in the congruences (19), (20), (21), (22) and (23) respectively we obtain the first five congruences, (1) $-$ (5) of Theorem 3.

To prove the remaining congruences we turn to Corollary 2. Multiplying both sides of the congruence (5) of this corollary by 5, and adding the result to each of the other congruences one by one we have respectively the following congruences modulo 11,

$$
\begin{aligned}
&{}^{363}_{176}p(n\qquad) + {}^{363}_{55}p(n-22) + 5\,{}^{121}p(n-5) \\
(24) \qquad &\equiv -(n^2 + 4n + 4)\sigma_5(n) - (3n^3 + 5n^2 - 4n - 4)\sigma_3(n) \\
&\quad + (4n^4 + 4n^3 - 3n^2 + n + 4)\sigma(n)\ ,
\end{aligned}
$$

$$
\begin{aligned}
&-{}^{363}_{154}p(n-1) - {}^{363}_{88}p(n-12) + 5\,{}^{121}p(n-5) \\
(25) \qquad &\equiv -(n^2 + n + 3)\sigma_5(n) - (3n^3 + 4n^2 - 3n + 2)\sigma_3(n) \\
&\quad + (4n^4 + n^3 - 5n^2 + 5n + 5)\sigma(n)\ ,
\end{aligned}
$$

$$
\begin{aligned}
&-{}^{363}_{143}p(n-2) - {}^{363}_{22}p(n-35) + 5\,{}^{121}p(n-5) \\
(26) \qquad &\equiv -(n^2 - 2n + 1)\sigma_5(n) - (3n^3 + 3n^2 - n - 5)\sigma_3(n) \\
&\quad + (4n^4 - 2n^3 + 2n^2 + 4n + 3)\sigma(n)\ ,
\end{aligned}
$$

$$-\frac{363}{77}p(n-15)+\frac{363}{44}p(n-26)+5^{121}p(n-5)$$

$$(27) \qquad \equiv -(n^2+3n+5)\sigma_5(n)-(3n^3+n^2-5n-1)\sigma_3(n) + (4n^4+3n^3-n^2+3n-2)\sigma(n)\ ,$$

$$\frac{363}{110}p(n-7)-\frac{363}{11}p(n-40)+5^{121}p(n-5)$$

$$(28) \qquad \equiv -(n^2+5n-2)\sigma_5(n)-(3n^3-2n^2+2n-3)\sigma_3(n) + (4n^4+5n^3-4n^2-2n+1)\sigma(n)\ .$$

By writing $11n+9$, $11n+5$, $11n+12$, $11n+15$ and $11n+14$ respectively in (24), (25), (26), (27) and (28) we obtain the last five congruences (6) $-$ (10) of Theorem 3.

10. Proof of Theorem 4. This theorem is based upon an artifice which depends upon the following simple congruence which can be established easily from first principles.

$$(29) \qquad \sigma_k(11n)\equiv\sigma_k(n) \pmod{11},\ k>0\ .$$

We shall illustrate the procedure adopted by proving the last congruence of the theorem. Writing $11n$ for n in the last congruence (6) of the basic Theorem 0 we have on using the above relation the following

$$(30) \qquad \frac{363}{110}p(11n-7)-\frac{363}{11}p(11n-40) \equiv -\sigma_7(n)+2\sigma_5(n)+3\sigma_3(n)+\sigma(n) \pmod{11}\ .$$

Subtracting (6) of Theorem 0 from (30), and then writing $11n+11$ for n we arrive at the desired result. Other congruences of the theorem are similarly established.

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