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AN APPROXIMATION THEORY FOR ELLIPTIC QUADRATIC FORMS ON HILBERT SPACES: APPLICATION TO THE EIGENVALUE PROBLEM FOR COMPACT QUADRATIC FORMS

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A theory for an elliptic quadratic form $J(x)$ defined on a Hilbert space $\mathfrak{A}$ has been given by Hestenes. A fundamental part of this theory is concerned with the signature s and nullity n of $J(x)$ on $\mathfrak{A}$. These indices are used to develop a generalized Sturm-Lionville Theory and a Local Morse Theory. In this paper the theory of Hestenes is extended to elliptic quadratic forms $J(x; \sigma)$ defined on $\mathfrak{A}(\sigma)$ where σ is a member of the metric space (Σ, ρ) and $\mathfrak{A}(\sigma)$ denotes a closed subspace of $\mathfrak{A}$. A fundamental part of this extension is concerned with inequalities dealing with the signature $s(\sigma)$ and nullity $n(\sigma)$ of $J(x; \sigma)$ on $\mathfrak{A}(\sigma)$, where σ is in a ρ neighborhood of a fixed point σ_0 in Σ.

It is noted that the hypothesis for these inequalities is sufficiently weak so as to include many mathematical problems. In the second part of this paper these results are applied to the study of eigenvalue problems for compact quadratic forms. A significant result is that the nth eigenvalue, $\lambda_n(\sigma)$, is a ρ continuous function of σ. Comparison theorems are given for completeness. This work is a generalization of the eigenvalue theory of A. Weinstein.

The inequality results may also be used to study focal point problems and numerical approximation problems associated with linear self adjoint systems of ordinary or partial differential equations.

2. Preliminaries. The basic theory of Hilbert spaces, strong and weak convergence, and linear operators and quadratic forms is given in References [2] and [3]. The fundamental Hilbert space is denoted by $\mathfrak{A}$; subspaces by $\mathscr{B}, \mathscr{C}, \mathscr{D}, \cdots$; elements of $\mathfrak{A}$ by the letters $x, y, z, \cdots$; scalars by $a, b, c, \cdots$. The inner product is denoted by (x, y); the norm by $\|x\|$; strong convergence by $x_q \Rightarrow x_0$; weak convergence by $x_q \rightarrow x_0$. We will assume that subspaces of $\mathfrak{A}$ are closed and the scalars are real. The latter assumption is for convenience; the complex case holds equally well.

A real valued function $L(x)$ defined on $\mathfrak{A}$ is said to be a *linear form* if it is linear and continuous. A real valued function $Q(x, y)$ defined on $\mathfrak{A} \times \mathfrak{A}$ is a *bilinear form* if, for each y in $\mathfrak{A}$, $Q(x, y)$ and

$Q(y, x)$ are linear forms in x. If $x_q \to x_0$ and $y_q \to y_0$ imply $Q(x_q, y_q) \to Q(x_0, y_0)$ then $Q(x, y)$ is *compact*. If $Q(x, y) = Q(y, x)$ then $Q(x) = Q(x, x)$ is the *quadratic form* associated with the bilinear form $Q(x, y)$. We assume throughout this paper that bilinear forms satisfy $Q(x, y) = Q(y, x)$.

$Q(x)$ is *positive* (*negative, nonpositive, nonegative*) on $\mathfrak{A}$ if $Q(x) > 0$ ($Q(x) < 0$, $Q(x) \leqq 0$, $Q(x) \geqq 0$) for $x \neq 0$ in $\mathfrak{A}$. $Q(x)$ is *positive definite* on $\mathfrak{A}$ if there exists a positive number h such that $Q(x) \geqq h \| x \|^2$ on $\mathfrak{A}$. $Q(x)$ is *compact* if $x_q \to x_0$ implies $Q(x_q) \to Q(x_0)$. $Q(x)$ is *weakly lower semicontinuous* (wlsc) if $x_q \to x_0$ implies $\liminf_{q=\infty} Q(x_q) \geqq Q(x_0)$.

Two vectors x any y in $\mathfrak{A}$ are Q *orthogonal* if $Q(x, y) = 0$. The vector x is Q *orthogonal* to $\mathscr{B}$ if y in $\mathscr{B}$ implies $Q(x, y) = 0$. The set of all vectors Q orthogonal to $\mathscr{B}$ is the Q *orthogonal* complement, denoted by $\mathscr{B}^Q$. $\mathscr{B}$ and $\mathscr{C}$ are Q *orthogonal* if each x in $\mathscr{B}$ is Q orthogonal to $\mathscr{C}$. A vector x is a Q *null* vector of $\mathscr{B}$ if x in $\mathscr{B} \cap \mathscr{B}^Q$. $\mathscr{B}_0$ will denote the set of Q null vectors of $\mathscr{B}$.

The *signature* (index) of $Q(x)$ on $\mathscr{B}$ is the dimension of a maximal, linear subclass $\mathscr{C}$ of $\mathscr{B}$ on which $Q(x)$ is negative. The *nullity* of $Q(x)$ on $\mathscr{B}$ is the dimension of $\mathscr{B}_0 = \mathscr{B} \cap \mathscr{B}^Q$. Finally $J(x)$ is an *elliptic form* on $\mathfrak{A}$ if $J(x)$ is wlsc on $\mathfrak{A}$, and $x_q \Rightarrow x_0$ whenever $x_q \to x_0$ and $J(x_q) \to J(x_0)$.

We note the following results for elliptic forms [3]: A quadratic form $J(x)$ is elliptic on $\mathfrak{A}$ if and only if there exists a finite dimensional subspace $\mathscr{B}$ of $\mathfrak{A}$ such that $J(x)$ is positive definite on the orthogonal complement of $\mathscr{B}$. A quadratic form $J(x)$ is elliptic on $\mathfrak{A}$ if and only if there exists a positive definite form $P(x)$ and a compact form $K(x)$ such that $J(x) = P(x) - K(x)$. Furthermore $K(x)$ can be chosen nonnegative on $\mathfrak{A}$. A positive elliptic form is positive definite.

Theorems 1 and 2 have been given in [3].

THEOREM 1. *The signature of $Q(x)$ on $\mathscr{B}$, if finite, is given by each of the following quantities*:

(a) *the dimension of a maximal subspace $\mathscr{C}$ in $\mathscr{B}$ on which $Q(x) < 0$*;

(b) *the dimension of a maximal subspace $\mathscr{D}$ in $\mathscr{B}$ on which $Q(x) \leqq 0$ and $\mathscr{D} \cap \mathscr{B}_0 = 0$*;

THEOREM 2. *If the sum $m = s + n$ of the signature s and nullity n of $Q(x)$ on $\mathscr{B}$ is finite, it is given by each of the following quantities*:

(a) *the dimension of a maximal subspace $\mathscr{C}$ in $\mathscr{B}$ in which $Q(x) \leqq 0$*;

(b) *the least integer* k *such that there exists* k *linear forms* $L_1, \cdots, L_k$ *on* $\mathscr{B}$ *with* $Q(x) > 0$ *for all* x *in* $\mathscr{B}$ *satisfying* $L_\alpha(x) = 0 (\alpha = 1, \cdots, k)$.

3. Fundamental Inequalities. The purpose of this section is to state and derive fundamental inequalities which relate the signature and nullity of an Elliptic Form on a closed subspace of $\mathfrak{A}$ to "approximating" Elliptic Forms on "approximating" closed subspaces.

The main results are contained in Theorems 6 and 7. Theorem 8 is a combination of these two theorems. Theorem 11 is an extension of Theorem 7 to the metric space $M = E^1 \times \Sigma$. Continuity of the nth eigenvalue, $\lambda_n(\sigma)$, follows immediately from Theorem 11.

Let Σ be a metric space with metric ρ. A sequence $\{\sigma_r\}$ in Σ converges to σ_0 in Σ, written $\sigma_r \to \sigma_0$, if $\lim_{r=\infty} \rho(\sigma_r, \sigma_0) = 0$. For each σ in Σ let $\mathfrak{A}(\sigma)$ be a closed subspace of $\mathfrak{A}$ such that

(1a) If $\sigma_r \to \sigma_0$, x_r in $\mathfrak{A}(\sigma_r)$, $x_r \to y_0$ then y_0 is in $\mathfrak{A}(\sigma_0)$;

(1b) If x_0 is in $\mathfrak{A}(\sigma_0)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\rho(\sigma, \sigma_0) < \delta$, there exists x_σ in $\mathfrak{A}(\sigma)$ satisfying $\| x_0 - x_\sigma \| < \varepsilon$.

LEMMA 3. *Condition* (1b) *is equivalent to the following*: *Let* $\mathscr{B}(\sigma_0)$ *be a subspace of* $\mathfrak{A}(\sigma_0)$ *of dimension* h *and* $\varepsilon > 0$. *There exists* $\delta > 0$ *such that whenever* $\rho(\sigma_0, \sigma) < \delta$, *there exists a subspace* $\mathscr{B}(\sigma)$ *of* $\mathfrak{A}(\sigma)$ *of dimension* h *with the property that if* x_0 *is a unit vector in* $\mathscr{B}(\sigma_0)$ *there exists* x_σ *in* $\mathscr{B}(\sigma)$ *such that* $\| x_0 - x_\sigma \| < \varepsilon$.

Clearly this condition implies (1b) with $h = 1$. Conversely let $x_1, \cdots, x_h$ be an orthonormal basis for $\mathscr{B}(\sigma_0)$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $\rho(\sigma_0, \sigma) < \delta$ then $x_{1\sigma}, \cdots, x_{h\sigma}$ is in $\mathfrak{A}(\sigma)$ with $\| x_k - x_{k\sigma} \|^2 < \varepsilon/h$.

Assume that usual summation conventions with $k, l = 1, \cdots, h$. Letting $x_0 = b_k x_k$ and $x_\sigma = b_k x_{k\sigma}$ where $b_k b_k = 1$ we have

$$
\begin{aligned}
\| x_0 - x_\sigma \|^2 = \| b_k(x_k - x_{k\sigma}) \|^2 &\leqq (| b_k | \, \| x_k - x_{k\sigma} \|)^2 \\
&\leqq (b_k b_k)(\| x_l - x_{l\sigma} \| \, \| x_l - x_{l\sigma} \|) \leqq h(\varepsilon/h) = \varepsilon \, .
\end{aligned}
$$

This concludes the proof of the lemma.

The "approximation" hypothesis for forms are now stated.

For each σ *in* Σ *let* $J(x; \sigma)$ *be a quadratic form defined on* $\mathfrak{A}(\sigma)$ *with* $J(x, y; \sigma)$ *the associated bilinear form.* Let $s(\sigma)$ and $n(\sigma)$ be the index and nullity of $J(x; \sigma)$ on $\mathfrak{A}(\sigma)$. For $r = 0, 1, 2, \cdots$ let x_r be in $\mathfrak{A}(\sigma_r)$, y_r in $\mathfrak{A}(\sigma_r)$ such that: if $x_r \to x_0$, $y_r \Rightarrow y_0$ and $\sigma_r \to \sigma_0$ then

(2a) $\lim_{r=\infty} J(x_r, y_r; \sigma_r) = J(x_0, y_0; \sigma_0)$;

(2b) $\liminf_{r=\infty} J(x_r; \sigma_r) \geqq J(x_0; \sigma_0)$; and

(2c) $\lim_{r=\infty} J(x_r; \sigma_r) = J(x_0; \sigma_0)$ implies $x_r \Rightarrow x_0$.

LEMMA 4. *Assume condition* (2a) *holds. Let σ_0 be given. Then there exists $\delta > 0$, $M > 0$ such that $\rho(\sigma, \sigma_0) < \delta$ implies $|J(x, y; \sigma)| \leqq M\|x\| \|y\|$ for all x, y in $\mathfrak{A}(\sigma)$.*

Suppose the conclusion does not hold. Then for $r = 1, 2, \cdots$ we may choose σ_r in Σ and x_r, y_r in $\mathfrak{A}(\sigma_r)$ such that $\|x_r\| = \|y_r\| = 1$, $\rho(\sigma_r, \sigma_0) < 1/r$ and $a_r^2 = |J(x_r, y_r; \sigma_r)| > r$.

Now $\bar{x}_r = x_r/a_r \Rightarrow 0$ and $\bar{y}_r = y_r/a_r \Rightarrow 0$ so by

(2a) $$1 = J(\bar{x}_r, \bar{y}_r; \sigma_r) \longrightarrow J(0, 0; \sigma_0) = 0 .$$

This contradiction establishes the result.

THEOREM 5. *If* (2a) *and* (2c) *hold then either $J(x; \sigma)$ or $-J(x; \sigma)$ satisfy* (2b).

Suppose the conclusion does not hold. Then there exists sequences $\{\sigma_r\}$, $\{y_r\}$ and $\{z_r\}$ $(r = 0, 1, 2, \cdots)$ such that $\sigma_r \rightarrow \sigma_0$; y_r, z_r in $\mathfrak{A}(\sigma_r)$; $y_r \rightarrow y_0$, $z_r \rightarrow z_0$; and

$$\lim_{r=\infty} J(y_r; \sigma_r) = A < J(y_0; \sigma_0) ,$$
$$\lim_{r=\infty} J(y_r, z_r; \sigma_r) = B, \text{ and}$$
$$\lim_{r=\infty} J(z_r; \sigma_r) = C > J(z_0; \sigma_0)$$

where A, B, and C are real numbers by Lemma 4. Thus the equation

$$[A - J(y_0; \sigma_0)]a^2 + 2a[B - J(y_0, z_0; \sigma_0)] + [C - J(z_0; \sigma_0)] = 0$$

has two distinct real roots a_1, a_2. For $i = 1, 2$ and $r = 0, 1, 2, \cdots$ let $x_{ri} = a_i y_r + z_r$ so that $x_{ri} \rightarrow x_{0i}$. By the definition of a_i,

$$\begin{aligned} J(x_{ri}; \sigma_r) &= J(y_r; \sigma_r)a_i^2 + 2a_i J(y_r, z_r; \sigma_r) + J(z_r; \sigma_r) \longrightarrow Aa_i^2 + 2Ba_i + C \\ &= J(y_0; \sigma_0)a_i^2 + 2a_i J(y_0, z_0; \sigma_0) + J(z_0; \sigma_0) = J(x_{0i}; \sigma_0) \end{aligned}$$

so that from (2c) $x_{ri} \Rightarrow x_{0i}$ $(i = 1, 2)$. Since $a_1 \neq a_2$ then $y_q \Rightarrow y_0$ and $z_q \Rightarrow z_0$. Finally from (2a) we have

$$A = \lim_{r=\infty} J(y_r; \sigma_r) = J(y_0; \sigma_0) > A .$$

This contradiction establishes the theorem.

THEOREM 6. *Assume conditions* (1a), (2b) *and* (2c) *hold. Then* *for any* σ_0 *in* Σ *there exists* $\delta > 0$ *such that* $\rho(\sigma_0, \sigma) < \delta$ *implies*

$$s(\sigma) + n(\sigma) \leqq s(\sigma_0) + n(\sigma_0) . \tag{3}$$

Assume the conclusion is false. Then there exists a sequence $\{\sigma_r\}$ with $\sigma_r \rightarrow \sigma_0$ and $s(\sigma_r) + n(\sigma_r) > s(\sigma_0) + n(\sigma_0)$. Let $k = s(\sigma_0) + n(\sigma_0) + 1$. For $r = 1, 2, \cdots$ there exists k orthonormal vectors $x_{1r}, x_{2r}, \cdots, x_{kr}$ in $\mathfrak{A}(\sigma_r)$ with $J(x; \sigma_r) \leqq 0$ on span $\{x_{1r}, \cdots, x_{kr}\}$. For each $p = 1, \cdots, k$ the sequence $\{x_{pr}\}$ is bounded in $\mathfrak{A}$ and hence has a weakly convergent subsequence, which we may assume to be $\{x_{pr}\}$, such that $x_{pr} \rightarrow x_p$. By (1a) x_p is in $\mathfrak{A}(\sigma_0)$.

Assume the usual repeated index summation convention with $p = 1, \cdots, k$. Let $b = (b_1, \cdots, b_k)$ be arbitrary, set $y_0 = b_p x_p$ and $y_r = b_p x_{pr}$. Since $y_r \rightarrow y_0$ we have by (2b)

$$J(y_0; \sigma_0) \leqq \liminf_{r=\infty} J(y_r; \sigma_r) \leqq 0 .$$

Thus $x_1, \cdots, x_k$ is a linear dependent set, for if not by Theorem 2, $k - 1 = s(\sigma_0) + n(\sigma_0) \geqq k$.

Choose $b \neq 0$ such that $y_0 = b_p x_p = 0$; also choose $y_r = b_p x_{pr}$. We note $y_r \rightarrow y_0 = 0$ and

$$0 = J(0; \sigma_0) \leqq \liminf_{r=\infty} J(y_r; \sigma_r) \leqq \limsup_{r=\infty} J(y_r; \sigma_r) \leqq 0 .$$

Hence $J(y_r; \sigma_r) \rightarrow 0 = J(0; \sigma_0)$ so that $y_r \Rightarrow 0$ by (2c).

Finally $0 = \lim_{r=\infty} \| y_r \|^2 = b_p b_p \neq 0$. This contradiction establishes the theorem.

THEOREM 7. *Assume conditions* (1b) *and* (2a) *hold. Then for any* σ_0 *in* Σ *there exists* $\delta > 0$ *such that* $\rho(\sigma_0, \sigma) < \delta$ *implies*

$$s(\sigma_0) \leqq s(\sigma) . \tag{4}$$

Let $\mathscr{B}(\sigma_0)$ be a maximal subspace of $\mathfrak{A}(\sigma_0)$ such that $J(x; \sigma_0) < 0$ on $\mathscr{B}(\sigma_0)$. Let $x_1, \cdots, x_h$ be a basis for $\mathscr{B}(\sigma_0)$. By Lemma 3 and conditions (1b) and (2a) there exists a basis $x_{1\sigma}, \cdots, x_{h\sigma}$ for $\mathscr{B}(\sigma)$ such that if $x_\sigma = a_p x_{p\sigma}$ and

$$A_{pq}(\sigma) = J(x_{p\sigma}, x_{q\sigma}; \sigma)$$

then

$$F(a, \sigma) = J(x_\sigma; \sigma) = a_p a_q A_{pq}(\sigma) \quad (p, q = 1, \cdots, h; p, q \text{ summed})$$

is a continuous function of σ at σ_0.

By the usual arguments for quadratic forms we may choose $M < 0$ and $\delta > 0$ such that

$$F(a, \sigma_0) \leqq 2Ma_pa_p$$

and

$$F(a, \sigma) = F(a, \sigma_0) + (A_{pq}(\sigma) - A_{pq}(\sigma_0))a_pa_q \leqq M\, a_pa_p$$

where $\rho(\sigma_0, \sigma) < \delta$. This completes the proof.

Combining Theorems 6 and 7 we obtain

THEOREM 8. *Assume conditions* (1) *and* (2) *hold. Then for any* σ_0 *in* Σ *there exists* $\delta > 0$ *such that* $\rho(\sigma, \sigma_0) < \delta$ *implies*

$$s(\sigma_0) \leqq s(\sigma) \leqq s(\sigma) + n(\sigma) \leqq s(\sigma_0) + n(\sigma_0)\ . \tag{5}$$

COROLLARY 9. *Assume* $\delta > 0$ *has been choosen such that* $\rho(\sigma, \sigma_0) < \delta$ *implies equation* (5) *holds. Then if* $\rho(\sigma, \sigma_0) < \delta$ *we have*
(6a) $n(\sigma) \leqq n(\sigma_0)$,
(6b) $n(\sigma) = n(\sigma_0)$ *implies* $s(\sigma) = s(\sigma_0)$ *and* $m(\sigma) = m(\sigma_0)$, *and*
(6c) $n(\sigma_0) = 0$ *implies* $s(\sigma) = s(\sigma_0)$ *and* $n(\sigma) = 0$.

This result follows at once from Theorem 8. As a further result we have

COROLLARY 10. *The set* $\{\sigma \in \Sigma \mid n(\sigma) = 0\}$ *is open. The set* $\{\sigma \in \Sigma \mid n(\sigma) \neq 0\}$ *is closed.*

As an example of these results we will extend Theorem 8 to a result for the metric space $M = E^1 \times \Sigma$. This result will be fundamental for the continuity of the nth eigenvalue $\lambda^n(\sigma)$. Thus assume $M = I \times \Sigma$, I an open interval of E^1, is a metric space with metric d defined by

$$d(\mu_1, \mu_2) = |\lambda_2 - \lambda_1| + \rho(\sigma_2, \sigma_1)$$

for any pair of points $\mu_1 = (\lambda_1, \sigma_1)$, $\mu_2 = (\lambda_2, \sigma_2)$ in M. Let $s(\mu) = s(\lambda, \sigma)$, $n(\mu) = n(\lambda, \sigma)$ be the index and nullity of $J(x; \mu)$ on $\mathfrak{A}(\mu)$; let $m(\mu) = m(\lambda, \sigma) = s(\lambda, \sigma) + n(\lambda, \sigma)$. Theorem 8 and Corollary 9 hold with the obvious modifications.

THEOREM 11. *Let conditions* (1), (2) *be satisfied with* $\mu = (\lambda, \sigma)$ *in* M *replacing* σ *in* Σ. *For fixed* σ *let the signature* $s(\lambda, \sigma)$ *be a monotone function of* λ *such that* $s(\lambda + 0, \sigma) = s(\lambda - 0, \sigma)$ *implies* $n(\lambda, \sigma) = 0$. *Let* $\mu_0 = (\lambda_0, \sigma_0)$ *in* M *be given such that* $s(\lambda_0 - 0, \sigma_0) = n$, $s(\lambda_0 + 0, \sigma_0) = m$. *Then there exists* $\delta_0 > 0$ *such that* $|\lambda - \lambda_0| \leqq \delta_0$ *and* $\rho(\sigma, \sigma_0) \leqq \delta_0$ *imply that* $s(\lambda, \sigma)$ *is between* n *and* m.

Assume $s(\lambda, \sigma)$ is monotone increasing on an interval I and hence $n \leqq m$. Choose $\delta > 0$ so small that $s(\lambda, \sigma_0) = n$ for $(\lambda_0 - 2\delta, \lambda_0) \subset I$ and $s(\lambda, \sigma_0) = m$ for $(\lambda_0, \lambda_0 + 2\delta) \subset I$. By assumption $n(\lambda_0 - \delta, \sigma_0) = n(\lambda_0 + \delta, \sigma_0) = 0$. Finally choose δ_0, $0 < \delta_0 \leqq \delta$, such that $\rho(\sigma, \sigma_0) < \delta_0$ implies (5) holds for both $\mu_0 = (\lambda_0 - \delta, \sigma_0)$ and $\mu_0 = (\lambda_0 + \delta, \sigma_0)$. By (6c), $s(\lambda_0 - \delta, \sigma) = n$ and $s(\lambda_0 + \delta, \sigma) = m$ for all σ such that $\rho(\sigma, \sigma_0) < \delta_0$. The theorem now follows by the monotone condition.

4. Eigenvalue Theory. The purpose of this section is to apply the theory of § 3 to the study of eigenvalue problems for compact quadratic forms. Our work is motivated by (and at times duplicates) the methods and results of Hestenes [3]. Of particular significance are Theorems 20 and 21 which give sufficiency conditions for the continuity of the nth eigenvalue. Theorem 22 and 23 are comparison theorems. They follow directly from "signature inequalities" given in Reference [3].

In this section we assume Σ is a metric space with metric ρ. For each σ in Σ let $\mathfrak{A}(\sigma)$ be a closed subspace of $\mathfrak{A}$, $J(x; \sigma)$ an elliptic form defined on $\mathfrak{A}(\sigma)$, and $K(x; \sigma)$ a compact form on $\mathfrak{A}(\sigma)$.

We assume conditions (1) and (2) are satisfied and that $\sigma_r \to \sigma_0$, x_r in $\mathfrak{A}(\sigma_r)$, x_0 in $\mathfrak{A}(\sigma_0)$, $x_r \to x_0$ imply $K(x_r; \sigma_r) \to K(x_0; \sigma_0)$.

Let $M = E^1 \times \Sigma$ be the metric space with metric d defined above (after Corollary 10). For each $\mu = (\lambda, \sigma)$ in M define $\mathfrak{A}(\mu) = \mathfrak{A}(\sigma)$ and

$$H(x; \mu) = J(x; \lambda, \sigma) = J(x; \sigma) - \lambda K(x; \sigma) \tag{7}$$

on the space $\mathfrak{A}(\mu)$. Finally let $s(\mu) = s(\lambda, \sigma)$, $n(\mu) = n(\lambda, \sigma)$, and $m(\mu) = m(\lambda, \sigma)$ denote the index, nullity, and sum of the index and nullity of $H(x; \mu)$ on $\mathfrak{A}(\mu)$.

THEOREM 12. *Conditions* (1) *and* (2) *hold with* μ *replacing* σ *and* H *replacing* J.

Since $\mathfrak{A}(\mu) = \mathfrak{A}(\sigma)$ conditions (1) hold. For (2a) let x_r, y_r in $\mathfrak{A}(\mu_r)$, $r = 0, 1, 2, \cdots$ with $x_r \to x_0$ and $y_r \Rightarrow y_0$. Then

$$\begin{aligned} H(x_r, y_r; \mu_r) - H(x_0, y_0; \mu_0) = {} & \{J(x_r, y_r; \sigma_r) - J(x_0, y_0; \sigma_0)\} \\ & + \lambda_0[K(x_0, y_0; \sigma) - K(x_r, y_r; \sigma_r)] \\ & + (\lambda_0 - \lambda_r)K(x_r, y_r; \sigma_r) \ . \end{aligned}$$

If $r \to \infty$ the first term goes to 0 since (2) holds on Σ, the third term goes to 0 as $K(x_r, y_r; \sigma_r)$ is bounded, and the second term goes to 0 by the equality

$$\begin{aligned} 2[K(x_0, y_0; \sigma_0) - K(x_r, y_r; \sigma_r)] = K(x_0 + y_0; \sigma_0) - K(x_r + y_r; \sigma_r) \\ - K(x_0; \sigma_0) - K(y_0; \sigma_0) \\ + K(x_r; \sigma_r) + K(y_r; \sigma_r) \; . \end{aligned}$$

For (2b) let $\underline{\lim}\, A_r$ denote $\liminf_{r=\infty} A_r$ then

$$\begin{aligned} \underline{\lim}\; H(x_r; \mu_r) &= \underline{\lim}\, [J(x_r; \sigma_r) - \lambda_r K(x_r; \sigma_r)] \\ &\geqq \underline{\lim}\, J(x_r; \sigma_r) - \lim_{r=\infty} \lambda_r K(x_r; \sigma_r) \geqq J(x_0; \sigma_0) - \lambda_0 K(x_0; \sigma_0) \\ &= H(x_0; \mu_0) \; . \end{aligned}$$

For (2c) if $x_r \rightarrow x_0$, $\lim H(x_r; \mu_r) = H(x_0; \mu_0)$ then

$$\begin{aligned} J(x_0; \sigma_0) - \lambda_0 K(x_0; \sigma_0) = H(x_0; \mu_0) = \lim_{r=\infty} H(x_r; \mu_r) = \lim_{r=\infty} J(x_r; \sigma_r) \\ - \lim_{r=\infty} \lambda_r K(x_r; \sigma_r) \end{aligned}$$

so that $J(x_0; \sigma_0) = \lim_{r=\infty} J(x_r; \sigma_r)$. Since (2c) holds on Σ, we have $x_r \Rightarrow x_0$. This complete the proof of the theorem.

Theorem 13 now follows immediately from Theorem 8.

THEOREM 13. *For any $\mu_0 = (\lambda_0, \sigma_0)$ in M there exists $\delta > 0$ such that if $\mu = (\lambda, \sigma)$, $d(\mu, \mu_0) < \delta$ then*

(8) $\quad s(\lambda_0, \sigma_0) \leqq s(\lambda, \sigma) \leqq s(\lambda, \sigma) + n(\lambda, \sigma) \leqq s(\lambda_0, \sigma_0) + n(\lambda_0, \sigma_0)$.

COROLLARY 14. *Assume $\delta > 0$ has been chosen such that $\mu = (\lambda, \sigma)$, $d(\mu, \mu_0) < \delta$ implies inequalities* (8) *hold. Then if $d(\mu, \mu_0) < \delta$ we have*

(9a) $n(\lambda, \sigma) \leqq n(\lambda_0, \sigma_0)$,

(9b) $n(\lambda, \sigma) = n(\lambda_0, \sigma)$ implies $s(\lambda, \sigma) = s(\lambda_0, \sigma_0)$ and $m(\lambda, \sigma) = m(\lambda_0, \sigma_0)$, and

(9c) $n(\lambda_0, \sigma_0) = 0$ implies $s(\lambda, \sigma) = s(\lambda_0, \sigma_0)$ and $n(\lambda, \sigma) = 0$.

Corollaries 14 and 15 follow immediately from Theorem 13.

COROLLARY 15. *The set $\{\mu$ in $M \mid n(\mu) = 0\}$ is open. The set $\{\mu$ in $M \mid n(\mu) \neq 0\}$ is closed.*

THEOREM 16. *Let σ_0 in Σ be given and let Λ_0 be a nonempty compact subset of $\{\lambda \mid n(\lambda, \sigma_0) = 0\}$. Then there exists $\varepsilon > 0$ such that λ_0 in $\Lambda_{0\varepsilon}$ and $\rho(\sigma, \sigma_0) < \varepsilon$ imply*

(10) $$s(\lambda_0, \sigma) = s(\lambda_0, \sigma_0) \; , \quad n(\lambda_0, \sigma) = n(\lambda_0, \sigma_0) = 0$$

where $\Lambda_{0\varepsilon}$ is the ε-neighborhood of Λ_0.

Let λ_0 in Λ_0 and set $\mu_0 = (\lambda_0, \sigma_0)$. By Corollary 14 there exists $\delta = \delta(\lambda_0) > 0$ such that $\mu = (\lambda, \sigma)$ and $d(\mu, \mu_0) < \delta$ imply conditions

(10). By the usual arguments for compact sets there exists $\delta > 0$ such that (10) holds whenever $d(\mu, \mu_0) < \delta$ for any λ_0 in Λ_0. This completes the proof.

COROLLARY 17. *Let λ^* be real and σ_0 in Σ such that $n(\lambda^*, \sigma_0) = s(\lambda^*, \sigma_0) = 0$. Then there exists $\varepsilon > 0$ such that $\rho(\sigma, \sigma_0) < \varepsilon$ and $|\lambda - \lambda_0| < \varepsilon$ imply $n(\lambda, \sigma) = s(\lambda, \sigma) = 0$.*

Let σ_0 in Σ be given. A real number λ_0 is an *eigenvalue* (characteristic value) of $J(x; \sigma_0)$ relative to $K(x; \sigma_0)$ on $\mathfrak{A}(\sigma_0)$ if $n(\lambda_0, \sigma_0) \neq 0$. The number $n(\lambda_0, \sigma_0)$ is its *multiplicity*. An eigenvalue λ_0 will be counted the number of times equal to its multiplicity. If λ_0 is an eigenvalue and $x_0 \neq 0$ in $\mathfrak{A}(\sigma_0)$ such that $J(x_0, y; \sigma_0) = \lambda_0 K(x_0, y; \sigma_0)$ for all y in $\mathfrak{A}(\sigma_0)$ then x_0 is an *eigenvector* corresponding to λ_0.

Assume J, K, and $\mathfrak{A}$ are independent of σ, that is, consider a fixed Elliptic Form $J(x)$ and a fixed compact form $K(x)$ on a fixed space $\mathfrak{A}$. Results for this case (Theorem 18) have been given by Hestenes [3].

THEOREM 18. *Assume $x \neq 0$ in $\mathfrak{A}$, $K(x) \leqq 0$ implies $J(x) > 0$. Then there exists λ^* such that $J(x; \lambda^*)$ is positive definite on $\mathfrak{A}$.*

If $\lambda_0 \geqq \lambda^*$ there exists $\varepsilon = \varepsilon(\lambda_0)$ such that

(11a) $s(\lambda) = s(\lambda_0)$, $n(\lambda) = 0$ $(\lambda_0 - \varepsilon < \lambda < \lambda_0)$ *and*

(11b) $s(\lambda) = s(\lambda_0) + n(\lambda_0)$, $n(\lambda) = 0$ $(\lambda_0 < \lambda < \lambda_0 + \varepsilon)$.

If $\lambda_0 \leqq \lambda^*$ there exists $\varepsilon = \varepsilon(\lambda_0)$ such that

(12a) $s(\lambda) = s(\lambda_0) + n(\lambda_0)$, $n(\lambda) = 0$ $(\lambda - \varepsilon < \lambda < \lambda_0)$ and

(12b) $s(\lambda) = s(\lambda_0)$, $n(\lambda) = 0$ $(\lambda_0 < \lambda < \lambda_0 + \varepsilon)$.

If $\lambda^* \leqq \lambda' < \lambda''$ then $s(\lambda'') - s(\lambda')$ is equal to the number of characteristic values on $\lambda' \leqq \lambda < \lambda''$; if *$\lambda'' < \lambda' \leqq \lambda^*$ then $s(\lambda'') - s(\lambda')$ is equal to the number of characteristic values on $\lambda'' < \lambda \leqq \lambda'$.*

If $\lambda^* \leqq \lambda' < \lambda''$ then $s(\lambda'') + n(\lambda'') - s(\lambda')$ is equal to the number of characteristic values on $\lambda' \leqq \lambda \leqq \lambda''$; if *$\lambda'' < \lambda' \leqq \lambda^*$ then $s(\lambda'') + n(\lambda'') - s(\lambda')$ is equal to the number of characteristic values on $\lambda'' \leqq \lambda \leqq \lambda'$.*

It is instructive to describe the graph of λ versus $s(\lambda)$. By Theorem 18 this graph is a step function with a finite or countably infinite number of intervals; each interval has the associated nonnegative integer value $s(\lambda)$. The number λ^* is not unique. It may be chosen to be any interior point of the interval on which $s(\lambda) = 0$. Note that $s(\lambda)$ is a nondecreasing function on (λ^*, ∞) and nonincreasing on $(-\infty, \lambda^*)$; it is continuous from the right if $\lambda < \lambda^*$ and from the left it $\lambda^* < \lambda$. The discontinuities in $s(\lambda)$ are points at which $n(\lambda) \neq 0$; in fact the jump at λ is $n(\lambda)$.

For convenience we will denote the kth eigenvalue greater than λ^* by λ_{k-1}, the kth eigenvalue less than λ^*, by λ_k. If σ_0 in Σ is such that Theorem 18 holds we use the notation $\lambda_k(\sigma_0)$ and $\lambda_{-k}(\sigma_0)$.

THEOREM 19. *Let σ_0 in Σ be given and assume $J(x;\sigma_0)>0$ whenever $x\neq 0$ in $\mathfrak{A}(\sigma_0)$, $K(x;\sigma_0)\leqq 0$. Then there exists $\eta>0$ such that $\rho(\sigma,\sigma_0)<\eta$ implies $J(x;\sigma)>0$ whenever $x\neq 0$ in $\mathfrak{A}(\sigma)$, $K(x;\sigma)<0$.*

In addition there exists λ^* and $\delta>0$ such that $\mu=(\lambda,\sigma)$, $\mu_0=(\lambda^*,\sigma_0)$, $d(\mu,\mu_0)<\delta$ imply $H(x;\mu)>0$ on $\mathfrak{A}(\mu)$.

If the first result is not true, we may choose sequences $\{\sigma_q\}$, $\{x_q\}$ such that $\sigma_q\to\sigma_0$, x_q in $\mathfrak{A}(\sigma_q)$, $\|x_q\|=1$, $K(x_q;\sigma_q)\leqq 0$, and $J(x_q;\sigma_q)\leqq 0$. Since $\{x_q\}$ is bounded there exist y_0 in $\mathfrak{A}$ and a subsequence $\{x_{q_n}\}$, which we assume to be $\{x_q\}$ such that $x_q\to y_0$. By (1) y_0 is in $\mathfrak{A}(\sigma_0)$.

We claim $y_0=0$. If not $K(y_0;\sigma_0)=\lim_{q=\infty}K(x_q;\sigma_q)\leqq 0$ implies $J(y_0;\sigma_0)>0$ which is impossible as

$$0\geqq\limsup_{q=\infty}J(x_q;\sigma_q)\geqq\liminf_{q=\infty}J(x_q;\sigma_q)\geqq J(y_0;\sigma_0)\ .$$

Thus $J(y_0;\sigma_0)=0=\lim_{q=\infty}J(x_q;\sigma_q)$ and by (2c) $x_q\Rightarrow 0$. The contradiction $1=\lim_{q=\infty}\|x_q\|=\|0\|=0$ establishes the first result.

For the second result; by Theorem 18 there exists λ^* such that $H(x;\mu_0)>0$ on $\mathfrak{A}(\mu_0)$. Thus $n(\lambda^*,\sigma_0)=s(\lambda^*,\sigma_0)=0$. The result now follows by Corollary 17.

THEOREM 20. *Let σ_0 in Σ be given such that $J(x;\sigma_0)>0$ whenever $x\neq 0$ in $\mathfrak{A}(\sigma_0)$, $K(x;\sigma_0)\leqq 0$. Assume λ', $\lambda''(\lambda'<\lambda'')$ are not eigenvalues of σ_0 and there exists k eigenvalues of σ_0 on (λ',λ''). Then there exists $\varepsilon>0$ such that $\rho(\sigma,\sigma_0)<\varepsilon$ implies there are exactly k eigenvalues of σ on (λ',λ'').*

In fact if $\lambda_n(\sigma_0)\leqq\lambda_{n+1}(\sigma_0)\leqq\cdots\leqq\lambda_{n+k-1}(\sigma_0)$ are the k eigenvalues of σ_0 on (λ',λ'') then $\lambda_n(\sigma)\leqq\lambda_{n+1}(\sigma)\leqq\cdots\leqq\lambda_{n+k-1}(\sigma)$ are the k eigenvalues of σ on (λ',λ'').

We may assume $\lambda^*(\sigma_0)\leqq\lambda'<\lambda''$; if $\lambda'<\lambda^*(\sigma_0)<\lambda''$ we consider the two intervals $\lambda'\leqq\lambda\leqq\lambda^*(\sigma_0)$ and $\lambda^*(\sigma_0)\leqq\lambda\leqq\lambda''$ separately. Assume $s(\lambda',\sigma_0)=n$ then by Theorem 18, $s(\lambda'',\sigma_0)=n+k-1$, $n(\lambda',\sigma_0)=n(\lambda'',\sigma_0)=0$. By Corollary 14 there exists $\delta>0$ such that if $\rho(\sigma,\sigma_0)<\delta$ then $n(\lambda',\sigma)=n(\lambda'',\sigma)=0$, $s(\lambda',\sigma)=n$ and $s(\lambda'',\sigma)=n+k-1$. The result follows from Theorem 18 by taking $\varepsilon=\min(\delta,\eta)$ where η given in Theorem 19.

COROLLARY 21. *If the* nth *eigenvalue* $\lambda_n(\sigma)$ $(n = 0, \pm 1, \pm 2, \cdots)$ *exists for* $\sigma = \sigma_0$ *it exists in a neighborhood of* σ_0 *and is a continuous function of* σ.

We note that the continuity of the nth eigenvalue also follows from Theorem 11 as the hypothesis of Theorem 18 implies the hypothesis of Theorem 11.

Theorems 22 and 23 are concerned with comparison theorems and eigenvalue problem. These results have been given in Reference [2] and are included for completeness.

THEOREM 22. *Let* $\mathfrak{A}^*$ *be a subspace of* $\mathfrak{A}$, $J(x) > 0$ *whenever* $x \neq 0$ *and* $K(x) \leqq 0$, *and* λ^* *be given as in Theorem* 18. *Let* $\{\lambda_i\}$, $\{\lambda_i^*\}$ $(i = 0, \pm 1, \pm 2, \cdots)$ *be the eigenvalues of* $J(x)$ *relative to* $K(x)$ *on* $\mathfrak{A}$ *and* $\mathfrak{A}^*$ *respectively. If the* kth *eigenvalues* λ_k, λ_k^* *exist* $(k = 0, \pm 1, \pm 2, \cdots)$ *we have*

$$\lambda_k \leqq \lambda_k^* \quad (k = 0, 1, 2, \cdots) \quad \text{and} \tag{13a}$$

$$\lambda_k \geqq \lambda_k^* \quad (k = -1, -2, -3, \cdots) . \tag{13b}$$

Strict inequality holds for any k $(k = 0, \pm 1, \pm 2, \cdots)$ *such that the* $J(x; \lambda_k)$ *null vectors of* $\mathfrak{A}$ *and* $\mathfrak{A}^*$ *are disjoint.*

If $\mathfrak{A} \ominus \mathfrak{A}^*$ *has finite dimension* e *then*

$$\lambda_k \leqq \lambda_k^* \leqq \lambda_{k+e} \quad (k = 0, 1, 2, \cdots) \text{ and} \tag{14a}$$

$$\lambda_k \geqq \lambda_k^* \geqq \lambda_{k-e} \quad (k = -1, -2, -3, \cdots) . \tag{14b}$$

THEOREM 23. *Let* $J^*(x)$ *and* $K^*(x)$ *be a second pair of elliptic and compact forms on* $\mathfrak{A}$ *such that* $J^*(x) > 0$ *whenever* $x = 0$, $K^*(x) < 0$. *Let* $J^*(x; \lambda) = J^*(x) - \lambda K^*(x)$ *and assume for any real* λ *that* $J(x; \lambda) \leqq 0$ *whenever* $J^*(x; \lambda) \leqq 0$. *Then there exists* λ^* *such that both* $J^*(x; \lambda^*)$ *and* $J(x; \lambda^*)$ *are positive definite on* $\mathfrak{A}$.

Let $\{\lambda_k\}$, $\{\lambda_k^*\}$ $(k = 0, \pm 1, \pm 2, \cdots)$ *be the eigenvalues of* $J(x)$ *relative to* $K(x)$ *on* $\mathfrak{A}$ *and* $J^*(x)$ *relative to* $K^*(x)$ *on* $\mathfrak{A}$ *respectively. Then inequalities* (13) *hold. If* $J(x; \lambda) < 0$ *whenever* $x \neq 0$ *and* $J^*(x; \lambda) < 0$ *then inequalities* (13) *hold with strict inequality.*

5. An Example. In this section we show that condition (1) and (2) include the hypothesis of the eigenvalue theory of A. Weinstein [1]. Thus many physical problems, including those of vibrating membranes and plates, may be handled by our methods. In a subsequent paper we will indicate how the values $\lambda_n(\sigma)$ may be found by numerical methods.

The assumptions of Weinstein are now given. Gould [1] contains the most complete discussion of this theory as well as a complete list of references.

Let $\mathscr{L}$ be a closed subspace of $\mathfrak{A}$ and $\mathscr{L}^n$ $(n = 1, 2, \cdots)$ be a sequence of closed subspaces of $\mathfrak{A}$. Let P and P^n be the respective projections of $\mathfrak{A}$ onto $\mathscr{L}$ and $\mathscr{L}^n$. The sequence $\{\mathscr{L}^n\}$ converges to $\mathscr{L}$, written $\mathscr{L}^n \to \mathscr{L}$ if $P^n x \Rightarrow Px$ for all x in $\mathfrak{A}$. The sequence $\{\mathscr{L}^n\}$ *converges downward* to $\mathscr{L}$, written $\mathscr{L}^n \searrow \mathscr{L}$, if $\mathscr{L}^n$ converges to $\mathscr{L}$ and $\mathscr{L}^{n+1} \subset \mathscr{L}^n$. The sequence $\{\mathscr{L}^n\}$ *converges upward* to $\mathscr{L}$, written $\mathscr{L}^n \nearrow \mathscr{L}$, if $\mathscr{L}^n$ converges to $\mathscr{L}$ and $\mathscr{L}^n \subset \mathscr{L}^{n+1}$.

THEOREM 24. *If* $\mathscr{L}^n \searrow \mathscr{L}$ *then* $\mathscr{L} = \bigcap \mathscr{L}^n$. *If* $\mathscr{L}^n \nearrow \mathscr{L}$ *then* $\mathscr{L} = \overline{\bigcup \mathscr{L}^n}$.

If $\mathscr{L}^n \searrow \mathscr{L}$ then x in $\mathscr{L}$ implies x in $\mathscr{L}^n$ for $n = 1, 2, \cdots$ so that x in $\bigcap \mathscr{L}^n$. Conversely if x in $\mathscr{L}^n$ for each n then $P^n x = x$ and hence $x = \lim_{n=\infty} P^n x = Px$ so that x in $\mathscr{L}$.

If $\mathscr{L}^n \nearrow \mathscr{L}$ and x not in $\mathscr{L}$ then $Px \neq x$. Thus there exists $\alpha > 0$ such that

$$\alpha < \| x - Px \| = \lim_{n=\infty} \| x - P^n x \|$$

so that x is not a limit point of $\bigcup \mathscr{L}^n$ i.e., x is not in $\overline{\bigcup \mathscr{L}^n}$. Conversely if x in $\mathscr{L}$ then $P^n x \Rightarrow Px = x$. Thus given $\varepsilon > 0$ there exists m such that $\| x - P^m x \| < \varepsilon$ with $P^m x$ in $\mathscr{L} \subset \bigcup \mathscr{L}^n$; hence x is in $\overline{\bigcup \mathscr{L}^n}$.

A correspondence between Weinstein's setting and § 3 is now given: Let $\Sigma = \{x \in E^1 \mid x = 1/n \ (n = 1, 2, \cdots) \text{ and } 0\}$ with the usual metric. For $n = 1, 2, 3, \cdots$ we set $\mathfrak{A}(1/n) = \mathscr{L}^n$ and $\mathfrak{A}(0) = \mathscr{L}$ where $\mathscr{L}$, $\mathscr{L}^n (n = 1, 2, 3, \cdots)$ are subspaces of $\mathfrak{A}$ and $\mathscr{L}^n \nearrow \mathscr{L}$. A correspondence between the Generalized Raleigh-Ritz Method and § 3 is obtained in the same manner except that $\mathscr{L}^n \searrow \mathscr{L}$.

THEOREM 25. *Assume* $\{\mathscr{L}^n\}$ *satisfies* $\mathscr{L}^n \searrow \mathscr{L}$ or $\mathscr{L}^n \nearrow \mathscr{L}$. *Then* $n \to \infty$, x_n *in* $\mathscr{L}^n$, $x_n \to y_0$ *implies* y_0 *in* $\mathscr{L}$.

Assume $\mathscr{L}^n \searrow \mathscr{L}$. Let m be an arbitrary fixed integer. If $n \geqq m$ then x_n in $\mathscr{L}^m$. It follows that y_0 is in $\mathscr{L}^m$. By Theorem 24, y_0 is in $\bigcap \mathscr{L}^m = \mathscr{L}$.

Assume $\mathscr{L}^n \nearrow \mathscr{L}$. By Theorem 24 $\mathscr{L} = \overline{\bigcup \mathscr{L}^n}$ and hence x_n in $\mathscr{L}$ for $n = 1, 2, \cdots$. This implies, y_0 in $\mathscr{L}$.

THEOREM 26. *Assume* $\{\mathscr{L}^n\}$ *satisfies* $\mathscr{L}^n \searrow \mathscr{L}$ *or* $\mathscr{L}^n \nearrow \mathscr{L}$. *Then given any* x_0 *in* $\mathscr{L}$ *and* $\varepsilon > 0$ *there exists a fixed integer* n_0 *such that if* $n > n_0$ *there exists* x_n *in* $\mathscr{L}^n$ *satisfying* $\| x_0 - x_n \| < \varepsilon$.

If $\mathscr{L}_n \searrow \mathscr{L}$ then x_0 in $\mathscr{L} = \bigcap \mathscr{L}^n$ and we take $x_n = x_0$ in $\mathscr{L}^n$ $(n = 1, 2, \cdots)$. If $\mathscr{L}^n \nearrow \mathscr{L}$ then x_0 in $\mathscr{L} = \overline{\bigcup \mathscr{L}^n}$. Thus there exists $\bar{x}$ in $\bigcup \mathscr{L}^n$ such that $\| x_0 - \bar{x} \| < \varepsilon$ and an m such that $\bar{x}$ in $\mathscr{L}^m$. The result follows by taking $n_0 = m$.

We remark that the spaces $\{\mathscr{L}^n\}$ are chosen by Weinstein in a more restrictive manner than that above. In particular for the case $\mathscr{L}^n \searrow \mathscr{L}$ they satisfy $\mathscr{L}^n = \mathscr{L}^0 \ominus \text{span}\{p_1, \cdots, p_n\}$ when $\{p_k\}$ is a complete orthonormal sequence in $\mathscr{L}^0 \ominus \mathscr{L}$. In the case $\mathscr{L}^n \nearrow \mathscr{L}$ they satisfy $\mathscr{L}^n = \mathscr{L}^0 \oplus \text{span}\{p_1, \cdots, p_m\}$ where $\{p_k\}$ is a complete orthonormal sequence in $\mathscr{L} \ominus \mathscr{L}^0$.

We note that inequalities (14) with $e = 1$ include the comparison (or separation) results of Weinstein contained in [1; pp. 77].

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