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QUASI-PROJECTIVE AND QUASI-INJECTIVE MODULES

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This paper contains results which are needed to prove a decomposition theorem for quasi-projective modules over left perfect rings.

An R-module M is called quasi-projective if and only if for every R-module A, every R-epimorphism $q: M \to A$, and every R-homomorphism $f: M \to A$, there is an $f' \in \mathrm{End}_R(M)$ such that the diagram

$$\begin{tikzcd} & M \arrow[dl, dashed, "f'"'] \arrow[d, "f"] & \\ M \arrow[r, "q"] & A \arrow[r] & 0 \end{tikzcd}$$

commutes, that is, $q \circ f' = f$. An R-module M is called quasi-injective if and only if for every R-module A, every R-monomorphism $j: A \to M$, and R-homomorphism $f: A \to M$, there is an $f' \in \mathrm{End}_R(M)$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "j"] \arrow[d, "f"'] & M \arrow[dl, dashed, "f'"] \\ & M & \end{tikzcd}$$

commutes.

The first section of this paper contains results which are needed to prove a decomposition theorem for quasi-projective modules over left perfect rings (Theorem 1.10). This decomposition is a characterization for quasi-projective modules over left perfect rings. A ring is left perfect if a projective cover (the dual concept of injective envelope) exists for every left R-module [4, p. 467]. It is known, for example, that left Artinian rings are left perfect [4, p. 467]. Some of the propositions are stated for semiperfect rings which are rings such that every finitely generated module has a projective cover [4, p. 471].

In the second section the decomposition for quasi-projective modules is used to obtain a decomposition for quasi-injective modules over a special class of rings. For these rings this decomposition characterizes quasi-injective modules. This decomposition theorem (Theorem 2.5) is specialized to the cases where the ring is quasi-Frobenius and where it is a finite dimensional algebra.

It will be assumed that all rings have an identity and that the modules are unital. Modules will be left R-modules, and homomorphisms will be R-homomorphisms unless otherwise stated. When S is the centralizer of ${}_RM$ in the sense of Jacobson [8], the notation will be abused and be written $S = \mathrm{End}_R(M)$. Actually, S operates on the right is anti-isomorphic to $\mathrm{End}_R(M)$. The radical will mean the Jacobson radical and be denoted by N. A direct sum of card(I) copies of M will be written M^I unless card$(I) = n < \infty$, and then M^n will be used in place of M^I. Also $\sum_{i=1}^{k} \oplus M_i^{g(i)}$ is a direct sum where $M_i^{g(i)}$ is $g(i)$ copies of M, and $g(i)$ can be any cardinal number. If $g(i) = 0$, then $M_i^{g(i)} = 0$.

I wish to thank Professor Azumaya who suggested that I investigate quasi-projective modules.

1. Quasi-projective modules. The goal of this section is to prove Theorem 1.10 which is a characterization of quasi-projective modules over left perfect rings. The first proposition to be presented was proved by Wu and Jans.

PROPOSITION 1.1. *Let R be a semi-perfect ring. Then M is a finitely generated, indecomposable, quasi-projective module if and only if $M = Re/Je$ where e is an indecomposable idempotent, and J is an ideal of R* [12, Thm. 3.1].

PROPOSITION 1.2. *Let R be a semi-perfect ring. If $Re/Je \neq 0$ where e is an indecomposable idempotent and J is an ideal, then $Je = J'e$ where J' is an ideal contained in the radical N.*

Proof. The module Ne is small in Re [4, p. 473]. Since Re is indecomposable and the projective cover of Re/Ne, Re/Ne is indecomposable. It is known that R/N is completely reducible if R is semi-perfect [4, Thm. 2.1]. Thus Re/Ne is simple, and Ne is maximal in Re. Now $Je \subseteqq Ne$ because Ne is both maximal and small in Re. Let $J' = J \cap N$.

PROPOSITION 1.3. *If M is quasi-projective and has a projective cover $P \xrightarrow{\pi} M \longrightarrow 0$ and if $P = \Sigma \oplus P_a$ ($a \in I$, and indexing set), then $M = \sum \oplus M_a$ and $P_a \xrightarrow{\pi_a} M_a \longrightarrow 0$ is the projective cover of M_a where $\pi_a = \pi \mid P_a$.*

Proof. The proof for the finite case [12, Prop. 2.4] will work here also.

PROPOSITION 1.4. *Let* $P_a \xrightarrow{\pi_a} M_a \longrightarrow 0$ *be the projective cover of* M_a *where* $a \in I$, *an indexing set. If* $f(Ker\ \pi_a) \subseteqq Ker\ \pi_b$ *for every* $a, b \in I$ and $f \in Hom_R(P_a, P_b)$, *then* $\Sigma \oplus M_a$ *is quasi-projective.*

Proof. It is sufficient to show that $\Sigma \oplus Ker\ \pi_a$ is an $\mathrm{End}_R(\Sigma \oplus P_a)$-module [12, Prop. 1.1]. Let q_c be the projection of $\Sigma \oplus P_a$ onto P_c and $f \in \mathrm{End}_R(\Sigma \oplus P_a)$. We will be done if we show $f(Ker\,\pi_b) \subseteqq \Sigma \oplus \mathrm{Ker}\,\pi_a$. Let $x \in \mathrm{Ker}\,\pi_b$. Since $q_a \circ (f|P_b) \in \mathrm{Hom}_R(P_b, P_a)$, $f(x) = (q_{a_1} \circ f)(x) + \cdots + (q_{a_n} \circ f)(x) \in \mathrm{Ker}\,\pi_a + \cdots + \mathrm{Ker}\,\pi_{a_n} \subseteqq \Sigma \oplus \mathrm{Ker}\,\pi_a$.

REMARK. If I is finite or R is left perfect, then the converse is true, that is, if $\Sigma \oplus M_a$ is quasi-projective, then $f(\mathrm{Ker}\,\pi_a) \subseteqq \mathrm{Ker}\,\pi_b$ for every $a, b \in I$ and $f \in \mathrm{Hom}_R(P_a, P_b)$.

COROLLARY 1.5. If M is quasi-projective and has a projective cover, then M^I is quasi-projective.

PROPOSITION 1.6. *If* M_1 *and* M_2 *are quasi-projective and have projective covers* P_1 *and* P_2 *which are isomorphic and* $M_1 \oplus M_2$ *is quasi-projective, then* $M_1 \cong M_2$.

Proof. The proof of the dual theorem for quasi-injective modules [7, Prop. 2.4] can be dualized.

Bass has shown [4, p. 473] that if R is a left perfect ring and P is a projective module, then $P = \Sigma \oplus Re_i$ where Re_i/Ne_i is simple and e_i is an idempotent in R. This result will be stated in a different form in the next proposition.

PROPOSITION 1.7. *Let* R *be left perfect. Then* P *is projective if and only if* $P = \sum_{i=1}^{k} \oplus (Re_i)^{g(i)}$ *where* Re_i *is the projective cover of a simple module,* e_i *is an indecomposable idempotent,* k *is the number of non-isomorphic simple modules, and* $Re_i \not\cong Re_j$ *if* $i \neq j$.

Proof. Ne_i is small in Re_i [4, p. 473]. Hence Re_i is the projective cover of Re_i/Ne_i and is indecomposable by Proposition 1.3. Since R/N is left Artinian [4, p. 467], and the simple R-modules and the simple R/N-modules are the same, there are only a finite number of nonisomorphic simple modules. Also, simple modules are isomorphic if and only if their projective covers are isomorphic.

PROPOSITION 1.8. *Let* R *be semi-perfect and* M *be a finitely generated, quasi-projective module. Then* M *is indecomposable* (*non-zero*) *if and only if* $End_R(M)$ *is a local ring.*

Proof. (i) If M is not indecomposable, then $\mathrm{End}_R(M)$ has a nonzero idempotent e which is different from the identity. Since neither e nor $1 - e$ is a unit, $\mathrm{End}_R(M)$ is not a local ring.

(ii) If M is indecomposable, then $M = Re/Je$ where J' is an ideal of R and e is an indecomposable idempotent. Thus $M = R^*e^*$ where $R^* = R/J$. R^* is semi-perfect [4, Lemma 2.2]. Since M is indecomposable as an R^*-module, e^* is an indecomposable idempotent. In addition $\mathrm{End}_R(M) = \mathrm{End}_{R^*}(R^*e^*) = e^*R^*e^*$. Finally, $e^*R^*e^*$ is a local ring because R^* is semi-perfect and e^* is indecomposable [10, p. 76].

LEMMA 1.9. *Let R be semi-perfect and $1 = e_1 + \cdots + e_n$ where $e_1, \cdots e_n$ are orthogonal, indecomposable idempotents. If*

$$Re_1/J_1e_1 \oplus Re_2/J_2e_2 \oplus \cdots \oplus Re_m/J_me_m$$

is quasi-projective where J_i, $i = 1, \cdots, m$, is an ideal, then there is an ideal J such that $Je_i = J_ie_i$ for $i = 1, \cdots, m$.

Proof. The projective cover of $\sum_{1=i}^{m} \oplus Re_i/J_ie_i$ is

$$\sum_{i=1}^{m} \oplus Re_i \xrightarrow{\pi} \sum_{i=1}^{m} \oplus Re_i/J_ie_i \longrightarrow 0$$

where $\mathrm{Ker}\,\pi = \sum_{i=1}^{m} \oplus J_ie_i$. Since $\mathrm{End}_R(\sum_{i=1}^{m} \oplus Re_i) = \sum_{j=1}^{m} \oplus_{i=1}^{m} \oplus e_iRe_j$, it follows that $J_ie_i \cdot e_iRe_j \subseteqq J_je_j$ for $i, j = 1, \cdots, m$ [12, Prop. 2.2]. Let

$$J = \sum_{i=1}^{m} \oplus J_ie_i \oplus \left(\sum_{i=1}^{m} \oplus J_ie_i\right) R \left(1 - \sum_{i=1}^{m} e_i\right).$$

Then J is an ideal because $R = \sum_{j=1}^{n} \oplus \sum_{i=1}^{n} \oplus e_iRe_j$. Also, $Je_i = J_ie_i$ for $i = 1 \cdots, m$.

REMARKS. 1. The proof for Lemma 1.9 remains valid if any subcollection of $e_1, \cdots e_n$ is used rather than the first m of them.

2. The result that for a semi-perfect ring $1 = e_1 + \cdots + e_n$ where $e_1, \cdots, e_n$ are orthogonal indecomposable idempotents can be found in [10].

THEOREM 1.10. *Let R be left perfect. Then M is a quasi-projective module if and only if*

$$M = \sum_{i=1}^{k} \oplus (Re_i/Je_i)^{g(i)}$$

where J is an ideal, $e_1, \cdots, e_k$ are indecomposable idempotents, the number of nonisomorphic simple R-modules is k, and $Re_1, \cdots, Re_k$ are the corresponding nonisomorphic projective covers. In addition *the decomposition is unique up to automorphism.*

Proof. (i) Let M be quasi-projective. If $M = 0$, then we can choose $J = R$ and be done. If $M \neq 0$, let $P \to M \to 0$ be the projective cover of M. By Proposition 1.7 $P = \sum_{i=1}^{k} \oplus (Re_i)^{g(i)}$ where $Re_1, \cdots, Re_k$ are the nonisomorphic indecomposable projective covers of all the simple modules. By Proposition 1.3 $M = \sum_{i=1}^{k} \oplus \sum_{a \in I_i} \oplus M_{ai}$ where $\operatorname{card}(I_i) = g(i)$. Proposition 1.6 shows that $M_{ai} \cong M_{bi}$ for every $a, b \in I_i$. From Proposition 1.1 $M_{ai} = Re_i/J_ie_i$ with J_i an ideal and e_i an indecompotent. As a result of Lemma 1.9 and the remark following it, there is an ideal J such that $Je_i = J_ie_i$ for $i = 1, \cdots, k$.

(ii) Conversely, if $M = \sum_{i=1}^{k} \oplus (Re_i/Je_i)^{g(i)}$ with the same notation as in the statement of the theorem and $J \neq R$, then Propositions 1.2 and 1.4 show that M is quasi-projective. If $J = R$, then $M = 0$ and is, of course, quasi-projective.

(iii) Uniqueness. Using Proposition 1.8 and a generalized Krull-Remark-Schmidt theorem which was proved by Azumaya [1, Thm. 1], we have the following result: if $\sum_{a \in A} \oplus M_a$ and $\sum_{b \in B} \oplus M'_b$ are two decompositions of quasi-projective module into indecomposable, modules, then there is a 1 to 1, onto mapping $f\colon A \to B$ such that $M_a \cong M'_{f(a)}$.

REMARKS. 1. Theorem 1.10 is true for semi-perfect rings if M is finitely generated.

2. If M is nonzero in Theorem 1.10, then J can be chosen is the radical of the ring.

2. Quasi-injective modules.

In the first section a decomposition theorem for quasi-projective modules was obtained. The motivation for attempting to prove this proposition came from a paper by Harada on quasi-injective modules [7]. Now Theorem 1.10 will be used to obtain a characterization for quasi-injective modules over left Artinian rings which have a finitely generated, lower distinguished (contains an isomorphic copy of every simple module), and injective module. This class of rings includes quasi-Frobenius rings and finitely generated algebras over commutative Artinian rings [2].

PROPOSITION 2.1. Let R *be left Artinian. Then* R *has a finitely generated, lower distinguished, injective module if and only if the injective envelope of every simple module is finitely generated.*

Proof. (Given by G. Azumaya). Assume Q is finitely generated, lower distinguished, and injective. Let $Q_1, \cdots, Q_k$ be the non-isomorphic injective envelopes of all the simple modules. Then

$Q = \sum_{i=1}^{k} \oplus Q_i^{h(i)}$ where $0 \leqq h(i) < \infty$ [2, Thm. 11, p. 268]. Since Q is lower distinguished, $h(i) \neq 0$ for each $i = 1, \cdots, k$. It follows that each Q_i is finitely generated. The converse is clear.

PROPOSITION 2.2. *If R is left Artinian and has a finitely generated, lower distinguished, injective module, then every indecomposable quasi-injective module is finitely generated.*

Proof. Let M be indecomposable and quasi-injective, and let Q be its injective envelope. Q is indecomposable [7, Proposition 2.3], so it is the injective envelope of a simple module [2, Thm. 1, p. 268]. Hence, Q is finitely generated by Proposition 2.1. Since R is left Noetherian, M is finitely generated.

REMARK. If R is perfect, then every indecomposable quasi-projective module is finitely generated by Proposition 1.7.

The following proposition was proved by Azumaya for the class of rings in the last two propositions and will be stated without giving his proof.

PROPOSITION 2.3. (Duality Theorem). *Let R be a left Artinian ring which has a finitely generated, injective, and lower distinguished module Q, and let $S = End_R(Q)$. Then for any finitely generated left R-module X, $X^* = \mathrm{Hom}_R(X, Q)$ is a finitely generated right S-module and $(X^*)^* = \mathrm{Hom}_S(X^*, Q) = {}_RX$. The same is true for finitely generated S-modules* [2, Thm. 8, p. 262].

PROPOSITION 2.4. *If R is left Noetherian and M is quasi-injective, then M^I is quasi-injective.*

Proof. Let Q be the injective, envelope of M. Since R is left Noetherian, Q^I is the injective envelope of M^I. With this result and a theorem of Johnson and Wong [9, Thm. 1.1], a procedure which is similar to the one found in the proof of Proposition 1.4 can be used to see that M^I is quasi-injective.

THEOREM 2.5. *Let R be left Artinian and have a finitely generated, lower distinguished, and injective module Q. Then M is quasi-injective if and only if*

$$M = \sum_{i=1}^{k} \oplus (\mathrm{Hom}(e_iS/e_iJ, Q))^{g(i)}$$

where $S = \mathrm{End}_R(Q)$, e_i is an indecomposable idempotent in S for $i = 1, \cdots, k$, J is an ideal of S, the number of nonisomorphic simple

R-modules is k, and for $i \neq j$ $e_iS \not\cong e_jS$. This decomposition is unique up to automorphism.

Proof. If $M = 0$, we can choose $J = S$. Thus we will assume that M is a nonzero quasi-injective module. It is known that if R is left Artinian, then it is left Noetherian and has only a finite number of simple R-modules. Harada has shown that for left Noetherian rings $M = \sum \oplus M_a$ where the M_a's are indecomposable quasi-injective modules and that this decomposition is unique up to automorphism [7, Prop. 2.5]. If Q_a is the injective envelope of M_a, then it is the injective envelope of a simple module (see proof of Prop. 2.2). By the dual theorem of Proposition 1.6 and the result that nonisomorphic simple modules have nonisomorphic injective envelopes, $M = \sum_{i=1}^{k} \oplus M_i^{g(i)}$ and $M_i \not\cong M_j$ for $i \neq j$.

As a result of Proposition 2.2, M_i is finitely generated. By the Duality Theorem $\mathrm{Hom}_R(M_i, Q)$ is a finitely generated, indecomposable, quasi-projective, right S-module. Also, S is right Artinian [2, Thm. 6, p. 259]. Hence, $\mathrm{Hom}_R(M_i, Q) = e_iS/e_iJ_i$ where e_i is an indecomposable idempotent in S, and J_i is an ideal of S. Since $\sum_{g(i)\neq 0} \oplus M_i$ is a direct summand of M it is quasi-injetive. It follows that $\mathrm{Hom}\,(\sum_{g(i)\neq 0} \oplus M_i, Q) = \sum_{g(i)\neq 0} \oplus \mathrm{Hom}(M_i, Q) = \sum_{g(i)\neq 0} \oplus e_iS/e_iJ_i$ and is quasi-projective. For $i \neq j$ $M_i \not\cong M_j$, so $e_iS \not\cong e_jS$. By Lemma 1.9 and a remark following it we can choose $J_i = J$ for $g(i) \neq 0$. In addition $M_i = \mathrm{Hom}\,(\mathrm{Hom}\,(M_i, Q), Q) = \mathrm{Hom}_S\,(e_iS/e_iJ, Q)$.

(ii) Suppose $M = \sum_{i=1}^{k} \oplus (\mathrm{Hom}_S\,(e_iS/e_iJ, Q)^{g(i)}$ with the same notation as in the statement of the theorem. Let $M' = \sum_{g(i)\neq 0} \oplus \mathrm{Hom}_S(e_iS/e_iJ, Q)$. Then $\mathrm{Hom}_R\,(M', Q) = \sum_{g(i)\neq 0} \oplus e_iS/e_iJ$ which is quasi-projective by Theorem 1.10. Thus M' is quasi-injective. Let $m = \max\{g(i)\}_{i=1,\dots,k}$ and $M'' = (M')^m$. Proposition 2.4 gives us that M'' is quasi-injective. Therefore the direct summand M is quasi-injective.

COROLLARY 2.6. *Let R be quasi-Frobenius. Then M is quasi-injective if and only if*

$$M = \sum_{i=1}^{k} \oplus (\mathrm{Hom}_R\,(e_iR/e_iJ, R))^{g(i)}.$$

Proof. R being quasi-Frobenius implies R is left Artinian, self injective, lower distinguished, and finitely generated [2, Thm. 6, p. 259]. Also, $R = \mathrm{End}_R(R)$.

COROLLARY 2.7. *Let R be a finitely generated algebra over a commutative Artinian ring K. Then M is quasi-injective if and only if*

$$M = \sum_{i=1}^{k} \oplus (\mathrm{Hom}_K(e_iR/e_iJ, F))^{g(i)}$$

where F is the K-injective envelope of $K/\mathrm{rad}\ K$.

Proof. R has a finitely generated, lower distinguished, injective module Q such that $R = \mathrm{End}_R(Q)$ [2, Prop. 19, p. 273]. The functors $\mathrm{Hom}_K(\cdot, F)$ and $\mathrm{Hom}_R(\cdot, Q)$ are naturally equivalent for finitely generated R-modules [2, Thm. 20, 275].

COROLLARY 2.8. *Let R be a finite dimensional algebra over a field K. Then M is quasi-injective if and only if*

$$M = \sum_{i=1}^{k} \oplus (\mathrm{Hom}_K (e_iR/e_iJ, K))^{g(i)}.$$

Proof. $K = F$ in Corollary 2.7.

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