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THE PRIMARY DECOMPOSITION THEORY FOR MODULES

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Necessary and sufficient conditions are given for the classical Lasker-Noether primary decomposition theory to exist on a module over an arbitrary commutative ring. Also investigation is conducted into the connection between the existence of the primay decomposition theory and the Artin-Rees property of a module.

In [1] we introduced a new technique for constructing decomposition theories for modules and we used it to give necessary and sufficient conditions for the Lesieur-Croisot tertiary decomposition theory to exist on a module over an arbitrary ring. By again making use of this technique we have obtained necessary and sufficient conditions for the classical Lasker-Noether primary decomposition theory to exist on a module over an arbitrary commutative ring. In the noncommutative case we have necessary and sufficient conditions for the primary theory to exist on an R-module M which has the property that nil ideals are nilpotent in each factor ring of $R/(0: M)$.

In [1], [4], and [7] investigation was conducted into the connection between the existence of the primary decomposition theory and the Artin-Rees property of a module. In § 2 we have obtained the following new results in this connection. For R commutative we show that if an R-module M is an Artin-Rees module and if each factor module of M is finite dimensional, then M has the primary decomposition theory. For R noncommutative the additional hypothesis is added that nil ideals are nilpotent in each factor ring of $R/(0: M)$. In addition we show that if an R-module M has the primary decomposition theory and if nil ideals are nilpotent in each factor ring of $R/(0: M)$, then M is an Artin-Rees module.

In § 3 we give some examples. In Example 1 we use our existence Theorem 1.7 to give an example of a module which has the primary decomposition theory but which is not Noetherian.

It should be brought to the attention of the reader that all the results in this paper which, for reasons of convenience, are stated for an R-module M where R is left Noetherian are also valid with the weaker assumption that nil ideals are nilpotent in each factor ring of $R/(0: M)$.

1. The primary theory. Throughout this paper, R will denote an arbitrary associative ring which does not necessarily have an identity. All R-modules will be assumed to be left R-modules and $\mathscr{M}$ will denote the category of all R-modules and R-homomorphisms.

The *primary radical* of an R-module M, denoted $p(M)$, is the intersection of all the prime ideals in R which contain the annihilator, $(0\colon M)$, of M. A submodule N of M is called *primary* [7, 8] if $N \neq M$ and each r in R, which annihilates a nonzero submodule of M/N, lies in $p(M/N)$. A finite set $\{N_i\colon i \in I\}$ of submodules of M is a *primary decomposition* of N in M if the following conditions are satisfied:

(1) $\bigcap_{i \in I} N_i = N$ and for no $i \in I$ is $\bigcap_{j \neq i} N_j \subseteqq N_i$;
(2) the N_i, $i \in I$ are primary submodules of M; and
(3) $p(M/N_i) \neq p(M/N_j)$ for $i \neq j$.

If each submodule of M has a primary decomposition in M, then M is said to have the *primary decomposition theory.*

An R-module S is said to be *p-stable* if $S \neq (0)$ and for each nonzero submodule N of S, then $p(N) = p(S)$. An ideal $\mathscr{P}$ in R is called an *associated ideal* of M if there exists a p-stable submodule S of M such that $\mathscr{P} = p(S)$. Denote the set of associated ideals of M by $P(M)$.

PROPOSITION 1.1. *Let M be an R-module where R is either commutative or left Noetherian. If $\mathscr{P} \in P(M)$ then $\mathscr{P}$ is prime.*

Proof. If R is commutative the result is Proposition 9.1 in [1]. If R is left Noetherian and $\mathscr{P} \in P(M)$ then there exists a p-stable submodule S of M such that $\mathscr{P} = p(S)$. Suppose that I, J are ideals of R such that $IJ \subseteqq \mathscr{P}$. Since $p(S)$ is a nil ideal modulo $(0\colon S)$ [3, p. 196], there exists a positive integer n such that $(IJ)^{n-1}S \neq (0)$ and $(IJ)^n S = (0)$ by Levitzki's theorem [3, p. 199]. If $J \not\subseteqq \mathscr{P} = p((IJ)^{n-1}S)$ then $J(IJ)^{n-1}S \neq (0)$. Hence $I \subseteqq \mathscr{P}$ since $(IJ)^n S = (0)$ and $\mathscr{P} = p(J(IJ)^{n-1}S)$.

A submodule N of M is said to be a *P-submodule* if $N \neq M$ and M/N is *P-stable*, i.e., there exists an ideal $\mathscr{P}$ in R such that for each nonzero submodule M'' of M/N, $P(M'') = \{\mathscr{P}\}$. A finite set $\{N_i\colon i \in I\}$ of submodules of M is a *P-decomposition* of N in M if the following conditions are satisfied:

(1) $\bigcap_{i \in I} N_i = N$ and for no $i \in I$ is $\bigcap_{j \neq i} N_j \subseteqq N_i$;
(2) the N_i, $i \in I$ are P-submodules of M; and
(3) $P(M/N_i) \neq P(M/N_j)$ for $i \neq j$.

If each submodule of M has a P-decomposition in M then M is said to have the *P-decomposition theory.* An R-module M is called *p-worthy* if each factor module M'' of M satisfies the following conditions:

(a) each nonzero submodule of M'' contains a p-stable submodule, and

(b) $P(M'')$ is finite.

In the terminology of [1] p is a radical function on $\mathscr{M}$ and P is the associated ideal function on $\mathscr{M}$ that is obtained from p. Therefore Theorem 4.10 in [1] shows that a necessary and sufficient condition for M to have the P-decomposition theory is that M be p-worthy. The condition that M be p-worthy is not a sufficient condition for M to have the primary decomposition theory. See Example 4 in [1]. We proceed with the task of finding necessary and sufficient conditions for the existence of the primary theory.

LEMMA 1.2. *Let N be a submodule of an R-module M. Then N is primary if M/N is p-stable.*

Proof. Suppose that $r \in R$ annihilates a nonzero submodule M'' of M/N. Then $r \in p(M'') = p(M/N)$. Hence N is primary.

THEOREM 1.3. *Let R be an arbitrary ring and let M be an R-module. Sufficient conditions for M to have the primary decomposition theory are the following*:

(1) *M is p-worthy, and*

(2) *S a P-submodule of M implies that M/S is p-stable.*

Proof. It follows from Theorem 4.10 in [1] that each submodule N of M has a P-decomposition $\{N_i : i \in I\}$ in M. Since each N_i is a P-submodule of M, each M/N_i is p-stable. Hence each N_i is primary by Lemma 1.2. Moreover $p(M/N_i) \neq p(M/N_j)$ for $i \neq j$ because $P(M/N_i) \neq P(M/N_j)$. Therefore $\{N_i : i \in I\}$ is a primary decomposition of N in M.

Let (r) denote the principal ideal in R which is generated by r in R.

LEMMA 1.4. *Let M be a nonzero R-module where R is either commutative or left Noetherian. If $r \in p(M)$ then there exists a nonzero submodule N of M such that $(r)N = (0)$.*

Proof. The $p(M)$ is a nil ideal modulo $(0: M)$. If R is commutative then indeed (r) is nilpotent modulo $(0: M)$. If R is left Noetherian then (r) is nilpotent modulo $(0: M)$ by Levitzki's theorem. In either case there exists a positive integer n such that $(r)^{n-1}M \neq (0)$ and $(r)^n M = (0)$. Therefore (r) annihilates the nonzero submodule $(r)^{n-1}M$ of M.

PROPOSITION 1.5. *Let M be an R-module where R is either* *commutative or left Noetherian and let N be a submodule of M. Then N is primary if and only if M/N is p-stable.*

Proof. The "if" follows from 1.2.

Suppose that N is primary. If M'' is a nonzero submodule of M/N then $p(M/N) \subseteqq p(M'')$. If $r \in p(M'')$ then Lemma 1.4 shows that (r) annihilates a nonzero submodule of M/N. Wherefore $r \in p(M/N)$ since N is primary. Therefore M/N is p-stable.

PROPOSITION 1.6. *Let M be an R-module where R is either commutative or left Noetherian and let N be a submodule of M. If $\{N_i: i \in I\}$ is a primary decomposition of N in M then $\{N_i: i \in I\}$ is a P-decomposition of N.*

Proof. Proposition 1.5 yields that each M/N_i is p-stable. Hence each N_i is a P-submodule of M because $P(M/N_i) = \{p(M/N_i)\}$. Moreover $P(M/N_i) \neq P(M/N_j)$ for $i \neq j$ since $p(M/N_i) \neq p(M/N_j)$. Therefore $\{N_i: i \in I\}$ is a P-decomposition of N in M.

THEOREM 1.7. *Let M be an R-module where R is either commutative or left Noetherian. Necessary and sufficient conditions for M to have the primary decomposition theory are the following:*

(1) *M is p-worthy, and*

(2) *P-submodules of M are primary.*

Proof. The sufficiency follows from Theorem 1.3.

In order to prove the necessity assume that M has the primary decomposition theory. Then M has the P-decomposition theory by Proposition 1.6. Therefore M is p-worthy by Theorem 4.10 in [1].

Suppose that N is a P-submodule of M. Then there exists a primary decomposition $\{N_i: i = 1, 2, \cdots, k\}$ of N in M. Moreover $\{N_i: i = 1, 2, \cdots, k\}$ is a P-decomposition of N in M. Since N is a P-submodule and $P(M/N) = \bigcup_{i=1}^{k} P(M/N_i)$ [1, Proposition 4.5], we have that $k = 1$. Consequently $N = N_1$ and so N is primary.

As a corollary we get the following well-known theorem.

COROLLARY 1.8. *Let R be a commutative ring and let M be a Noetherian R-module. Then M has the primary decomposition theory.*

Proof. The fact that M is p-worthy follows from Proposition 5.6 and Lemma 9.2 in [1]. That P-submodules of M are primary follows from Proposition 9.3 in [1].

2. Artin-Rees modules. An R-module M is said to be an *Artin-Rees module* [4, 7] if it has the *Artin-Rees property*, i.e., for each submodule N of M, ideal I in R, and positive integer n, there exists a positive integer h such that $I^h M \cap N \subseteqq I^n N$. It should be noted that submodules and factor modules of Artin-Rees modules are Artin-Rees modules. An R-module M is said to be *finite dimensional* [2] over R if each direct sum of nonzero submodules of M has only a finite number of terms. An R-module U is *uniform* if $U \neq (0)$ and each pair of nonzero submodules of U has nonzero intersection. A submodule E of M is called *essential* if E has nonzero intersection with each nonzero submodule of M.

LEMMA 2.1. *Let M be an R-module where R is either commutative or left Noetherian. If M is an Artin-Rees module then P-submodules of M are primary.*

Proof. It is sufficient to show that if (0) is a P-submodule of M then (0) is primary. Suppose that $r \in R$ annihilates a nonzero submodule N of M. Since (0) is a P-submodule of M, there exists an ideal $\mathscr{P}$ in R such that $P(N) = P(M) = \{\mathscr{P}\}$. Hence there exists a p-stable submodule S of N such that $\mathscr{P} = p(S)$. Whence $r \in \mathscr{P}$.

We claim that $(0: (r)) = \{m \in M: (r)m = 0\}$ is an essential submodule of M. If W is a nonzero submodule of M then $P(W) = \{\mathscr{P}\}$. Hence there exists a nonzero submodule W' of W such that $\mathscr{P} = p(W')$. Since $r \in \mathscr{P}$ there is a nonzero submodule W'' of W' such that $(r)W'' = (0)$ by 1.4. Therefore $(0: (r)) \cap W \supseteqq W'' \neq (0)$ and so $(0: (r))$ is essential.

Now we apply the Artin-Rees property to $(0: (r))$, (r), and $n=1$ to obtain an h such that $(r)^h M \cap (0: (r)) \subseteqq (r)(0: (r)) = (0)$. Thus $(r)^h M = (0)$ and so $r \in p(M)$. Therefore (0) is primary.

THEOREM 2.2. *Let M be an R-module where R is either commutative or left Noetherian. Sufficient conditions for M to have the primary decomposition theory are the following*:

(1) *M is p-worthy, and*

(2) *M is an Artin-Rees module.*

Proof. The result follows from Theorem 1.7 since M Artin-Rees guarantees that P-submodules of M are primary.

We will now proceed to show that a module M over either a commutative or left Noetherian ring has the primary decomposition theory provided that M is an Artin-Rees module and each factor module of M is finite dimensional.

LEMMA 2.3. *Let M be an R-module where R is either commutative or left Noetherian. If M is an Artin-Rees module then uniform submodules are p-stable.*

Proof. Let U be a uniform submodule of M and let V be a nonzero submodule of U. Indeed $p(U) \subseteqq p(V)$. Let $r \in p(V)$. By Lemma 1.4 there exists a nonzero submodule N of V such that $(r)N = (0)$. The Artin-Rees property of U applied to N, (r), and $n = 1$, produces an h such that $(r)^h U \cap N \subseteqq (r)N = (0)$. Since U is uniform, we have that $(r)^h U = (0)$. Hence $r \in p(U)$. Therefore $p(V) = p(U)$ and so U is p-stable.

LEMMA 2.4. *Let M be an R-module where R is either commutative or left Noetherian. If M is a finite dimensional Artin-Rees module then $P(M)$ is finite.*

Proof. From [2, Th. 3.3] we have that there exists an essential submodule E of M of the form $E = U_1 \oplus U_2 \oplus \cdots \oplus U_n$, where each U_i is uniform. Moreover $P(M) = P(E)$ [1, Proposition 2.5] and $P(E) = \bigcup_{i=1}^{n} P(U_i)$ [1, Proposition 2.4]. Since uniform submodules are p-stable $P(U_i)$ consists of a single ideal. Therefore $P(M)$ is finite.

PROPOSITION 2.5. *Let M be an R-module where R is either commutative or left Noetherian. If each factor module of M is finite dimensional and if M is an Artin-Rees module then M is p-worthy.*

Proof. Let M'' be a factor module of M. Since M'' is a finite dimensional Artin-Rees module, each nonzero submodule of M'' contains a uniform [2, Lemma 3.1]; hence, p-stable submodule. Furthermore $P(M'')$ is finite. Therefore M is p-worthy.

THEOREM 2.6. *Let M be an R-module where R is either commutative or left Noetherian. Sufficient conditions for M to have the primary decomposition theory are the following:*

(1) *each factor module of M is finite dimensional, and*

(2) *M is an Artin-Rees module.*

Proof. The result follows immediately from Theorem 2.2 and Proposition 2.5.

Neither the hypothesis that each factor module of M be finite dimensional nor the hypothesis that M be an Artin-Rees module can be deleted. See Example 2 and [1, Example 4].

THEOREM 2.7. *Let R be a left Noetherian ring. If an R-module M has the primary decomposition theory then M is an Artin-Rees module.*

Proof. Let N be a submodule of M, I an ideal in R, and n a positive integer. Consider I^nN. Since $I^n \subseteqq (I^nN: N)$, we have that $I^n \subseteqq \mathscr{P}$ for each $\mathscr{P}$ in $P(N/I^nN)$.

Proposition 1.6 shows that M has the P-decomposition theory. Then Proposition 11.1 in [1] produces a submodule X of M such that $X \cap N = I^nN$ and $P(M/X) = P(N/I^nN)$.

The submodule X of M has a primary decomposition $(M_1, M_2, \cdots, M_k\}$ in M which is also a P-decomposition of X in M. If $P\{M/M_i) = \{\mathscr{P}_i\}$ for $i = 1, 2, \cdots, k$, then $P(M/X) = \{\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_k\}$ by Proposition 4.5 in [1]. Because $P(M/X) = P(N/I^nN)$ we have that $I^n \subseteqq \mathscr{P}_i$ for each i.

Since M_i is a primary submodule of M, we have that M/M_i is p-stable (1.5) and hence $\mathscr{P}_i = p(M/M_i)$ for each i. Whence $\mathscr{P}_i$ is a nil ideal modulo $(M_i: M)$. By Levitzki's theorem we have that for each i, there exists an h_i such that $\mathscr{P}_i^{h_i} \subseteqq (M_i: M)$. Accordingly there is an h such that $\mathscr{P}_i^h M \subseteqq M_i$ for each i. Thus $I^{nh}M \subseteqq M_1 \cap M_2 \cap \cdots \cap M_k = X$. So $I^{nh}M \cap N \subseteqq X \cap N = I^nN$. Therefore M is an Artin-Rees module.

As this theorem shows, a module M over a left Noetherian ring which has the primary decomposition theory has the Artin-Rees property. Example 1 shows that we cannot also deduce that each factor module of M is finite dimensional.

We have the following well-known theorem as a corollary.

COROLLARY 2.8. *Let R be a commutative ring and let M be an R-module such that $RM = M$. If M is Noetherian then M is an Artin-Rees module.*

Proof. By using a technique similar to the one used in the proof of Theorem 2 in [6, p. 180], we can show that $R/(0: M)$ is Noetherian. Also M has the primary decomposition theory by Corollary 1.8. Consequently M is Artin-Rees.

The following theorem is an extension of the classical Krull "Intersection Theorem".

THEOREM 2.9. *Let R be a left Noetherian ring and let M be an R-module such that*

(1) *M is p-worthy, and*

(2) *P-submodules of M are primary. For an ideal I of R, set* $N = \bigcap_n I^n M$. *Then* $IN = N$.

Proof. Theorem 1.7 shows that M has the primary decomposition theory. Hence M is an Artin-Rees module by Theorem 2.7. Accordingly, there exists an h such that $I^h M \cap N \subseteqq IN$. Thus $N = I^h M \cap N \subseteqq IN \subseteqq N$ and so $N = IN$.

As a corollary we get the following well-known theorem.

COROLLARY 2.10. *Let R be a commutative ring and let M be a Noetherian R-module such that* $RM = M$. *For an ideal I of R, set* $N = \bigcap_n I^n M$. *Then* $IN = N$. *Furthermore, if I is contained in the Jacobson radical of R then* $N = (0)$.

Proof. Again $R/(0: M)$ is Noetherian and M has the primary decomposition theory. Therefore the result follows from 2.9.

3. Examples. The following provides an example of a module M over a commutative ring which has the primary decomposition theory and is an Artin-Rees module. However there exist factor modules of M which are not finite dimensional. Hence M is not Noetherian.

EXAMPLE 1. Let Z be the ring of rational integers and let $\mathscr{P}_i$, $i = 1, 2, \cdots, n$ be nonzero proper prime ideals in Z. Consider the Z-module $M = \sum_{j=1}^{\infty} \oplus N_j$ where each $N_j = Z/\mathscr{P}_i$ for some i. From well-known properties of semisimple modules it follows that each section, i.e., submodule of a factor module, of M is isomorphic to a direct sum of a subset of $\{N_j : j = 1, 2, \cdots\}$. By making use of this fact we will show that M is p-worthy and that P-submodules of M are primary. Then it will follow from Theorem 1.7 that M has the primary decomposition theory.

Let M'' be a factor module of M. Then $P(M'') \subseteqq P(M)$. Since $P(Z/\mathscr{P}_i) = \{\mathscr{P}_i\}$, it follows from [1, Proposition 2.4] that $P(M) = \bigcup_{j=1}^{\infty} P(N_j) = \{\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_n\}$. Hence $P(M'')$ is finite. That each nonzero submodule of M'' contains a p-stable submodule follows from the fact that each nonzero section of M is isomorphic to a direct sum of a nonempty subset of $\{N_j : j = 1, 2, \cdots\}$. Therefore M is p-worthy.

Suppose that N is a P-submodule of M with say $P(M/N) = \{\mathscr{P}_i\}$. If $z \in Z$ annihilates a nonzero submodule of M/N then $z \in \mathscr{P}_i$. Since $P(M/N) = \{\mathscr{P}_i\}$ and M/N is isomorphic to a direct sum of a subset of $\{N_j : j = 1, 2, \cdots\}$, we have that M/N is isomorphic to a direct sum of copies of $Z/\mathscr{P}_i$. Hence $z \in p(M/N)$. Thence N is primary.

Consequently M has the primary decomposition theory. That M is an Artin-Rees module now follows from Theorem 2.7.

The following is an example of an Artin-Rees module which does not have the primary decomposition theory.

EXAMPLE 2. Let Z be the ring of rational integers and let (p) be the ideal in Z which is generated by the prime number p. Consider the Z-module $M = \sum_p^{\oplus} Z/(p)$. The submodule (0) does not have a primary decomposition in M since $P(M) = \bigcup\{(p): p \text{ prime}\}$ is not finite.

We claim that M is an Artin-Rees module. It is immediate that each submodule N of M has the form $N = \sum_{p \in S}^{\oplus} Z/(p)$ where $S \subseteqq \{p: p \text{ prime}\}$. By using this fact we can show that for each submodule N of M, ideal I in Z, and positive integer n, $I^n M \cap N \subseteqq I^n N$. Therefore M is Artin-Rees.

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