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QUASIFIBRATION AND ADJUNCTION

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This paper is concerned with the preservation of quasifibration under pair adjunction of Hurewicz fibrations, and the preservation of weak homotopy equivalence under pair adjunction of identity maps.

The foundations of the theory of quasifibrations were laid by Dold and Thom in their paper [3] in which they proved that the homotopy groups of the infinite symmetric product of a space X were naturally isomorphic with the integral homology groups of X. Another application was soon given by Dold and Lashof [2] generalizing Milnor's construction [9] of a universal principal fibre bundle with given structure group and their results were further generalized by Stasheff [13], Milgram [8], Steenrod [15] and Stasheff [14]. Other applications of quasifibrations occur in [6], [5] and [1]. Since, as a generalisation of fibration, quasifibration has a serious deficiency (it is not preserved under pull-back) one may well ask why it has proved to be so useful. A study of the papers referred to reveals that it is essentially the behaviour of quasifibration with respect to adjunction which is involved. However, the relevant arguments mostly rely on a basic lemma of [3] (lemma 2.10) and proceed ad hoc.

Let $p: P \to P'$, $t: T \to T'$, $q: Q \to Q'$ be *pairs* (i.e., continuous maps) and let $\phi = (f, f'): p \to t$, $\gamma = (g, g'): p \to q$ be pair maps and consider the push-out diagram

(0.1)
$$\begin{tikzcd} p \arrow[r, "\gamma"] \arrow[d, "\phi"'] & q \arrow[d, dashed] \\ t \arrow[r, dashed] & r \end{tikzcd}$$

in the category of pairs. γ is a *weak homotopy equivalence of fibres* (WHEF) if, for each $x \in P'$, the induced map

$$g'': p^{-1}(x) \longrightarrow q^{-1}(g'x)$$

is a weak homotopy equivalence. We shall prove the following theorem.

THEOREM 0.2. *If f' is a closed cofibration, if t is a fibration, if p is the pull-back of t over f', if q is a quasifibration and if γ is a* WHEF *then r is a quasifibration.*

In most of the constructions of quasifibrations in the papers cited the situation is or is equivalent to the special case of 0.2 in which t is a trivial fibration.

That 0.2 does not remain valid if "quasifibration" is replaced by "Hurewicz fibration" can be seen from the example suggested by the diagram.

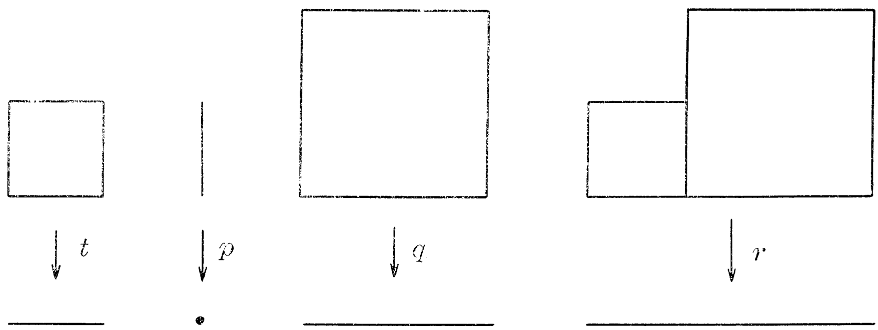

FIGURE 1

Let $1_P\colon P \to P$ denote the identity pair. Replacing p by 1_P and t by 1_T we shall obtain the following analogue of 0.2 for weak homotopy equivalences.

THEOREM 0.3. *If f is a closed cofibration, if $\phi = (f, f)\colon 1_P \to 1_T$ and if q is a weak homotopy equivalence then r is a weak homotopy equivalence.*

0.3 is particularly useful when the range and domain of r are not 1-connected for then a proof via relative homology isomorphisms and the J.H.C. Whitehead theorem breaks down.

1. Mapping cylinders. We recall that the space $R = R(f, g)$ obtained by completing the push-out diagram

(1.1)
$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "f"'] & Q \arrow[d, dashed, "\bar{f}"] \\ T \arrow[r, dashed, "\bar{g}"] & R \end{tikzcd}$$

is that obtained from the topological sum $T \vee Q$ (disjoint union) of T and Q by factoring out by the relation

$$fx \sim gx \qquad (x \in P) .$$

We shall denote the equivalence class of $y \in T \vee Q$ by $\{y\}$. Let i_0, i_1: $P \to P \times I$ be the maps given by

$$i_0x = (x, 0), i_1x = (x, 1) \qquad (x \in P) .$$

We recall that the *mapping cylinder* of f is the space $Z_f = R(f, i_1)$. The projection of $P \times I$ on to P induces a retraction $b: Z_f \to T$ and if we set $i = \bar{f} \cdot i_0$, we have $b \cdot i = f$ and $b \cdot \bar{i}_1 = 1_T$. The space $Z(f, g) = R(i, g)$ has been called [10] the mapping cylinder of the cotriad $T \xleftarrow{f} P \xrightarrow{g} Q$ $(TfPgQ)$. The cotriad map $(b, 1_P, 1_Q): (Z_f iPgQ) \to (TfPgQ)$

$$\begin{array}{ccccc} Z_f & \xleftarrow{\quad i \quad} & P & \xrightarrow{\quad g \quad} & Q \\ {\scriptstyle b}\big\downarrow & & {\scriptstyle 1_P}\big\downarrow & & \big\downarrow{\scriptstyle 1_Q} \\ T & \xleftarrow{\quad f \quad} & P & \xrightarrow{\quad g \quad} & Q \end{array}$$

induces a map $k(f, g): Z(f, g) \to R$ and we have:

LEMMA 1.2. *If f (or g) is a cofibration then $k(f, g)$ is a homotopy equivalence.*

Proof. If f is a cofibration then $(1_P, b): i \to f$ is a homotopy equivalence of pairs. A proof for the reduced case but equally valid in our situation is given in [4; pp. 17, 18]. Then 1.2 is a consequence of the following lemma, the proof of which being straightforward is omitted. Let $(t, p, q): (TfPgQ) \to (T'f'P'g'Q')$ be a cotriad map. Then there is an induced map

$$r: R(f, g) \longrightarrow R(f', g')$$

and we remark that r completes diagram 0.1. We have

LEMMA 1.3. *If (t, p, q) is a cotriad homotopy equivalence then r is a homotopy equivalence.*

REMARK 1.4. In view of [11; Satz 3], the conclusion of 1.2 remains valid if f has instead the weak homotopy extension property (WHEP).

The cotriad map (t, p, q) also induces a map $r_z: Z(f, g) \to Z(f', g')$ and we have:

LEMMA 1.5. *If t, p, q are homotopy equivalences then r_z is a homotopy equivalence. If, further, f and f' (or f and g') have the* WHEP *then r is a homotopy equivalence.*

The proof of the first assertion of 1.5 will also be omitted since it is equivalent to a special case of [10; 4.6], the proof of the dual [10; 2.6] having been given in full. The first assertion combined with 1.2 yields the second assertion.

Similarly the pair map $(p, t): f \to f'$ induces a diagram

(1.6)
$$\begin{tikzcd}
P \arrow[r, "i"] \arrow[d, "p"] \arrow[dr, phantom, "\xrightarrow{\iota}"] & Z_f \arrow[r, "b"] \arrow[d, "z"] \arrow[dr, phantom, "\xrightarrow{\beta}"] & T \arrow[d, "t"] \\
P' \arrow[r, "i'"] & Z_{f'} \arrow[r, "b'"] & T'
\end{tikzcd}$$

where $\iota = (i, i'): p \to z$, $\beta = (b, b'): z \to t$ and $\beta \cdot \iota = \phi$, which in turn yields the push-out diagram

$$\begin{tikzcd}
p \arrow[r, "\gamma"] \arrow[d, "\iota"'] & q \arrow[d, dashed] \\
z \arrow[r, dashed] \arrow[d, "\beta"'] & r_z \arrow[d, dashed, "\chi = (k, k')"] \\
t \arrow[r, dashed] & r\,.
\end{tikzcd}$$

We have

LEMMA 1.7. *If t is a fibration, if f' is a cofibration, if p is the pull-back of t over f' and if γ is a* WHEF *then χ is a* WHEF.

Proof of 1.7. For any map $m: M \to M'$ and any $x \in M'$, let $F(m, x)$ denote the fibre of m lying above x. Let $y \in Z(f', g')$ and let

$$k'': F(r_z, y) \longrightarrow F(r, k'y)$$

be the map induced by χ. There are three cases. If $y = \{x\}$ where $x \in Q'$ and $x \neq g'x'$ for any $x' \in P'$ then clearly $F(r_z, y)$ and $F(r, k'y)$ are homeomorphic with $F(q, x)$. Similarly if $y = \{x\}$, where $x \in T'$ and $x \neq f'x'$ for any $x' \in P'$ then $F(r_z, y)$ and $F(r, k'y)$ are homeomorphic. Suppose that

$$y = \{x, s\}(x \in P', s \in I) \text{ and let}$$
$$[x] = \{x' \in P' \mid g'x' = g'x\}\,.$$

Since f', being a cofibration, is injective [16], there is a push-out diagram

$$\begin{tikzcd}
p^{-1}[x] \arrow[r, "g''"] \arrow[d, "f''"] & F(q, g'x) \arrow[d, dashed, "h"] \\
t^{-1}(f'[x]) \arrow[r, dashed] & F(r, k'y)\,.
\end{tikzcd}$$

Then f'' is actually an equivalence: it is injective since f is injective, and surjective since P is a pull-back. It follows that h is an equivalence. If $0 \leqq s < 1$ then $F(r_z, y) \approx F(p, x)$ and k'' is equivalent to $g | F(p, x)\colon F(p, x) \to F(q, g'x)$ which is a weak homotopy equivalence. Similarly if $s = 1$ then k'' is an equivalence. Hence χ is a WHEF.

2. **Proper cofibration.** A map $i\colon A \to X$ is a *cofibration* if for any map $f\colon X \to Y$ and any homotopy $g_t\colon A \to Y$ such that $g_0 = f \cdot i$ there exists a homotopy $f_t\colon X \to Y$ such that $f_0 = f$ and $g_t = f_t \cdot i$. If, further, $i(A)$ is a closed subset of X then i is a *closed* cofibration. The following lemma is [16; Th. 2].

LEMMA 2.1. *i is a closed cofibration if and only if there exist*

(i) *a neighborhood U of $i(A)$ and a homotopy $H\colon U \times I \to X$ such that $H(x, 0) = x$, $H(a, t) = a$, $H(x, 1) \in \mathrm{i}(A)$ $(x \in U, a \in \mathrm{i}(A), t \in I)$;*

(ii) *a map $u\colon X \to I$ such that $i(A) = u^{-1}(0)$ and $ux = 1$ for all $x \in X - U$.*

If U, H and u exist and satisfy the further conditions $U = u^{-1}([0, 1))$, $H(U \times I) \subseteqq U$ then i is a *proper* cofibration. If $f\colon P \to T$ is any map, we remark that the associated map $i\colon P \to Z_f$ is a proper cofibration, for we may set $u\{x, t\} = t$, $u\{y\} = 1$, $H(\{x, t\}, s) = \{x, ts\}$ $(x \in P, y \in T, s, t \in I)$.

Let $f, g, \bar{f}, \bar{g}$ be as in diagram 1.1. We have

LEMMA 2.2. *If f is a proper cofibration then so is $\bar{f}$.*

Proof. Let H and u represent f as a proper cofibration and let $\bar{u}\colon R \to I$ be such that

$$\bar{u}\{x\} = ux \ (x \in T),\ \bar{u}\{y\} = 0 \ (y \in Q) \ .$$

Then $\bar{u}$ is well-defined and continuous and if we let $\bar{U} = \bar{u}^{-1}([0, 1))$ and let $\bar{H}\colon \bar{U} \times I \to R$ be such that

$$\bar{H}(\{x\}, s) = \{H(x, s)\},\ \bar{H}(\{y\}, s) = \{y\} (x \in U, y \in Q, s \in I) \ ,$$

then $\bar{H}$ and $\bar{u}$ represent $\bar{f}$ as a proper cofibration. We remark further that there is a commutative diagram:

(2.3)
$$\begin{tikzcd} U \times I \arrow[r, "H"] \arrow[d, "(\bar{g} | U) \times 1_I"'] & T \arrow[d, "\bar{g}"] \arrow[dr, "u"] & \\ \bar{U} \times I \arrow[r, "\bar{H}"] & R \arrow[r, "\bar{u}"'] & I \end{tikzcd} .$$

Now let $\phi = (f, f') = p \to t$ be a pair map. ϕ is a *proper cofibration* if H and u (resp. H' and u') represent f (resp. f') as a proper cofibration and are such that the following diagram is commutative.

$$\begin{tikzcd}
U \times I \arrow[r, "H"] \arrow[dd, "(t \mid U) \times 1_I"] & T \arrow[dd, "t"] \arrow[dr, "u"] & \\
 & & I \\
U' \times I \arrow[r, "H'"] & T' \arrow[ur, "u'"'] &
\end{tikzcd} \tag{2.4}$$

Thus the map $(f, \bar{f})\colon g \to \bar{g}$ in the situation of lemma 2.2 yields an example of proper cofibration. It is easy to check that another example is the pair map $\iota\colon p \to z$ in diagram 1.6. If ϕ is a proper cofibration note that, since $f(P) = u^{-1}(0)$ and $f'(P') = u'^{-1}(0)$, it is necessarily the case that $f(P)$ is saturated with respect to t. One may also prove that z is equivalent to a retract of $t \times 1_I$, from which it follows that ϕ is a cofibre map in the sense of Eckmann-Hilton [4; p 74]. As we shall not rely on these facts the details are omitted. We shall however require the following pair analogue of 2.2 concerning the push-out diagram

$$\begin{tikzcd}
p \arrow[r, "r"] \arrow[d, "\phi"'] & q \arrow[d, "\bar{\phi}"] \\
t \arrow[r, "\bar{r}"] & r\,.
\end{tikzcd}$$

LEMMA 2.5. *If ϕ is a proper cofibration then so is $\bar{\phi}$.*

Proof. Let H, H', u, u' represent ϕ as a proper cofibration, let $\bar{H}$ and $\bar{u}$ be defined as in the proof 2.2 and let $\bar{H}'$ and $\bar{u}'$ be given by analogous formulae. Then in view of 2.2 it is only necessary to check the commutativity of

$$\begin{tikzcd}
\bar{U} \times I \arrow[r, "\bar{H}"] \arrow[dd, "(r \mid \bar{U}) \times 1_I"] & R \arrow[dd, "r"] \arrow[dr, "\bar{u}"] & \\
 & & I \\
\bar{U}' \times I \arrow[r, "\bar{H}'"] & R' \arrow[ur, "\bar{u}'"'] &
\end{tikzcd} \tag{2.6}$$

However this follows easily from the commutativity of diagram 2.4 and the relevant definitions.

Our chief reason for introducing the concept of proper cofibration is the following lemma.

LEMMA 2.7. *If t is a fibration, if q is a quasifibration, if ϕ:*

$p \to t$ is a proper cofibration, if ϕ is a WHEF *and if γ is a* WHEF *then r is a quasifibration.*

Proof. Since $\bar{\phi}$ is a proper cofibration, $\bar{f}$ and $\bar{f}'$ are cofibrations and hence injective maps. Moreover $\bar{f}(Q)$ is saturated with respect to r and since q is a quasifibration it follows that $\bar{f}'(Q') = \bar{u}'^{-1}(0) = B'$ (say) is distinguished in the sense of Dold-Thom [3]. We claim that $\bar{U}'$ is also distinguished, for $\bar{H}$ and $\bar{H}'$ define homotopies $D_s\colon \bar{U} \to \bar{U}$, $d_s\colon \bar{U}' \to \bar{U}'$ with $D_0 = 1$, $D_1(\bar{U}) \subseteqq \bar{f}(Q) = E'$ (say), $D_s(E') \subseteqq E'$, $d_0 = 1$, $d_1(\bar{U}') \subseteqq B'$, $d_s(B') \subseteqq B'$. In view of [3; 2.10] it will be sufficient to prove, for each $x \in \bar{U}'$ and $i \geqq 0$, that

$$D_{1*}\colon \pi_i(F(r, x)) \approx \pi_i(F(r, d_1x)) \,. \tag{2.8}$$

2.8 is certainly satisfied if $x \in B'$. Suppose that $x \in \bar{U}' - \bar{f}'(Q')$. Then $x = \{y\}$ where $y \in U' - f'(P')$ and $\bar{\gamma}$ induces a homeomorphism $\bar{g}''\colon F(t, y) \to F(r, x)$. Let $k_s\colon U \to U$ and $k'_s\colon U' \to U'$ be the homotopies associated with H and H' respectively. Then commutativity in 2.4 yields $t \cdot k_s = k'_s \cdot t$ so that if $y' = k'_1 y$ we have a map

$$(k_1 \,|\, F(t, y))\colon F(t, y) \longrightarrow F(t, y') \,.$$

Commutativity in 2.3′ yields $\bar{g}' \cdot k'_s = d_s \cdot \bar{g}'$, hence $\bar{g}'y' = d_1 x$ and we have an induced map

$$(\bar{g} \,|\, F(t, y'))\colon F(t, y') \longrightarrow F(r, d_1x) \,.$$

Further, commutativity in 2.3 yields $\bar{g} \cdot k_s = D_s \cdot \bar{g}$ so that we have commutativity in

$$\begin{array}{ccc} F(t, y) & \xrightarrow{k_1 | F(t, y)} & F(t, y') \\ {\scriptstyle \bar{g}''} \big\downarrow & & \big\downarrow {\scriptstyle \bar{g} | F(t, y')} \\ F(r, x) & \xrightarrow{D_1 | F(r, x)} & F(r, d_1x) \end{array} \quad .$$

But t is a fibration, hence $k_{1*}\colon \pi_i(F(t, y) \approx \pi_i(F(t, y'))$, by [12; 2.8.13]. Since $\bar{g}''$ is a homeomorphism it will be sufficient to prove $\bar{g}_*\colon \pi_i(F(t, y')) \approx \pi_i(F(r, d_1x))$. Since $d_1x = \bar{g}'y' \in B'$, there exists $x' \in P'$ such that $f'x' = y'$ and hence we have an induced diagram

$$\begin{array}{ccc} F(p, x') & \longrightarrow & F(q, g'x') \\ \big\downarrow & & \big\downarrow {\scriptstyle \bar{f} | F(q, g'x')} \\ F(t, y') & \xrightarrow{\bar{g} | F(t, y')} & F(r, d_1x) \end{array} \quad .$$

Since $\bar{f}$ is injective and $\bar{f}(Q)$ is saturated with respect to r, $\bar{f} | F(q, g'x')$ is a homeomorphism and the remaining arrows induce homotopy isomorphisms since γ and ϕ are both WHEF. To complete the proof of 2.7 we may observe that $R' - B'$ is open, for $\bar{f}'$ is a closed cofibration. Moreover $R' - B'$ is distinguished since $(\bar{g}, \bar{g}')$ induces an equivalence of a restriction of t with $r|r^{-1}(R' - B')$. Similarly $\bar{U}' \cap (R' - B')$ is distinguished and hence 2.7 follows from [3; Satz 2.2].

We may also complete the proof of 0.2, for it follows from 2.7 that r_z is a quasifibration. By 1.2, k and k' are homotopy equivalences and, by 1.7, χ is a WHEF. A standard argument using the 5-lemma now shows that r is a quasifibration.

Michael McCord has shown [7] that many of the proofs of [3] can be modified so as to apply to weak homotopy equivalences. Let $p: E \to B$ be a map and let U be a subset of B. Then U is *distinguished* in the sense of McCord if $p|p^{-1}(U): p^{-1}(U) \to U$ is a weak homotopy equivalence. We shall need the following simple analogue of [3; 2.10].

LEMMA 2.9. *Let $p: E \to B$ be a continuous map onto B. Let B' be distinguished and let $E' = p^{-1}(B')$. If there exist homotopies $D_t: E \to E$, $d_t: B \to B$ with $D_0 = 1_E$, $D_t(E') \subseteqq E'$, $D_1(E) \subseteqq E'$, $d_0 = 1_B$, $d_t(B') \subseteqq B'$, $d_1(B) \subseteqq B'$ and with $p \cdot D_1 = d_1 \cdot p$, then B itself is distinguished.*

Proof. For any $x \in B$ and any $y \in p^{-1}(x)$ we have a commutative diagram

$$\begin{array}{ccc} \pi_i(E, y) & \xrightarrow{D_{1*}} & \pi_i(E', D_1 y) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle (p|E')_*} \\ \pi_i(B, x) & \xrightarrow{d_{1*}} & \pi_i(B', d_1 x) \end{array} .$$

Since three of the arrows are isomorphisms, the fourth is also.

A weak homotopy equivalence analogue of 2.7 is as follows.

LEMMA 2.10. *If $f: P \to T$ is a proper cofibration, if $\phi = (f, f): 1_P \to 1_T$, if q is a weak homotopy equivalence and if $\gamma: 1_P \to q$ is any map then r is a weak homotopy equivalence.*

Proof. ϕ is certainly a proper cofibration. As in the proof of 2.7, but this time using 2.9, we may show that $\bar{U}'$ is distinguished. Again as in the proof of 2.7, $R' - B'$ and $\bar{U}' \cap (R' - B')$ are distinguished. Thus 2.10 follows from [7; Th. 6].

Proof of 0.3. 2.10 implies that r_z is a weak homotopy equivalence. Thus 0.3 follows by two applications of 1.2.

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