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ON THE GROWTH OF ENTIRE FUNCTIONS OF BOUNDED INDEX

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A class E of entire functions of zero order and with widely spaced zeros has been defined and it is proved that if $f \in E$ then $f', f'', \cdots \in E$. Furthermore f is of index one. This class includes many functions which are both of bounded index and arbitrarily slow growth. If f is any transcendental entire function then there is an entire function g of unbounded index with the same asymptotic behavior. When f is of infinite order then it is of unbounded index and we simply take $g = f$. When f is of finite order we give the construction for g.

DEFINITION 1. An entire function $f(z)$ is said to be of bounded index if there exists an integer M, independent of z, such that

$$\left|\frac{f^{(n)}(z)}{n!}\right| \leqq \max_{0 \leqq s \leqq M} \left\{\left|\frac{f^{(s)}(z)}{s!}\right|\right\}$$

for all n and all z. The least such integer M is called the index of $f(z)$.

Although functions of bounded index have been the object of a number of recent investigations (cf: [3], [5], [6], [7]–[9]), little is known about their properties, and most of the following natural questions seem to require further study.

I. *What are the growth properties of functions of bounded index*:

(a) can they increase arbitrarily rapidly,

(b) can they increase arbitrarily slowly,

(c) is it possible to derive the boundedness (or the unboundedness) of the index from the asymptotic properties of the logarithm of the maximum modulus of $f(z)$, i.e., $\log M(r, f)$?

II. *Classes of functions of bounded index*:

(a) find classes of functions of bounded index,

(b) is the sum (or product) of two functions of bounded index also of bounded index?

Question I(a) was settled by Shah [8] who proved that the growth of functions of bounded index is at most of the exponential type of order one. (See also Lepson [6].) Shah [8] and Lepson [6] have constructed functions of arbitrarily slow growth and of unbounded index.

In the present note we derive a simple answer to Question I(b) from the consideration of

Functions with widely spaced zeros. Let $f(z)$ be an entire function of genus zero, and let $\{a_j\}_{j=1}^{\infty}$ be the sequence of its zeros. We say that $f(z)$ has widely spaced zeros if the zeros $\{a_j\}$ are all simple and

$$|a_1| \geqq a = 5, \ |a_{n+1}| \geqq a^n |a_n| \qquad (n = 1, 2, 3, \cdots) .$$

Using this definition we prove

THEOREM 1. *Let $f(z)$ have widely spaced zeros. Then, for all z,*

$$|f^{(n)}(z)| < \max\{|f(z)|, |f'(z)|\} \qquad (n = 2, 3, 4, \cdots) .$$

COROLLARY 1.1. *Functions with widely spaced zeros are of bounded index.*

COROLLARY 1.2. *There exist functions of bounded index and of arbitrarily slow growth.*

Corollary 1.1 may also be considered as a contribution to Question II(a). Corollary 1.2 answers Question I(b). Other contributions, due to separate efforts of the present authors, will be found elsewhere. In [9] Shah proves that all solutions of certain classes of linear differential equations are of bounded index. In his doctoral dissertation, Pugh shows that the functions

$$F_\sigma(z) = \prod_{j=1}^{\infty}\left(1 + \frac{z}{j^\sigma}\right) \qquad (\sigma > 8) ,$$

and

$$f_q(z) = \prod_{j=0}^{\infty}(1 - q^j z) \qquad \left(0 < q < \frac{1}{16}\right) ,$$

are of bounded index. As a contribution to II(b), Pugh [7] has shown that the sum of two functions of bounded index need not be of bounded index.

Our second result clarifies one aspect of Question I(c). We prove

THEOREM 2. *Let $f(z)$ be any transcendental entire function of finite order. It is always possible to find an entire function $g(z)$, of unbounded index such that*

$$\log M(r, f) \sim \log M(r, g) \qquad (r \rightarrow \infty) .$$

Choosing $f(z)$ to be of bounded index, we see that it is always possible to find *functions of unbounded index with the same asymptotic behavior as* $f(z)$.

The authors gratefully acknowledge the help of Professor Albert Edrei who suggested the class of functions with widely spaced zeros, and indicated the connection between Theorem 2 and the results of [2].

1. Successive derivatives of functions with widely spaced zeros.

LEMMA 1. *Let* $f(z)$ *be an entire function with widely spaced zeros* $\{a_j\}_{j=1}^{\infty}$. *Let* $\{b_j\}_{j=1}^{\infty}$ $(|b_j| \leqq |b_{j+1}|)$, *be the zeros of* $f'(z)$.
Then

$$\frac{|a_{n+1}|}{b} < |b_n| \leqq |a_{n+1}|, \qquad (n \geqq 2, b = 1.6), \tag{1.1}$$

and

$$\left(1 + \frac{2R + d}{a}\right)|a_1| < |b_1| \leqq |a_2|, \; (R = 2.4, d = 10^{-3}, |a_1| \geqq a = 5). \tag{1.2}$$

Proof. In §§1-3, we shall write $1.6 = b, 2.4 = R, 10^{-3} = d$, $1 + (2R + d)/a = 1.9602 = c$. Put

$$g_n(z) = \sum_{j=1}^{n} \frac{1}{z - a_j}, \qquad (n \geqq 1),$$

and

$$h_n(z) = \frac{f'(z)}{f(z)} - g_n(z) = \sum_{j=n+1}^{\infty} \frac{1}{z - a_j}. \tag{1.3}$$

Our proof of the lemma depends on obvious applications of Rouché's theorem [4, p. 254].

Let $z = re^{i\theta}$ and

$$|a_n| < r < |a_{n+1}|, \qquad (n \geqq 1). \tag{1.4}$$

Clearly

$$\operatorname{Re}(zg_n(z)) = \sum_{j=1}^{n} \frac{\operatorname{Re}(r^2 - z\bar{a}_j)}{|z - a_j|^2} \geqq \sum_{j=1}^{n} \frac{r}{r + |a_j|}$$

and hence

$$|g_n(z)| \geqq \sum_{j=1}^{n} \frac{1}{r + |a_j|} .$$

In particular by the definition of widely spaced zeros we have

$$(1.5) \qquad |g_n(z)| \geqq \frac{n}{|a_{n+1}| + |a_n|} \geqq \frac{n}{|a_{n+1}|}\frac{25}{26}, \qquad (n \geqq 2),$$

$$(1.6) \qquad \left|g_n\left(\frac{|a_{n+1}|}{b}e^{i\theta}\right)\right| \geqq 2\left(\frac{|a_{n+1}|}{b} + |a_2|\right)^{-1}$$

$$> \frac{3}{|a_{n+1}|}, \qquad (n \geqq 2).$$

For $h_n(z)$ we have

$$\left|h_n\left(\frac{|a_{n+1}|}{b}e^{i\theta}\right)\right| \leqq \left(|a_{n+1}| - \frac{|a_{n+1}|}{b}\right)^{-1} + \left(|a_{n+2}| - \frac{|a_{n+1}|}{b}\right)^{-1} + \cdots$$

$$(1.7) \qquad < \frac{b}{b-1}\frac{1}{|a_{n+1}|} + \frac{1.25}{|a_{n+2}| - (|a_{n+1}|/b)}$$

$$< \frac{2.8}{|a_{n+1}|} \qquad (n \geqq 2).$$

Now in the disc

$$(1.8) \qquad |z| \leqq \frac{|a_{n+1}|}{b},$$

$g_n(z)$ has n poles, and, by the theorem of Gauss-Lucas [10, p. 6], exactly $(n-1)$ zeros. The function $h_n(z)$ is regular in the disc (1.8), and by (1.6) and (1.7)

$$|g_n(z)| > |h_n(z)|, \qquad \left(n \geqq 2, |z| = \frac{|a_{n+1}|}{b}\right).$$

Hence, by Rouché's theorem

$$g_n(z) + h_n(z) = \frac{f'(z)}{f(z)}$$

has exactly $(n-1)$ zeros in the disc (1.8).

We have thus proved

$$(1.9) \qquad \frac{|a_{n+1}|}{b} < |b_n|, \qquad (n \geqq 2).$$

Similarly, for

$$(1.10) \qquad r = |z| = \gamma|a_n|, \qquad (1 < \gamma < 1.01, n \geqq 2)$$

we have

$$\begin{aligned} |h_n(z)| &< (|a_{n+1}| - \gamma|a_n|)^{-1} + (|a_{n+2}| - \gamma|a_n|)^{-1} + \cdots \\ &\leqq (|a_{n+1}| - \gamma|a_n|)^{-1} + (1.1)(|a_{n+2}| - \gamma|a_n|)^{-1} \\ &\leqq (\gamma|a_n| + |a_1|)^{-1} < |g_n(z)| . \end{aligned}$$

Again by Rouché's theorem $f'(z)/f(z)$ has exactly $(n-1)$ zeros in any disc with center at the origin and a radius r satisfying (1.10). Hence

$$|b_{n-1}| < \gamma|a_n| \qquad (n \geqq 2) ,$$

and letting $\gamma \to 1+$, we obtain

$$|b_{n-1}| \leqq |a_n| \qquad (n \geqq 2) . \tag{1.11}$$

The second of the inequalities (1.2) also follows from (1.11).

We complete the proof of the lemma by showing that

$$|z| \leqq c|a_1| \tag{1.12}$$

implies

$$\left|\frac{f'(z)}{f(z)}\right| > 0 . \tag{1.13}$$

Thus $f'(z)$ will have no zeros in the disc (1.12) and, therefore

$$c|a_1| < |b_1| ,$$

which is the first of the inequalities (1.2).

In order to verify (1.13) notice that (1.12) and the definition of widely spaced zeros imply

$$\begin{aligned} \left|\frac{f'(z)}{f(z)}\right| &\geqq \frac{1}{|a_1|}\left\{\frac{1}{1+c} - \sum_2^\infty \frac{1}{a^{j(j-1)/2} - c}\right\} \\ &> 0 . \end{aligned}$$

This completes the proof of Lemma 1.

LEMMA 2. *If $f(z)$ has widely spaced zeros all the derivatives*

$$f'(z), f''(z), \cdots$$

have the same property.

Proof. It is sufficient to prove that if $f(z)$ has widely spaced zeros, the zeros of $f'(z)$ are also widely spaced. By (1.2)

$$9.801 \leqq c|a_1| < |b_1| . \tag{1.14}$$

By (1.1) and (1.2)

$$| b_n | \leqq | a_{n+1} | , \qquad (n \geqq 1)$$
$$\frac{1}{b} | a_{n+2} | < | b_{n+1} | , \qquad (n \geqq 1) .$$

Hence

$$\left| \frac{b_{n+1}}{b_n} \right| > \frac{| a_{n+2} |}{b \, | a_{n+1} |} \geqq \frac{a^{n+1}}{b} > a^n \qquad (n \geqq 1) . \tag{1.15}$$

The relations (1.14) and (1.15) show that the b's are widely spaced.

2. Minimum distance between a zero of $f(z)$ and a zero of $f'(z)$. The inequalities (1.1) do not preclude the possibility that $| a_{n+1} - b_n |$ be very small. In this section we show that

$$\inf_{\substack{1 \leqq j < \infty \\ 1 \leqq k < \infty}} | a_j - b_k | > 2R + d . \tag{2.1}$$

I. From now on, we denote the zeros of $f^{(k)}(z)$, in order of ascending moduli by $\{a_j^{(k)}\}_{j=1}^{\infty}$. By definition $a_n^{(0)} = a_n$ and $f^{(0)} \equiv f$.

II. We consider systematically the sets

$$D_k(\rho) = \bigcup_{j=1}^{\infty} \{z : | z - a_j^{(k)} | \leqq \rho\} \qquad (\rho > 0, k = 0, 1, \cdots) .$$

LEMMA 3. *If $f(z)$ has widely spaced zeros, and if $z \notin D_0(R)$, then*

$$\left| \frac{f'(z)}{f(z)} \right| < 1, \left| \frac{f''(z)}{f(z)} \right| < 1 . \tag{2.2}$$

Proof. The identities

$$\frac{d}{dz}\left(\frac{f'(z)}{f(z)}\right) = - \sum_{j=1}^{\infty} \frac{1}{(z - a_j)^2} = \frac{f''(z)}{f(z)} - \left(\frac{f'(z)}{f(z)}\right)^2$$

imply

$$\left| \frac{f''(z)}{f(z)} \right| \leqq \sum_{j=1}^{\infty} \frac{1}{| z - a_j |^2} + \left(\sum_{j=1}^{\infty} \frac{1}{| z - a_j |} \right)^2 \leqq 2 \left(\sum_{j=1}^{\infty} \frac{1}{| z - a_j |} \right)^2 .$$

Hence, the inequalities (2.2) follow from the single inequality

$$\sum_{j=1}^{\infty} \frac{1}{| z - a_j |} < \frac{\sqrt{2}}{2} . \tag{2.3}$$

If $z \notin D_0(R)$, and $| z | < | a_1 |$, then

$$|z - a_1| > R \tag{2.4}$$

and

$$|z-a_j| \geqq |a_j| - |z| > |a_j| - |a_1| > |a_1|(a^{j-1}-1) > \frac{a^j}{2}, \ (j \geqq 2) . \tag{2.5}$$

Hence

$$\sum_{j=1}^{\infty} \frac{1}{|z-a_j|} < \frac{1}{R} + 2\sum_{j=2}^{\infty}\frac{1}{a^j} < \frac{\sqrt{2}}{2} ,$$

so that (2.3) holds if $|z| < |a_1|$.

In general, the relations

$$|a_n| \leqq |z| < |a_{n+1}| \qquad (n \geqq 1), \ z \notin D_0(R)$$

imply

$$|z - a_j| \geqq |z| - |a_j| \geqq |a_n| - |a_{n-1}| > \frac{a^n}{2} \tag{2.6}$$

provided

$$n \geqq 2 , \qquad j < n . \tag{2.7}$$

Similarly, for $j > n + 1$

$$\begin{aligned} |z - a_j| \geqq |a_j| - |a_{n+1}| &> (a^{j-1} - 1)\,|a_{n+1}| \\ &> \frac{a^{j-1}}{2}\,|a_{n+1}| . \end{aligned} \tag{2.8}$$

Finally,

$$\frac{1}{|z-a_n|} + \frac{1}{|z-a_{n+1}|} \leqq \frac{1}{R} + (\max\{|z-a_n|, |z-a_{n+1}|\})^{-1} \tag{2.9}$$

with

$$\max\{|z-a_n|, |z-a_{n+1}|\} \geqq \frac{|a_{n+1}| - |a_n|}{2} > \frac{(a^n - 1)|a_n|}{2} . \tag{2.10}$$

Combining (2.6), (2.8), (2.9) and (2.10), we find, for $n \geqq 2$,

$$\begin{aligned} \sum_{j=1}^{\infty}\frac{1}{|z-a_j|} &< \frac{2(n-1)}{a^n} + \frac{1}{R} + \frac{2}{(a^n-1)|a_n|} + \\ \frac{2a}{|a_{n+1}|}\sum_{j=n+2}^{\infty}\frac{1}{a^j} &< \frac{2(n-1)}{a^n} + \frac{1}{R} + \frac{2}{(a^n-1)a} + \frac{2}{(a-1)a^{n+2}} . \end{aligned} \tag{2.11}$$

It is easily seen that (2.11) holds for $n = 1$ also and that (2.11) implies (2.3). Hence the lemma is proved.

LEMMA 4. *If* $z \in D_0(2R + d)$, *then* $f'(z) \neq 0$.

Proof. If $z \in D_0(2R + d)$, then for some n,

$$|z - a_n| \leqq 2R + d = 4.801 . \tag{2.12}$$

Hence, if $j < n$ and $n \geqq 2$,

$$\begin{aligned} |z - a_j| &\geqq |z| - |a_{n-1}| \geqq |a_n| - |a_{n-1}| - (2R + d) \\ &\geqq |a_n|\left(1 - \frac{1}{a} - \frac{2R + d}{a^2}\right) > \frac{6}{10}|a_n| . \end{aligned} \tag{2.13}$$

If $j > n$, then

$$\begin{aligned} |z - a_j| &\geqq |a_j| - |a_n| - (2R + d) \\ &> |a_j|\left(1 - \frac{1}{a} - \frac{2R + d}{a^2}\right) > \frac{6}{10}|a_j| . \end{aligned} \tag{2.14}$$

By (2.12), (2.13), and (2.14) we have, for $n \geqq 2$,

$$\begin{aligned} \left|\frac{f'(z)}{f(z)}\right| &\geqq \frac{1}{4.801} - \frac{10(n-1)}{6\,|a_n|} - \frac{10}{6}\sum_{j=n+1}^{\infty}\frac{1}{|a_j|} , \\ &\geqq \frac{1}{4.801} - \frac{1}{3}\frac{(n-1)}{a^{n(n-1)/2}} - \frac{5}{12}\frac{1}{a^{(n+1)n/2}} . \end{aligned} \tag{2.15}$$

Again, it is easily seen that (2.15) holds for $n = 1$ also. The expression on the right of (2.15) is positive and consequently in $D_0(2R + d)$, $f'(z) \neq 0$ unless $f(z) = 0$. On the other hand $f'(z) \neq 0$ if $f(z) = 0$ because all the zeros of $f(z)$ are simple. This completes the proof of Lemma 4.

3. Proof of Theorem 1. Because all the derivatives of $f(z)$ have widely spaced zeros, Lemmas 1 to 4 apply to all of the functions $f^{(k)}(z)$, $(k = 0, 1, 2, 3, \cdots)$. In particular Lemma 4 shows that the sets $D_{n-2}(R)$ and $D_{n-1}(R)$ are disjoint for $n \geqq 2$.

Hence, by Lemma 3, at least one of the two inequalities

$$\left|\frac{f^{(n)}(z)}{f^{(n-2)}(z)}\right| < 1, \ \left|\frac{f^{(n)}(z)}{f^{(n-1)}(z)}\right| < 1 \qquad (n \geqq 2) \tag{3.1}$$

must hold.

Thus, for all z

$$|f^{(n)}(z)| < \max\{|f^{(n-1)}(z)|, |f^{(n-2)}(z)|\} \qquad (n = 2, 3, 4, \cdots) . \tag{3.2}$$

Theorem 1 follows from (3.2) by an obvious induction over n.

4. Proof of Theorem 2. In this section we assume familiarity with the most elementary results and notations of Nevanlinna's theory of meromorphic functions.

Let $f(z)$ be a given entire, nonrational function of finite order. A theorem of Edrei and Fuchs [2; p. 384 and p. 390, formula (3.5)] asserts the existence of an entire function $h(z)$ such that $h(0) = 1$ and

$$N\left(r, \frac{1}{h}\right) \sim \log M(r, h) \sim \log M(r, f) \qquad (r \to +\infty) . \tag{4.1}$$

We take $g(z)$ to be of the form

$$g(z) = h(z)P(z) , \tag{4.2}$$

where

$$P(z) = \prod_{j=1}^{\infty} \left(1 + \frac{z}{d_j}\right)^j . \tag{4.3}$$

The quantities d_j are positive and satisfy the following conditions:

(i) $d_1 > e^2$, $d_{j+1} > d_j^2$ $(j = 1, 2, 3, \cdots)$;

(ii) for $t \geqq d_j$,

$$\frac{j(j+1)}{2} < \left\{\frac{\log M(t, f)}{\log t}\right\}^{1/2} .$$

Since $f(z)$ is not rational

$$\frac{\log M(t, f)}{\log t} \longrightarrow +\infty \qquad (t \to +\infty) \tag{4.4}$$

and hence it is possible to satisfy condition (ii).

Putting

$$n(t) = n\left(t, \frac{1}{P}\right) ,$$

we see that

$$n(t) = 0 \ \ (0 \leqq t < d_1),\ n(t) = \frac{k(k+1)}{2} \quad (d_k \leqq t < d_{k+1}) . \tag{4.5}$$

Hence, if

$$d_k \leqq t < d_{k+1} \qquad (k \geqq 1) \tag{4.6}$$

(4.5) and condition (i) imply

$$n(t) < 2^k < \log d_k \leqq \log t < t^{1/2} \qquad (k \geqq 1) . \tag{4.7}$$

By (4.6), (i) and (4.5)

$$t^2 < d_{k+1}^2 < d_{k+2} ,$$

$$\frac{n(t^2)}{n(t)} \leqq 1 + \frac{2}{k} . \tag{4.8}$$

By (4.6), (ii), (4.5) and (4.4)

$$n(t)\log t < \log M(t, f)\left\{\frac{\log t}{\log M(t, f)}\right\}^{1/2} = o(\log M(t, f)) \qquad (t \to \infty) . \tag{4.9}$$

By (4.1), (4.2) and the elements of Nevanlinna's theory

$$\begin{aligned}(1 + o(1))\log M(r, f) &= N\left(r, \frac{1}{h}\right) \leqq N\left(r, \frac{1}{g}\right)\\ &\leqq \log M(r, g) \leq \log M(r, h) + \log M(r, P)\\ &= \log M(r, f)\left\{1 + o(1) + \frac{\log M(r, P)}{\log M(r, f)}\right\} \qquad (r \to +\infty) .\end{aligned}$$

Hence, in order to obtain Theorem 2 it is sufficient to show that

$$\frac{\log M(r, P)}{\log M(r, f)} \longrightarrow 0 \qquad (r \to +\infty) \tag{4.10}$$

and to remark that $g(z)$ cannot be of bounded index because it has zeros of arbitrarily high multiplicity.

The relation (4.10) follows readily from the identity [1, p. 48]

$$\log M(r, P) = r\int_0^\infty \frac{n(t)}{t(t + r)}dt ,$$

which, in view of (4.7), (4.8) and (4.9), leads to

$$\begin{aligned}\log M(r, P) &< n(r)\log r + r\int_r^{r^2} \frac{n(r^2)}{t^2}dt + r\int_{r^2}^\infty t^{-3/2}dt\\ &= o(\log M(r, f)) \qquad (r \to +\infty) .\end{aligned}$$

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