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EXISTENCE OF TRICONNECTED GRAPHS WITH PRESCRIBED DEGREES

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Necessary and sufficient conditions for the existence of a p-connected (linear undirected) graph with prescribed degrees $d_1, d_2, \cdots, d_n$ are known for $p = 1, 2$. In this paper we solve this problem for $p = 3$.

Let $d_1, d_2, \cdots, d_n$ be positive integers and let $d_1 \leqq d_2 \leqq \cdots \leqq d_n$.

LEMMA. *If a triconnected graph G exists with degrees $d_1, d_2, \cdots, d_n$, then*

(1) $d_i \geqq 3$.

(2) $d_1, d_2, \cdots, d_n$ *is graphical*, i.e., *there exists a graph with these degrees.*

(3) $d_n + d_{n-1} \leqq m - n + 4$ *where* $2m = \sum_{i=1}^n d_i$.

(4) *If* $d_n + d_{n-1} = m - n + 4$, *then* $m \geqq 2n - 2$.

Proof. (1) and (2) are evident. To prove (3), let x_n, x_{n-1} be the vertices of G with degrees d_n and d_{n-1} respectively. Then the number of edges in $G - \{x_n, x_{n-1}\}$ is $m - (d_n + d_{n-1} - 1)$ or $m - (d_n + d_{n-1})$ according as x_n, x_{n-1} are adjacent or not adjacent in G. Also $G - \{x_n, x_{n-1}\}$ is connected, so (3) follows. If now $d_n + d_{n-1} = m - n + 4$, then

$$2m \geqq d_n + d_{n-1} + 3(n-2) = m + 2n - 2 .$$

This completes the proof of the lemma.

THEOREM. *Conditions* (1) *to* (4) *of the lemma are necessary and sufficient for the existence of a triconnected graph with degrees* $d_1, d_2, \cdots, d_n$.

Proof. Necessity was proved in the lemma.

To prove sufficiency, first let conditions (1), (3) be satisfied and let $d_n + d_{n-1} = m - n + 4 = n + \lambda$ where $2 \leqq \lambda \leqq n - 2$. Let k be the number of d_i such that $1 \leqq i \leqq n - 2$ and $d_i = 3$. Then define

$$e_i = d_i - 2 \text{ for } i = k + 1, \cdots, n - 2 .$$

Then we have

$$\sum_{i=1}^{n-2} d_i = 2m - d_n - d_{n-1} = 3n + \lambda - 8 ,$$

$$\sum_{i=k+1}^{n-2} e_i = 3n + \lambda - 8 - 3k - 2(n - 2 - k) = n + \lambda - k - 4 .$$

Define now $\eta = n - 2 - \lambda$ and $\varepsilon = k - \eta$. Then $\eta \geqq 0$, and $\varepsilon \geqq 2$ since

$$\begin{aligned} 2m &\geqq m - n + 4 + 3k + 4(n - 2 - k) \\ &= m + 3n - k - 4 \end{aligned}$$

and so

$$\lambda = m - 2n + 4 \geqq n - k .$$

Write now

$$e_i = \begin{cases} 1 & \text{for } i = 1, 2, \cdots, \varepsilon , \\ 2 & \text{for } i = \varepsilon + 1, \cdots, k , \\ d_i - 2 & \text{for } i = k + 1, \cdots, n - 2 . \end{cases}$$

Then $\sum_{i=1}^{n-2} e_i = 2(n - 3)$ and so there exists a tree T with degrees $e_1, \cdots e_{n-2}$, attained by the vertices $x_1, \cdots, x_{n-2}$, say, in that order [2]. Take two more vertices x_{n-1} and x_n and join them. Also join each of x_{n-1}, x_n to x_i for $i = 1, \cdots, \varepsilon, k + 1, \cdots, n - 2$. Of the η vertices $x_{\varepsilon+1}, \cdots, x_k$, join $d_{n-1} - 1 - \varepsilon - n + 2 + k$ to x_{n-1} and the rest ($d_n - 1 - \varepsilon - n + 2 + k$ in number) to x_n. Note that

$$d_{n-1} - 1 - \varepsilon - n + 2 + k = d_{n-1} - \lambda - 1 \geqq 0 .$$

The graph we thus obtain has degrees $d_1, \cdots, d_n$ and is triconnected since any vertex of T with degree in T less than 3 is joined to either x_{n-1} or x_n.

Next let conditions (1), (2) be satisfied and let

$$d_n + d_{n-1} \leqq m - n + 3 .$$

Then $d_n < m - n + 2$, so there exists a biconnected graph G with degrees $d_1, d_2, \cdots, d_n$ [2]. If G is not triconnected, let x_i, x_j be two vertices such that $G - \{x_i, x_j\}$ is disconnected. Let $C_1, C_2, \cdots$ be the components of $G - \{x_i, x_j\}$. By (1), $|C_g| \geqq 2$ for $g = 1, 2, \cdots$. Also by hypothesis,

$$m - d_i - d_j \geqq n - 3 ,$$

so it follows that one of the components, say C_1, contains a cycle.

We first prove that there exists an edge (x, y) in C_1 and two chains μ_1, μ_1' of G connecting x and y such that $(x, y), \mu_1, \mu_1'$ are disjoint except for x and y, and μ_1 is contained in C_1. Since G is biconnected, there exists a chain connecting x_i and x_j with all intermediate vertices in C_2.

If now two vertices x, y with degree two in C_1 are adjacent and belong to a cycle of C_1, the required edge is (x, y). So we may take that no two vertices of degree two in C_1 can belong to a block (on more than two vertices) and be adjacent. Let B be any block of C_1 which is not an edge. If some cycle of B has a chord (x, y), then (x, y) is the required edge. Otherwise, by the results of [1], two vertices y, z of degree two in B will be adjacent to a vertex x of degree three in B. If w is another vertex of B adjacent to x, then there is a chain connecting w to y in $B - \{x\}$. This chain together with (x, w) may be taken as μ_1. To get μ_1', go from x to z along (x, z), from z to x_i or x_j (through another block of C_1 at z if necessary), then to y. Thus (x, y) is the required edge.

Let now (x, y) be an edge of C_1 chosen as explained above. If C_2 is a tree, take any edge (u, v) of C_2. Then (u, v) is a chord of a cycle of G. If C_2 is not a tree, choose an edge (u, v) of C_2 such that there are chains μ_2, μ_2' of G connecting u and v, (u, v), μ_2, μ_2' are disjoint except for u, v, and μ_2 is contained in C_2.

We define $f_G(s, t)$ to be the number of components of $G - \{s, t\}$. Now we will make a modification on G so that the degrees of the vertices are unaltered, $f(x_i, x_j)$ decreases and $f(s, t)$ does not increase for any two vertices s and t.

First we associate with x, a subset $A(x)$ of $\{x_i, x_j\}$ by the following rule. $x_i \in A(x)$ if and only if there is a chain ν connecting x to x_i with all intermediate vertices in C_1 such that ν is disjoint with (x, y) and μ_1 except for x. Similarly $A(y)$ is defined. If C_2 is a tree, put $A(u) = A(v) = \{x_i, x_j\}$. Otherwise $A(u)$, $A(v)$ are defined in a manner similar to that of $A(x)$ and $A(y)$. Now $A(x)$, $A(y)$ are made nonempty by a proper choice of μ_1, and $A(u)$, $A(v)$ are made nonempty by a proper choice of μ_2 (in case C_2 is not a tree).

Now suppress the edges (x, y), (u, v) and join x to one of u, v and y to the other as follows. Join x to u if $A(x) \neq A(u)$ and $A(y) \neq A(v)$ whenever such a choice is possible. Let the new graph thus obtained be H. To be specific we take that x is joined to u in H.

First we show that H is biconnected. Obviously $G_1 = G - (x, y)$ is biconnected. Now we show that (u, v) is a chord of a cycle of G_1. If C_2 is a tree, then the cycle is

$$(u, x) + \mu_1[x, y] + (y, v) + [v, \cdots, p_1] + (p_1, x_i) + (x_i, p_2) + [p_2, \cdots, u]$$

where p_1, p_2 are suitable pendant vertices of C_2. Otherwise the cycle is

$$\mu_2[u, v] + \mu_2'[v, u]$$

where if μ_2' contains the edge (x, y), then (x, y) is replaced by $\mu_1[x, y]$ and the resulting cycle is made elementary.

Trivially now $f_G(x_i, x_j) = f_H(x_i, x_j) + 1$. Next we will show that

$$f_G(s, t) \geqq f_H(s, t) \tag{5}$$

for any two vertices s and t. For this it is enough to show that x, y are connected and u, v are connected in $H - \{s, t\}$.

First let $s = x_i$. Now x, y, u, v belong to a cycle in $H - \{x_i\}$, so (5) follows. So we may take $\{s, t\} \cap \{x_i, x_j\} = \emptyset$.

Now let $s = x$. Then to prove (5) it is enough to show that u, v are connected in $H - \{x, t\}$ when $t \neq u$ and $t \neq v$. This is evident if C_2 is a tree or $t \notin \mu_2$. So let $t \in \mu_2$ and C_2 be not a tree. If $A(u) \cap A(v) \neq \emptyset$, there is a chain connecting u, v in $H - \{x, t\}$. So we take without loss of generality $A(u) = x_j$ and $A(v) = x_i$. If now $x_j \in A(y)$, then u, v are connected through x_j and y in $H - \{x, t\}$. So we take $A(y) = x_i$. If $x_j \in A(x)$, then y would not have been joined to v, so $A(x) = x_i$. Now in G, x_j is connected to some vertex z of μ_1 by a chain with all intermediate vertices belonging to C_1 but not to μ_1. Now we obtain a chain connecting u, v in $H - \{x, t\}$ by going from u to x_j, x_j to z, z to y along μ_1, y to x_i, and x_i to v. Thus we may take $\{s, t\} \cap \{x_i, x_j, x, y\} = \emptyset$.

Next let $s = u$. If $t \notin \mu_1$, then (5) is trivial, so let $t \in \mu_1$. Suppose first that C_2 is a tree. Then we obtain a chain connecting x, y in $H - \{u, t\}$ by going from x to x_i or x_j, then to v through a suitable pendant vertex of C_2 and then to y. If C_2 is not a tree, the situation is similar to that of the preceding paragraph. Thus we take $\{s, t\} \cap \{x_i, x_j, x, y, u, v\} = \emptyset$.

If none of s, t belongs to μ_1, then (5) is trivial. So let $s \in \mu_1$.

Suppose now that C_2 is a tree. Then for any fixed vertex t, there are chains in $H - \{s, t\}$ from one of u, v to both x_i and x_j, and a chain from the other (of the vertices u, v) to x_i or x_j. Hence u, v are connected and (5) follows.

Suppose next that C_2 is not a tree. Obviously we may take $s \in \mu_1$ and $t \in \mu_2$. If now $A(x) \cap A(y) \neq \emptyset$ or $A(u) \cap A(v) \neq \emptyset$, then again (5) follows. So we may take $A(x) = x_i$, $A(y) = x_j$, $A(u) = x_j$ $A(v) = x_i$. Now we obtain a chain connecting x, y in $H - \{s, t\}$ by going from x to u, u to x_j, x_j to y. This proves (5) completely.

Now by a repeated application of the above procedure we reduce the graph until finally $f(s, t) = 1$ for any two vertices. The final graph has degrees $d_1, d_2, \cdots, d_n$ and is triconnected and this completes the proof of the theorem.

Perhaps necessary and sufficient conditions, similar to the conditions (1) to (4) above, for the existence of a p-connected graph with prescribed degrees $d_1, d_2, \cdots, d_n$ can be obtained for all $p \geqq 3$, but the authors have not yet succeeded in this.

References

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