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ON LEFT QF-3 RINGS

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In this paper the following results are proved:

(i) Three classes of left QF-3 rings are closed under taking left quotient rings respectively.

(ii) A subcategory of left modules having dominant dimensions $\geqq 2$ over a right perfect left QF-3 ring R is equivalent to a category of all left fRf-modules, where f is a suitable idempotent of R.

(iii) In case a left QF-3 ring is obtained as the endomorphism ring of a generator, dominant dimensions ($\geqq 2$) of modules are closely connected with the vanishing of Ext-functors.

(iv) Three classes of left and right QF-3 rings are identical in case of perfect rings.

Let R be an associative ring having an identity element 1 and denote by ${}_RR$ (resp. R_R) a left (resp. right) R-module R. To generalize the notion of QF-3 algebras [18] we shall make the following definitions:

(1) R is said to be left QF-3, if ${}_RR$ has a direct summand Re (e is an idempotent of R) which is a faithful, injective left ideal.

(2) R is said to be left QF-3$^+$, if the injective hull $E({}_RR)$ of ${}_RR$ is projective.

(3) R is said to be left QF-3$'$, if the injective hull $E({}_RR)$ of ${}_RR$ is torsionless in the sense of Bass [1].

Right QF-3, QF-3$^+$ and QF-3$'$ rings are defined in a similar fashion. It is obvious that the class of left QF-3$'$ rings is the most general class of the above three classes.

Our main purpose in this note is to introduce some generalizations of results for QF-3 algebras [11], [12], [15], [16], [17] and semi-primary QF-3 rings [4], [6], [13], [14] to the above generalized classes of rings.

We shall say that the dominant dimension of left (resp. right) R-module X, denoted by dom. dim ${}_RX$(resp. dom. dim X_R), is at least n, if there exists an injective resolution of X:

$$0 \longrightarrow X \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_n$$

such that all W_i, $1 \leqq i \leqq n$, are torsionless. Then it is clear that R is left (resp. right) QF-3$'$ if and only if dom. dim ${}_RR$ (resp. dom. dim R_R) $\geqq$ 1.

In §1 we shall show that each class defined as above is closed under taking quotient rings (not necessarily classical), that is, a left quotient ring S of R is left QF-3′, QF-3$^+$ or QF-3 if R is left QF-3′, QF-3$^+$ or QF-3 respectively. Further, S is the maximal left quotient ring of R if and only if $\mathrm{dom.\,dim}\,_S S \geqq 2$. This is a generalization of the results for QF-3 algebras by Morita [12], Tachikawa [17] and Mochizuki [11]. Then, as an immediate consequence we have that R and the double centralizer R' of any faithful right ideal of R are contained in the same class, and $\mathrm{dom.\,dim}\,_R R = 1$, if $R' \neq R$. Here it is to be noted that the double centralizer of a faithful left ideal of R is not necessarily left QF-3′, even if R is left QF-3′ (cf. §4, example 1).

In §2 we shall consider a left QF-3 ring R which has a faithful projective right ideal K and shall develop a proof in order to notice that the injectivity of K_R is not necessary to obtain some results in [14, Propositions 1.1 and 1.2]. Further, defining a special injective dimension closely connected to a fixed injective module, we shall prove that in the case R is a right perfect left QF-3 ring, there exists a suitable idempotent f of R such that the subcategory of left R-modules of dominant dimension at least two is equivalent to the category of all left fRf-modules. We shall remark also that the two characterizations for dominant dimension by Mueller [15] can be applied to an estimation of dominant dimensions of endomorphism rings of modules which are generators.

Recently, the characterization of Artinian QF-3 rings due to Wu, Mochizuki and Jans [19] suggested the notion of QF-3′ rings to Colby and Rutter [4] and Kato [7]. In [4] it was proved that semi-primary[1] left QF-3′ rings are not necessarily left QF-3$^+$, however "left QF-3′" implies "left QF-3" for semi-primary rings. Without the proof we shall state in §3 that the same result holds for perfect rings, since the proof in [4] is available for this case. Moreover, we shall prove by duality of modules and the result proved in the first part of §2 the notions of two sided QF-3′, QF-3$^+$ and QF-3 are identical for perfect rings.

1. Quotient rings of QF-3 rings. Let R be a ring with an identity element 1 and N a submodule of a left R-module M. M is said to be a rational extension of N in case $f(N) = 0$ implies $f = 0$ for $f \in \mathrm{Hom}_R(L, M)$, where L is any submodule of M containing N. Then, following Lambek [10], a ring S is said to be a left quotient ring of R if S contains R as a subring and if S is a rational extension of R as a left R-module. To begin with we shall prove

PROPOSITION 1.1 *Let S be a left quotient ring of R. If R is*

[1] In this case a ring R is said to be semi-primary if it contains a nilpotent ideal N with R/N semi-simple with minimum condition.

left QF-3′ (resp. QF-3⁺), *then* S *is left* QF-3′ (resp. QF-3⁺).

Proof[2]. Denote by I the injective hull $E({}_RR)$. Then, by the assumption $I \cong {}_RE({}_SS)$ and we have

$$I \xrightarrow{j} \textstyle\prod_\lambda R \xrightarrow{\varphi} \prod_\lambda S\ ,$$

$$\Big(\text{resp. } I \xrightarrow{j} \textstyle\sum_\lambda \oplus R \xrightarrow{\varphi} \sum_\lambda \oplus S\Big)$$

where φ is an R-monomorphism and j is the inclusion mapping.

Suppose s be an element of S such that $\varphi j(sx) \neq s(\varphi j(x))$ for some $x \in I$. Then we have a projection p_λ of $\prod_\lambda S$(resp. $\sum \oplus S$) onto S such that the R-homomorphism

$$s \longrightarrow p_\lambda(\varphi j(sx) - s(\varphi j(x))$$

of S into S is nonzero but has kernel containing R. However, this contradicts that ${}_RS$ is a rational extention of R. Hence φj is an S-monomorphism and consequently S is left QF-3′ (resp. left QF-3⁺).

PROPOSITION 1.2. *Let S be a left quotient ring of R. If R is left* QF-3, *then S is left* QF-3.

Proof. Let Re, $e^2 = e$ be a faithful projective, injective left ideal of R. Since ${}_RS$ is an essential extension of ${}_RR$ and $e^2 = e$, ${}_RSe$ is an essential extension ${}_RRe$ so $Se = Re$.

Next, we shall prove that ${}_SSe$ is injective. Let L be a left ideal of S and φ a left S-homomorphism of L into Se. Denote by $\hat{\varphi}$ the map of $L \cap R$ into $Se(= Re)$ which is the restriction of φ. Since ${}_RRe$ is injective, it follows by Baer's Criterion that there exists an element q of Re such that $\hat{\varphi}(l) = lq$ for all $l \in L \cap R$. Then we shall define a map $\bar{\Psi}$ of L into Se by putting $\bar{\Psi}(r) = \varphi(r) - rq$, for all $r \in S$. Now we shall suppose that $\bar{\Psi}(r_1) \neq 0$ for some nonzero element r_1 of L. $\bar{\Psi}(r_1)$ and r_1 are both elements of S. Since ${}_RS$ is a rational extension, there exists an element r_0 of R such that $r_0\bar{\Psi}(r_1) \neq 0$ and $r_0r_1 \in R$. Then $r_0r_1 \in R \cap L$ and hence $r_0\bar{\Psi}(r_1) = r_0\varphi(r_1) - r_0(r_1q) = \varphi(r_0r_1) - r_0r_1q = \hat{\varphi}(r_0r_1) - (r_0r_1)q = 0$. This is a contradiction. Thus $\bar{\Psi}(r) = 0$ for all $r \in L$. Hence a left S-homomorphism Φ of S into Se defined by $\Phi(r) = rq$ for all $r \in S$ is an extension of φ and by Baer's Criterion we obtain that ${}_SSe$ is injective.

It remains to prove the faithfulness of ${}_SSe$. Let q be a nonzero element of S. Then there exists an element d of R such that $dq \in R$ and $dq \neq 0$. Then there exists an element x of Re such that $dqx \neq 0$,

[2] The author is grateful to the referee who clarified the proofs in this section.

because ${}_R Re$ is faithful. Hence $qx \neq 0$, $x \in Se$, this implies that ${}_S Se$ is faithful.

Let us denote by Q the maximal left quotient ring of R. Then we have

PROPOSITION 1.3. *If R is left* QF-3′ *and S is a left quotient ring of R, then* dom. dim. ${}_S S \geqq 2$ *if and only if* $S = Q$.

Proof. It is known [9] that S is a subring of Q and Q can be imbedded into I by a Q-monomorphism j. Lambek proved in [9] that $j(Q) = \{x \in I \mid h(j(S)) = 0$ implies $h(x) = 0$ for $h \in \mathrm{Hom}_S(I, I)\}$. If dom. dim ${}_S S \geqq 2$, $I/j(S)$ is isomorphic to a submodule of a direct product of copies of S. Hence for $x \in I$ such that $x \notin j(S)$, there exists a S-homomorphism f of I into I with $f(x) \neq 0$ and $f(j(S)) = 0$. Thus by the remark above $x \notin j(Q)$ and consequently $S = Q$.

Conversely assume $S = Q$. Since the maximal left quotient ring of Q is itself, by the same reason we have $\bigcap_{h \in H_0} \mathrm{Ker}\, h = j(Q)$, where $H_0 = \{h \in \mathrm{Hom}_Q(I, I) \mid h(j(Q)) = 0\}$.

Thus,

$$0 \longrightarrow Q \xrightarrow{j} I \xrightarrow{\varphi} \prod_{h \in H_0} I$$

is exact, where $\varphi(x) = (\cdots, h(x), \cdots)$, $x \in I$. It follows that

$$\text{dom. dim}\, {}_S S = \text{dom. dim}\, {}_Q Q \geqq 2.$$

In case R is a finite dimensional QF-3 algebra, $E({}_R R)$ is similar to the unique minimal faithful left R-module and the double centralizers of these modules are isomorphic to each other. Thus Proposition 1.3 is a generalization of the result for QF-3 algebras by Tachikawa [17] and Mochizuki [11].

COROLLARY 1.4. *Let K be a faithful right ideal of R. If R is left* QF-3 (QF-3′ *or* QF-3$^+$), *then the double centralizer R' of K_R is also left* QF-3 (QF-3′ *or* QF-3$^+$ *respectively*).

Proof. Since every left quotient ring of R can be imbedded naturally into Q, we shall prove that R' is a left quotient ring. For this purpose it is sufficient to show that for any two elements $r'_1 \neq 0$ and r'_2 of R', there exists an element r of R such that $rr'_1 \neq 0$ and $rr'_2 \in R$. However, K is faithful, hence there exists an element $k \in K \subseteqq R$ such that $kr'_1 \neq 0$ and it is obvious $kr'_2 \in K \subseteqq R$.

COROLLARY 1.5. *Let R' be a double centralizer of a faithful right ideal of R. If* dom. dim ${}_R R \geqq 2$, *then* $R' = R$.

2. On left QF-3 rings. Throughout this section we shall assume that Re means always a faithful, projective, injective left ideal of R and Q is the maximal left quotient ring of R.

We shall denote by K a finitely generated, projective, faithful right ideal of R, by C the ring eRe, by D $\mathrm{End}_R(K_R)$ and by U the D-C-bimodule Ke respectively. Then we have

PROPOSITION 2.1. *Let R' be the double centralizer of K_R. If R is a left* QF-3 *ring having Re as a faithful, projective, injective left ideal, then it holds*

(1) ${}_{R'}[\mathrm{Hom}_D({}_DK_{R'}, {}_DU_C)]_C \cong {}_{R'}R'e_C$,

(2) ${}_DU$ is injective.

Proof. Since K_R is faithful, we shall identify a D-endomorphism of ${}_DK$ obtained by the right multiplication of an element r of R with r itself. Then it follows that $\mathrm{Hom}_D(K, U) = \mathrm{Hom}_D(K, Ke) \supseteqq Re$. On the other hand, $\mathrm{Hom}_D(K, U)$ $(= \mathrm{Hom}_D(K, Ke))$ is a subset of $R' = \mathrm{Hom}_D(K, K)$. Hence we have that $\mathrm{Hom}_D(K, U) \subseteqq R'e$. However, it is known by Proposition 1.2 that $Re = R'e$. Thus we have (1).

Next, assume that the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "\varphi"] & Y' \\ & U & \end{tikzcd}$$

is given, where the row is exact, Y, Y' are left D-modules and j, φ are left D-homomorphisms. Then we have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & {[\mathrm{Hom}_D(K, Y)]} \arrow[r, "\mathrm{Hom}(1, j)"] \arrow[d, "\mathrm{Hom}(1, \varphi)"] & {[\mathrm{Hom}_D(K, Y')]} \arrow[dl, dashed, "\Phi"] \\ & {[\mathrm{Hom}_D(K, U)]} . & \end{tikzcd}$$

By (1), ${}_R[\mathrm{Hom}_D(K, U)]$ is isomorphic to Re and hence injective. Therefore there exists a dotted R-homomorphism Φ so as to make the above diagram commutative. Further, we have the next commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & K \otimes_R \mathrm{Hom}_D(K, Y) \arrow[r, "1 \otimes \mathrm{Hom}(1, j)"] \arrow[d, "1 \otimes \mathrm{Hom}(1, \varphi)"] & K \otimes_R \mathrm{Hom}_D(K, Y) \arrow[dl, "1 \otimes \Phi"] \\ & K \otimes \mathrm{Hom}_D(K, U) . & \end{tikzcd}$$

Since K is a finitely generated, projective right R-module, the functor $K \otimes_R \mathrm{Hom}_D(K, -)$ is naturally equivalent to the identity functor on the category of left D-modules. It follows that ${}_DU$ is injective. This completes the proof.

Let ${}_R\mathscr{M}$ and ${}_D\mathscr{M}$ be categories of left R-modules and left D-modules respectively. We shall define covariant functors $S: {}_D\mathscr{M} \longrightarrow {}_R\mathscr{M}$ and $T: {}_R\mathscr{M} \longrightarrow {}_D\mathscr{M}$ by $S(Y) = {}_R[\mathrm{Hom}_D(K, Y)]$ for $Y \in {}_D\mathscr{M}$ and $T(X) = {}_D[K \otimes_R X]$ for $X \in {}_R\mathscr{M}$ respectively. Then, since K is finitely generated, projective it is well known that there exists a natural equivalence $\sigma: TS \longrightarrow 1_{{}_D\mathscr{M}}$, where $1_{{}_D\mathscr{M}}$ means the identity functor on ${}_D\mathscr{M}$ and $\sigma(Y)(r \otimes f) = f(r)$ for $r \in K$, $f \in \mathrm{Hom}_D(K, Y)$. On the other hand, there exists a natural transformation (not necessarily an equivalence) $\tau: 1_{{}_R\mathscr{M}} \longrightarrow ST$, where $1_{{}_R\mathscr{M}}$ means the identity functor on ${}_R\mathscr{M}$ and $[\tau(X)](r) = r \otimes x$, for $x \in X$, $r \in K$.

Assume that ${}_RW$ is isomorphic to a direct product $\prod_{\lambda \in \Lambda} Re^{(\lambda)}$, $Re^{(\lambda)} \cong Re$. Since K_R is finitely generated, projective, by (1) of Proposition 2.1 we have that

$$\begin{aligned} {}_RW &\cong \prod_{\lambda \in \Lambda} Re^{(\lambda)} \cong \prod_{\lambda \in \Lambda} \mathrm{Hom}_D(K, Ke^{(\lambda)}) \cong \mathrm{Hom}_D(K, \prod_{\lambda \in \Lambda} Ke^{(\lambda)}) \\ &\cong \mathrm{Hom}_D(K, K \otimes_R \prod_{\lambda \in \Lambda} Re^{(\lambda)}) \cong \mathrm{Hom}_D(K, K \otimes_R W) = ST(W) \end{aligned}$$

and the composite of all isomorphisms is $\tau(W)$.

Now, we shall introduce a special injective dimension closely connected with ${}_DU$. Let Y be a left D-module. Then we shall say that Y has U-injective dimention $\geqq n$ (denoted by U-inj. dim ${}_DY$), if there exists a following injective resolution of Y:

$$0 \longrightarrow Y \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n$$

such that all V_i, $1 \leqq i \leqq n$, are isomorphic to direct products of copies of U. It is to be noted that this notion can be defined for any injective module.

Then we have

PROPOSITION 2.2. *Let $\mathscr{A}$ be the category consisting of left R-modules X such that* dom. dim ${}_RX \geqq 2$ *and $\mathscr{B}$ the category of left D-modules Y such that* U-inj. dim ${}_DY \geqq 2$. *Then $\mathscr{A}$ and $\mathscr{B}$ are equivalent by S and T.*

Proof. If $X \in \mathscr{A}$, then we have a injective resolution of X:

$$0 \longrightarrow X \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \cdots, \quad \text{where} \quad W_i,\ i = 1, 2,$$

are torsionless and injective. Since ${}_RRe$ is faithful, ${}_RR$ is imbedded into a direct product of copies of Re. On the other hand, every torsionless, injective left R-module is imbedded into a direct product of copies of ${}_R R$. Thus, every torsionless injective left R-module is imbedded into a direct product of copies of Re. Hence, without loss of generality we can assume that W_i are isomorphic to $\prod_{\lambda \in \Lambda_i} Re^{(\lambda)}$, $i = 1, 2$, $Re^{(\lambda)} \cong Re$. It follows by the finitely generated projectivity and the faithfulness of K_R that U-inj. dim ${}_D T(X) \geqq 2$, if $X \in \mathscr{A}$. Conversely dom. dim ${}_R S(Y) \geqq 2$, if $Y \in \mathscr{B}$, by Proposition 2.1 (1). Now, by the exposition preceding this proposition, it is enough to prove that the restriction of $\tau\colon 1_{{}_R\mathscr{M}} \longrightarrow ST$ to $\mathscr{A}$ is an equivalence. However, from the above remark and the exposition, in the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & W_1 & \longrightarrow & W_2 \\ & & \downarrow{\scriptstyle \tau(X)} & & \downarrow{\scriptstyle \tau(W_1)} & & \downarrow{\scriptstyle \tau(W_2)} \\ 0 & \longrightarrow & ST(X) & \longrightarrow & ST(W_1) & \longrightarrow & ST(W_2) \end{array}$$

we know that $\tau(W_1)$ and $\tau(W_2)$ are left R-isomorphisms. Hence by the Five lemma it follows that $\tau(X)$ is a left R-isomorphism.

THEOREM 2.3. *Let R be a right perfect ring. If R is left* QF-3, *then there exists an idempotent f of R such that fR_R is faithful, and the category $\mathscr{A}$ consisting of all left R-modules of dominant dimension at least two is equivalent to the category ${}_{fRf}\mathscr{M}$ of all left fRf-module.*

Proof. Since R is a right perfect ring, every nonzero left R-module has nonzero socle and there exists at most a finite number of non-isomorphic irreducible left R-modules. Therefore we can assume that the socle S of Re is a direct sum of finite number of non-isomorphic irreducible left R-module. Then there is an idempotent f of R such that Rf/Nf is isomorphic to ${}_R S$, where N is the Jacobson radical of R.

Suppose that fR_R is not faithful. Then there exists an nonzero element r of R such that $fRr = 0$. Since Re is faithful, there exists an element x of Re such that $rx \neq 0$. Hence we have a nonzero submodule Rrx of Re. It follows that $Rrx \cap S = 0$, for Re is an essential extension of S. Therefore $fRrx \neq 0$. Thus we have that and $fRr \neq 0$, but this is a contradiction.

Now, it is known that we can set fR as K in Proposition 2.2 and fRf as D. It follows that ${}_{fRf}fRe$ is injective. Further, the condition that $Rf/Nf \cong {}_R S$ insure that ${}_{fRf}fRe$ is cogenerator. Hence every left fRf-module has fRe-injective dimension $\geqq 2$ (in fact $= \infty$). This completes the proof by Proposition 2.2.

By similar proofs as in [14, Proposition 2.6 and Theorem 2.4] and [13, Corollary 5.3] we have further

THEOREM 2.4. *Under the same assumption as in Theorem* 2.3, *the following facts hold:*

(1) *Let X be a left R-module.* $\mathrm{dom.\,dim}\,{}_RX \geqq 2$ *if and only if* $X \cong {}_R\mathrm{Hom}_D(fR, Y)$ *for a left D-module Y where $D = fRf$.*

(2) *If* $[\mathrm{End}_D(fR)]^\circ = R'$, *then* $R' = Q$ (= *the double centralizer of* $E({}_RR)$)[3].

(3) *If* $\mathrm{dom.\,dim}\,{}_RX = 1$, *then there exists a left R-module X' such that* $X' \supset X$ *and such that* $\mathrm{dom.\,dim}\,{}_RX' \geqq 2$. *Further,* $\mathrm{dom.\,dim}\,{}_RX'' = 1$ *and* $\mathrm{dom.\,dim}\,{}_RX'/X'' = 0$, *if* $X' \supset X'' \supset X$.

(4) *The following two conditions are equivalent* $(n \geqq 1)$.

I. $\mathrm{dom.\,dim}\,{}_RX \geqq n+1$

II. (a) ${}_R[\mathrm{Hom}_D(fR, fR \otimes_R X)] \cong {}_RX$,

(b) $\mathrm{Ext}_D^k(fR, fR \otimes_R X) = 0$ *for* $1 \leqq k \leqq n-1$.

Now for given left modules M and U over a ring D, we shall say that U is a M-cogenerator if for a nonzero left D-homomorphism f of M into itself there exists a left D-homomorphism φ of M into U such that the composite of f and φ is a nonzero left D-homomorphism. Then we prove

PROPOSITION 2.5. *Let M be a left module over a ring D such that* (i) ${}_DM$ *is a generator and* (ii) M *has an injective, M-cogenerator submodule* ${}_DU$. *Then the inverse D-endomorphism ring R of* ${}_DM$ *is left* QF-3. *Conversely, every left* QF-3 *ring is obtained in the above way for a suitable ring D and* ${}_DM$ *which satisfy* (i) *and* (ii).

Proof. Assume that ${}_DM$ satisfies the conditions (i) and (ii). Since ${}_DM$ is a generator, M_R is finitely generated, projective and hence ${}_R[\mathrm{Hom}_D({}_DM_R, {}_DU)]$ is injective. On the other hand, if we denote by e the projection of M onto U, ${}_R[\mathrm{Hom}_D({}_DM_R, {}_DU)]$ is Re and hence is projective. Let r be a nonzero element of R. By (ii) there exists an element $r' \in Re$ such that $rr' \neq 0$. Thus Re is faithful and R is left QF-3.

To prove the converse, we have only to take R as D, R as M and Re as U respectively.

In view of Proposition 2.5, it seems of interest to obtain a method by which we can calculate $\mathrm{dom.\,dim}\,{}_RR$ in case $R = [\mathrm{End}_D(M)]^0$ and

[3] (2) follows from (1) and Theorem 1.5 by putting $Y = fR$ and (2) implies that every right perfect left QF-3 ring R' of dominant dimension ≥ 2 is obtained as an inverse fRf-endomorphism ring of a generator-cogenerator fRf-module. (cf. Kato [8].)

${}_DM$ is a generator. However it was done already by Morita [13], when D is an Artinian ring. We shall remark here that his condition concerned with the self injective dimension can be replaced by another condition related to Ext-functors.

PROPOSITION 2.6. *Assume that D, M and R retain their meanings in Proposition* 2.5. *Then* I *implies* II, *where* I *and* II *are the following two conditions for a left R-module X*:

I. dom. dim ${}_RX \geqq n+1$ $(n \geqq 1)$.

II. (a) ${}_R[\mathrm{Hom}_D({}_DM_R, {}_DM \otimes_R X)] \cong {}_RX$

(b) $\mathrm{Ext}_D^k(P \oplus M, M \otimes_R X) = 0$ *for* $1 \leqq k \leqq n-1$ *and for every maximal left ideal P of D such that D/P is not isomorphic to any submodule of U.*

The two conditions are equivalent, if D is a right perfect ring. Similarly, if ${}_DU$ is a cogenerator, the two conditions are equivalent, provided we replace $P \oplus M$ in II (b) *by M.*

Proof. Let $S = \mathrm{Hom}_D({}_DM, -)$ and $T = M \otimes_R -$ be two covariant functors. Assume I. There exists an exact sequence

$$(1) \qquad 0 \longrightarrow X \longrightarrow W_1 \longrightarrow W_2 \longrightarrow \cdots \longrightarrow W_{n+1}$$

such that all W_i are direct products of $S(U)$[4]. Then, in the following commutative diagram:

$$(2) \qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & X & \longrightarrow & W_1 & \longrightarrow & W_2 & \longrightarrow & \cdots & \longrightarrow & W_{n+1} \\ & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} & & & & \downarrow{\scriptstyle \tau_{n+1}} \\ 0 & \longrightarrow & ST(X) & \longrightarrow & ST(W_1) & \longrightarrow & ST(W_2) & \longrightarrow & \cdots & \longrightarrow & ST(W_{n+1}) \end{array}$$

$\tau_1, \tau_2, \cdots, \tau_{n+1}$ are isomorphisms, becacuse M_R is finitely generated, projective. Hence τ is an isomorphism and this implies

II. (a) $\mathrm{Hom}_D({}_DM, {}_DM \otimes_R X) \cong {}_RX$.

On the other hand, we have the following exact sequence

$$(3) \qquad 0 \longrightarrow T(X) \longrightarrow T(W_1) \longrightarrow T(W_2) \longrightarrow \cdots \longrightarrow T(W_{n+1})$$

where all $T(W_i)$ are isomorphic to direct products of ${}_DU(=TS(U))$. Since $T(W_i)$ are injective, the sequence (3) can be consider as an injective resolution of $T(X)$. Hence by the exactness of the bottom sequence in (2) it follows that

[4] By Proposition 2.5 we know $Re \cong S(U)$.

(4) $\mathrm{Ext}_D^k(M, T(X)) = \mathrm{Ext}_D^k(M, M \underset{R}{\otimes} X) = 0, \qquad 1 \leqq k \leqq n-1.$

Let

(5) $$0 \longrightarrow T(X) \xrightarrow{\sigma_0} V_1 \xrightarrow{\sigma_1} V_2 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} V_{n+1}$$

be the minimal injective resolution of $T(X)$. Then, by [15, Lemma 1] V_i is isomorphic to a direct summand of a direct product $T(W_i)$ of U. Hence any irreducible submodule of V_i is isomorphic to a submodule of U. Thus by [15, Lemma 7] we have that $\mathrm{Ext}_D^k(B, T(X)) = \mathrm{Ext}_D^k(B, M \otimes_R X) = 0$ for all $0 \leqq k \leqq n$ and for all irreducible left D-modules B which are not isomorphic to submodules of U.

Now, let P be a maximal left ideal of D such that $D/P \cong B$. Then,

(6) $\mathrm{Ext}_D^k(P, M \underset{R}{\otimes} X) = \mathrm{Ext}_D^{k+1}(B, M \underset{R}{\otimes} X) = 0, \qquad 1 \leqq k \leqq n-1.$

Hence form (4) and (6) we obtain

II. (b) $\mathrm{Ext}_D^k(P \oplus M, M \otimes_R X) = 0, \qquad 1 \leqq k \leqq n-1.$

Conversely, assume II. Clearly II (b) implies (4) and (6). Since S is left exact, it follows from II (a) and (4) that

$$0 \longrightarrow X \xrightarrow{S(\sigma_0)\theta} S(V_1) \xrightarrow{S(\sigma_1)} S(V_2) \xrightarrow{S(\sigma_2)} \cdots \xrightarrow{S(\sigma_n)} S(V_{n+1})$$

is an injective resolution of X, where θ is a given isomorphism: $X \longrightarrow \mathrm{Hom}_D(M, M \otimes_R X)(= ST(X))$ in II (a).

Now, we shall assume that D is right perfect. Since every nonzero left D-module has a nonzero socle, each V_i is the injective hull of its socle. However, by [15, Lemma 7] it follows from (6) that B is not isomorphic to a submodule of V_i. Hence V_i is imbedded into a direct product of U and consequently $S(V_i)$ is imbedded into a direct product of $S(U)$. Thus dom. dim ${}_RX \geqq n+1$ by [15, Lemma 1].

In case ${}_DU$ is a cogenerator, V_i is imbedded into a direct product of U and the converse also holds[5] by [15, Lemma 1].

Especially, dom. dim ${}_RR$ is characterized only by the vanishing of $\mathrm{Ext}_D^k(P \oplus M, M)$, $k = 1, 2, \cdots$.

3. Perfect QF-3 rings. Following Thrall's paper [18] we shall say that R has a minimal faithful left module L if L is a faithful left R-module and if L appears as a direct summand of every faithful left R-module. It is clear that L is projective, and injective, and is isomorphic to some left ideal direct summand of R. Jans

[5] This proof can be regarded as a proof of Theorem 2.8, (4) for the case $R = [\mathrm{End}_D(fR)]^0$, since fRf is right perfect and ${}_DfR$ is a cogenerator.

proved in [5] that semi-primary ring has a minimal faithful left module, if $E(_RR)$ is projective.

The next propositions show the equivalence between notions of left QF-3 and left QF-3′ for right perfect rings and show a necessary and sufficient condition for perfect, left QF-3′ rings to be left QF-3$^+$.

PROPOSITION 3.1. *If R is a right perfect ring, then the following conditions are equivalent*:

(1) *$E(_RR)$ is torsionless, i.e., R is left* QF-3′.

(2) *R has a minimal faithful left module.*

(3) R *has a faithful, projective, injective ideal, i.e., R is left* QF-3.

PROPOSITION 3.2. *Let R be a perfect left* QF-3′ *ring. Then R is left* QF-3$^+$ *if and only if the socle of $_RR$ is finitely generated.*

For the proofs we shall refer to that of Proposition 2 and Theorem 2 in [4], which are known to be valid, if we consider that for right perfect rings, every nonzero left module has nonzero socle and for left perfect rings, every projective module is isomorphic to a direct sum of primitive left ideals.

PROPOSITION 3.3. *Let R be a perfect ring. If R is left and right* QF-3′, *then R is left and right* QF-3$^+$.

Proof. By Proposition 3.1 we may assume that R has faithful, projective, injective left ideal Re and right ideal fR, where e and f are idempotents of R. Then it is seen by Proposition 2.1, (2) fRe is an injective left fRf-module as well as an injective right eRe-module. Further, by Proposition 2.3, without loss of generality we can assume that fRe is a cogenerator in the category of all left fRf-modules and the category of all right eRe-modules respectively. By Proposition 2.1, (1) we have that $\mathrm{Hom}_{fRf}(fRe, fRe) \cong eRe^\circ{}_{eRe}$ and $\mathrm{Hom}_{eRe}(fRe, fRe) \cong {}_{fRf}fRf$ and hence $\mathrm{End}_{fRf}(fRe) \cong eRe^\circ$ and $\mathrm{End}_{eRe}(fRe) \cong fRf$. Therefore the fRe-duality between categories of finitely generated left fRf-modules and finitely generated right eRe-modules holds, and by Proposition 2.1, (1) we have that $\mathrm{Hom}_{fRf}(fR, fRe) \cong Re$ and $\mathrm{Hom}_{eRe}(Re, fRe) \cong fR$ which implies fR and Re are both fRe-reflexive in the sense of Cohn [3]. It follows by [12, Lemma 2.2] that the socle of fR is reflexive. Since fRf is a perfect ring, the socle of $_{fRf}fR$ is a nonzero submodule and hence it is isomorphic to a direct sum of a finite number of irreducible fRf-modules. Thus, for an integer n, we have an exact sequence:

$$0 \longrightarrow {}_{fRf}fR \longrightarrow \sum_{i=1}^{n} \oplus [fRe]. \quad \text{Hence}$$

$$\begin{array}{ccccc} 0 \longrightarrow & \mathrm{Hom}_{fRf}(fR, fR) & \longrightarrow & \sum_{i=1}^{n} \oplus [\mathrm{Hom}_{fRf}(fR, fRe)] \\ & \| & & \wr\| \\ & R' & & \sum_{i=1}^{n} \oplus [Re] \end{array}$$

is exact and consequently, $E({}_RR)(=E({}_RR'))$ can be imbedded into a direct sum of finite number of copies of Re. Thus $E({}_RR)$ is projective. By symmetry it can be proved that $E(R_R)$ is projective.

4. Examples. 1. As was remarked in the introduction, we shall give a left QF-3′ ring R such that the double centralizer of a faithful left ideal of R is not left QF-3′. For this purpose first we shall refer to

PROPOSITION 4.1. *Let R be a left primitive ring with a minimal ideal M. If R is left* QF-3′, *then R is left* QF-3 *and M is a faithful, projective, injective ideal.*

Proof. Since ${}_RM$ is imbedded into $E({}_RR)$, the injective hull $E({}_RM)$ of M is imbeded into $E({}_RR)$. Hence we have an exact sequence:

$$0 \longrightarrow E({}_RM) \xrightarrow{\varphi} \prod_{\nu \in \Lambda} R^{(\nu)},$$

where ${}_RR^{(\nu)} \cong R$, $\bigcap_{\nu \in \Lambda} \mathrm{Ker}\,\varphi_\nu = 0$, provided φ_ν are defined by $\varphi(x) = (\cdots, \varphi_\nu(x), \cdots)$, $x \in E({}_RM)$. Now, suppose $\mathrm{Ker}\,\varphi_\nu \neq 0$ for all φ_ν, $\nu \in \Lambda$. Since M is essential in $E({}_RM)$, $\mathrm{Ker}\,\varphi_\nu \cap M \neq 0$. It follows that $\mathrm{Ker}\,\varphi_\nu \supseteqq M$, because M is minimal. This implies $\bigcap_{\nu \in \Lambda} \mathrm{Ker}\,\varphi_\nu \supseteqq M \neq 0$, and this is a contradiction. Therefore $E({}_RM)$ can be imbedded into R, and hence $E({}_RM)$ is projective. Since R has a faithful, projective, injective ideal $E({}_RM)$, R is left QF-3.

On the other hand, ${}_RM$ is faithful and hence R can be imbedded into a direct product of copies of M. Hence, similarly as above we can prove that $E({}_RM)$ can be imbedded into M. However M is minimal and hence M is isomorphic to $E({}_RM)$. Thus M is projective and injective.

Let K be a field and V an infinite dimensional left K-vector space. Denote by F the inverse K-endomorphism ring of ${}_KV$. Then F is a primitive ring and has a pair of projective, minimal ideals M and N such that ${}_FM \cong {}_FV^* = \mathrm{Hom}_K({}_KV_F, {}_KK)$, $N_F \cong V_F$. Since N_F is projective, ${}_FM$ is injective and hence F is left QF-3. However F is not right QF-3′. For otherwise, it would follow by Proposition 4.1 that F is right QF-3 and N is injective. Then, by [6, Th. 1]

$$V_F \cong N \cong M^* = \mathrm{Hom}_K({}_F M_K, K_K) = V_F^{**},$$

but it is impossible, because $(V:K) = \infty$.

Now, consider the double centralizer F' of ${}_F M$. Since $[\mathrm{End}_F(M)]^0 = K$, it follows $F' = \mathrm{End}_K(M)$. Then, similarly as above we can prove that F' is not left QF-3′. Hence the double centralizer F' of a faithful left ideal M of a left QF-3 ring F need not be left QF-3′.

2. The following example shows that perferct left QF-3 rings are not necessarily semi-primary. Let K be a field, ${}_K V_s$ s-dimensional vector spaces over K, $s = 1, 2, \cdots$. Denote by M the direct sum of all V_s. Then every element of $[\mathrm{End}_K(M)]^0$ can be considered as a row finite matrix. Let A be a subring of $[\mathrm{End}_K M)]^0$ such that each element of A has the following matrix representation $\lambda E + \sum_s T_s$, where $\lambda \in K$, E is the identity matrix, T_s is the zero matrix for almost all s and T_s is written by

$$\sum_{\substack{p-1 \geqq i \geqq q+1 \\ p-2 \geqq j \geqq q \\ i > j}} \tau_{ij} C_{ij}, \quad \text{where} \quad p = (s+1)(s+2)/2,\ q = s(s+1)/2 \quad \text{and} \quad C_{ij}$$

means the matrix with 1 in the (i, j) position and 0's elsewhere and $\tau_{ij} \in K$. Then, T_s is a lower triangular matrix and every element of the Jacobson radical N of A is a sum of T_s's.

Now, consider a ring $R = Kc_{11} + M^* c_{21} + Kc_{31} + Ac_{22} + Mc_{32} + Kc_{33}$, where $M^* = {}_A[\mathrm{Hom}_K({}_K M_A, {}_K K_K)]_K$ and $+$ means the direct sum as K-modules, and the multiplication of c_{ij}, $1 \leqq i, j \leqq 3$, is same as that of matrix units and $mf = f(m)$ for $f \in M^*$, $m \in M$.

Then primitive idempotents of R are $c_{11} = e_1$, $c_{22} = e_2$ and $c_{33} = e_3$, and the Jacobson radical J of R is $M^* c_{21} + Kc_{31} + Nc_{22} + Mc_{32}$. Since $R/J = K \oplus K \oplus K$ and J is left and right T-nilpotent, R is perfect. However R is not semi-primary, for J is not nilpotent. Since $Re_1 \cong {}_R[\mathrm{Hom}_K({}_K e_3 R, {}_K K)]$, Re_1 is a faithful, projective, injective left ideal of R and hence R is left QF-3. On the other hand, by Proposition 3.2 R is not left QF-3$^+$, because the socle of ${}_R R$ is not finitely generated.

3. The next example shows that every right Artinian QF-3 ring is not necessarily left Artinian,[6] while every Artinian self-injective ring (i.e., quasi-Frobenius ring) is left Artinian.

Let Q and P be skewfields such that P is a subfield of Q and the right dimension of Q over P is finite and the left dimension of Q over P is infinite. The existence of such skewfields was proved by Cohn [2]. Similarly as in Example 2, consider a ring R such that

[6] It was proved by K. Morita [**13**, Th. 1.1] that for left or right Artinian rings "left QF-3" implies "right QF-3".

$Qc_{11} + Qc_{21} + Qc_{31} + Pc_{22} + Qc_{32} + Qc_{33}$, that is to say, R is a subring of the matrix ring Q_3. Then it is clear that R is right Artinian but not left Artinian. On the other hand, R is semi-primary and Rc_{11} and $c_{33}R$ are faithful, projective, injective, left and right ideals respectively.

4. The next example shows a non-perfect QF-3 ring.

Let K be the field of rational numbers and Z the ring of rational integers. Consider a subring R of the matrix ring K_3 such that $Kc_{11} + Kc_{21} + Kc_{31} + Zc_{22} + Kc_{32} + Kc_{33}$. It is clear that $E({}_RR)$(resp. $E(R_R)$) is isomorphic to the direct sum of 3-copies of a projective, injective ideal Rc_{11}(resp. $c_{33}R$). Hence R is QF-3, while R is not perfect.

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