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LOCAL ISOMETRIES OF FLAT TORI

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Let T_1 and T_2 be two flat tori (i.e., provided with a complete Riemannian metric of vanishing curvature). Since they are locally Euclidean each pair of points $P_1, P_2, P_i \in T_i$, has isometric neighborhoods. In general it is not possible, however, to join these separate isometries of neighborhoods to produce a single isometry $T_1 \to T_2$ or $T_2 \to T_1$; indeed there may not even exist a locally isometric map (of the whole surfaces). Necessary and sufficient conditions for the existence of such maps are deduced, making use of a recent conformal classification of maps between tori. As expected "ample" and nonample tori behave differently, and the determination of all local isometries leads to number-theoretic problems. Finally, for two given tori, the local isometries are compared with respect to homotopy by analyzing their effect on the fundamental groups.

Let $\boldsymbol{R}^+$ denote the positive reals, H the upper z-half-plane, and $SL(2, Z)$ the group of all 2×2 unimodular matrices with integral entries acting in the usual way as hyperbolic motions on H. The set of isometry classes of complete flat tori is parametrized by the 3-dimensional manifold $\boldsymbol{R}^+ \times (H/SL(2, Z))$. A point (r^2, τ) of this space represents the isometry class of the torus E^2/Γ, where Γ is the group of Euclidean motions generated by the translations

$$t_1(z) = z + r \quad \text{and} \quad t_2(z) = z + rh ,$$

with $h \in \tau$, (cf. [2]). Instead of "an isometry class of tori" we speak simply of "a torus". A torus $T = (r^2, \tau)$ is called *ample* if there exists $h \in \tau$ such that both $\Re h$ and $|h|^2$ are rational.

2. Riemannian covering maps. The following statements are generalizations of results obtained in [1] which can be similarly proved.

(i) For two tori $T_i = (r_i^2, \tau_i)$ there exist conformal covering maps $T_1 \to T_2$ if and only if two representatives $h_i \in \tau_i$ are equivalent under the action of the group $GL^+(2, Q)$ = group of 2×2 matrices with rational entries and positive determinant.

(ii) Lifting any conformal covering $T_1 \to T_2$ to the universal covering planes we obtain

$$F(z, C, D) = Cz + D , \tag{1}$$

with complex constants $C \neq 0$ and D.

(iii) For nonample T_i only

$$(2) \qquad C(\kappa) = \frac{r_2}{r_1}\kappa\ , \qquad \kappa = \pm 1, \pm 2, \cdots$$

are admissible values in (1).

(iv) For ample $T_i = (r_i^2, \tau_i)$ (2) is replaced by

$$(3) \qquad C(\kappa_1, \kappa_2) = \frac{r_2}{r_1}(\kappa_1 + \kappa_2 q'' s'' h_2)\ ,$$

where $h_2 \in \tau_2$, $h_1 = ah_2$, a an integer, $(\kappa_1, \kappa_2) \neq (0, 0)$ is a pair of arbitrary integers, and the integers q'', s'' are determined via the following relations,

$$2\Re h_2 = \frac{p}{q}\ , \qquad |h_2|^2 = \frac{r}{s}\ ,$$

p, $q > 0$, $r > 0$, $s > 0$ integers,

$$\begin{aligned} \text{g.c.d.}\ (p, q) &= \text{g.c.d.}\ (r, s) = 1\ , \\ g &= \text{g.c.d.}\ (q, s),\ q' = q/g,\ s' = s/g\ , \\ g' &= \text{g.c.d.}\ (a, q),\ a' = a/g',\ q'' = q/g'\ , \\ g'' &= \text{g.c.d.}\ (a', s'),\ a'' = a'/g'',\ s'' = s'/g''\ . \end{aligned}$$

The following materices are computable from these numbers.

$$\widetilde{T}_1 = \begin{pmatrix} a, & 0 \\ 0, & 1 \end{pmatrix}, \qquad \widetilde{T}_2 = \begin{pmatrix} a'ps'', & -a''q'r \\ q''s'', & 0 \end{pmatrix}$$

Our main result is

THEOREM 1. *For the existence of a local isometry $f: T_1 \to T_2$ the following conditions are necessary and sufficient:*

(1) *τ_1 and τ_2 are equivalent under $GL^+(2, Q)$;*

(2a) *If T_1 is nonample, then r_1/r_2 must be an integer;*

(2b) *If T_1 is ample, then $(r_1^2/r_2^2)a$ must be an integer N, and N must be representable by the quadratic form*

$$(4) \qquad \det\ (\kappa_1 \widetilde{T}_1 + \kappa_2 \widetilde{T}_2)$$

with suitable integers κ_1 and κ_2.

Proof. Since f is a conformal covering we have necessarily (1) by (i). The following identity is readily verified:

$$\frac{r_1^2}{r_2^2}|C|^2 a = \begin{cases} \det\ (\kappa \widetilde{T}_1) & \text{for } T_1 \text{ nonample} \\ \det\ (\kappa_1 \widetilde{T}_1 + \kappa_2 \widetilde{T}_2) & \text{for } T_1 \text{ ample}\ . \end{cases}$$

(The right hand side gives the number N of sheets of the covering f).

Together with the condition $|C| = 1$ for local isometry it leads to (2a) and (2b). The sufficiency follows from (iii) and (iv).

In both cases we have the following consequences. A flat torus can cover a countably infinite set of tori by local isometries. For $T_1 = T_2$ a local isometry is a global isometry, since $|C| = 1$ entails $N = 1$. In general the existence of a local isometry $T_1 \rightarrow T_2$ does not imply that there is also a local isometry $T_2 \rightarrow T_1$; this occurs if and only if both $r_1 = r_2$ and condition (1) are satisfied. (Then the tori still need not be globally isometric).

3. Homotopy classes. We show how the combination $\kappa_1 \widetilde{T}_1 + \kappa_2 \widetilde{T}_2$ controls also the deformation properties of our maps. If the constant D in (ii) is varied the map stays in the same homotopy class, but maps corresponding to different parameter values κ or (κ_1, κ_2) are not analytically homotopic (i.e., with analytic intermediately stages during the deformation), since the set of admissible values of C is discrete. We show that they are not even homotopic in the ordinary sense.

Since the fundamental group $\pi_1(T)$ of a torus is Abelian the set $\mathscr{H}$ of homotopy classes of continuous maps $T_1 \rightarrow T_2$ is in one-to-one correspondence with the set of all homomorphisms $\eta: \pi_1(T_1) \rightarrow \pi_1(T_2)$. Denoting by L_i and L_i' $(i = 1, 2)$ the path homotopy classes of two generating loops of $\pi_1(T_i)$, each such η is characterized by the integral matrix

$$\xi = \begin{pmatrix} \xi_4, & \xi_3 \\ \xi_2, & \xi_1 \end{pmatrix}$$

given by

$$\eta(L_1) = L_2^{\xi_1} L_2'^{\xi_2}, \ \eta(L_1') = L_2^{\xi_3} L_2'^{\xi_4} ;$$

hence $\mathscr{H}$ is parametrized by Z^4. The subset $\{\xi \in Z^4 : \det \xi \neq 0\}$ contains those points of Z^4 representing monomorphisms, hence it corresponds to the homotopy classes containing covering maps.

THEOREM 2. *The subset of Z^4 corresponding to homotopy classes which contain analytic maps consists of*

(a) 0 *only if τ_1 and τ_2 are nonequivalent under $GL^+(2, Q)$;*

(b) *the 1-dimensional sublattice spanned by $\widetilde{T}_1$ if τ_1 and τ_2 are equivalent under $GL^+(2, Q)$ and both are nonample;*

(c) *the 2-dimensional sublattice spanned by $\widetilde{T}_1$ and $\widetilde{T}_2$ if τ_1 and τ_2 are equivalent under $GL^+(2, Q)$ and both are ample.*

Proof. We prove only (c); (a) and (b) can be handled similarly. The generators L_i, L_i' of $\pi_1(T_i)$ are represented in E_i by the segments S_i, S_i' joining the origin to r_i and $r_i h_i$ respectively. The segments S_1

and S_1' are mapped by $F(z; C, 0)$ (cf. (ii)) into segments from the origin of E_2 to the points

$$\kappa_1 r_2 + \kappa_2 s'' q'' r_2 h_2$$

and

$$-\kappa_2 r a'' q' r_2 + (\kappa_1 a + \kappa_2 s'' p a') r_2 h_2 .$$

The former can be deformed into the two sides $\kappa_1 r_2$ and $\kappa_2 s'' q'' r_2 h_2$ of a parallelogram parallel to S_2 and S_2'. The first side represents κ_1 circuits of L_2, the second $\kappa_2 s'' q''$ contours of L_2'. Similarly for S_1'. Hence the homomorphism

$$f_*: \pi_1(T_1) \longrightarrow \pi_1(T_2)$$

induced by f is determined by

$$f_*(L_1) = L_2^{\kappa_1} L_2'^{\kappa_2 s'' q''}$$

and

$$f_*(L_1') = L_2^{-\kappa_2 r a'' q'} L_2'^{\kappa_1 a + \kappa_2 s'' p a'} .$$

This is equivalent to $\xi = \kappa_1 \widetilde{T}_1 + \kappa_2 \widetilde{T}_2$.

The determination of *all* local isometries for two given tori is easy for the nonample case. In the ample case it involves the number of ways in which $N = (r_1^2/r_2^2)a$ can be represented by the quadratic form (4). Since this form is positive definite we have, in conjunction with Theorem 2:

THEOREM 3. *The number of homotopy classes of local isometries between two flat tori is finite.*

We obtain an upper bound for this number as follows: From (3) we find

$$\Re C = \frac{r_2}{r_1}\left(\kappa_1 + \kappa_2 s'' \frac{p}{2g'}\right),$$

which shows that $\Re C$ has the form $(r_2/r_1)(\gamma/2g')$, with γ an integer. Substituting this in $|\Re C| \leqq |C| = 1$ leads to

$$|\gamma| \leqq 2g' \frac{r_1}{r_2} . \tag{5}$$

From $(\Im C)^2 = |C|^2 - (\Re C)^2$ we deduce

$$\kappa_2^2 q''^2 s''^2 (\mathfrak{X} h_2)^2 = \frac{r_1^2}{r_2^2} - \frac{\gamma^2}{4g'^2} \tag{6}$$

and

$$\kappa_1 = \frac{\gamma}{2g'} - \kappa_2 s'' \frac{p}{2g'} \, . \tag{7}$$

Each of the $2[2g'(r_1/r_2)] + 1$ integers γ compatible with (5) leads to at most two pairs (κ_1, κ_2) compatible with (6) and (7). Thus the number of homotopically different local isometries does not exceed $4[2g'(r_1/r_2)] + 2$.

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