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MEASURES ON COUNTABLE PRODUCT SPACES

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A regular conditional measure ν on a space Y relative to an outer measure μ on a space X is defined as a function on $X \times \mathscr{R}$ such that (1) for each $x \in X, \nu(x, \cdot)$ is an outer measure on Y and $\mathscr{R}$ is the family of subsets of Y which are (Carathéodory) measurable under each of the measures $\nu(x, \cdot), x \in X$, and (2) for each $\beta \in \mathscr{R}$ the function $\nu(\cdot, \beta)$ on X is μ integrable) i.e., $\int \nu(x, \beta)\mu\, dx \leqq \infty$).

Letting g be the function on the subsets of $Z = X \times Y$ defined by

$$g(\beta) = \iint I_\beta(x, y)\nu(x, \cdot)dy\, \mu\, dx\ ,$$

defining a covering family $\mathfrak{J}$ to consist of those rectangles $A \times B$ where A is μ measurable, $B \in \mathscr{R}$ and $g(A \times B) < \infty$ or those sets N such that $g(N) = 0$, we obtain the outer measure $\phi = (\mu \circ \nu)$ on Z generated by (the content) g and covering family $\mathfrak{J}$.

A system of regular conditional measures is a sequence begun by a measure ν_0 on a space X_1 and followed by regular conditional measures ν_i (relative to μ_i) on spaces X_{i+1} $(i=1, 2, \cdots)$ where $\mu_1 = \nu_0$ and $\mu_{i+1} = (\mu_i \circ \nu_i)$ for $i = 1, 2, \cdots$. Set $X = \prod_i X_i$, and for $x \in X$ write x^i for the point $(x_1, x_2, \cdots, x_i)$ which is the projection of x onto the space $X^i = \prod_{j=1}^i X_j$ and similarly write $S^i = \prod_{j=1}^i S_j$ whenever the sets S_j are subsets of X_j $(j = 1, \cdots, i)$.

For such a system of regular conditional measures a generalization of Tulcea's extension theorem for regular conditional probabilities holds, a Fubini-like theorem for integrable functions is obtained and finally, for topological spaces, a condition is given for the extension of inner regularity and almost Lindelöfness properties.

We let $\mathscr{R}_1$ be the family of ν_0 measurable sets and let $\mathscr{R}_i$ be the family of subsets of X_i which are measurable under each of the measures $\nu_{i-1}(x^{i-1}, \cdot)$, $x^{i-1} \in X^{i-1}$, and let $\mathfrak{J}_i$ be the family of subsets γ of X^i such that $\mu_i(\gamma) = 0$ or $\gamma = \alpha \times \beta$ where α is μ_{i-1} measurable and $\beta \in \mathscr{R}_i$ and $\mu_i(\gamma) < \infty$. Thus $\mathfrak{J}_i$ is the covering family which generates μ_i.

Now, writing $X_i^* = \prod_{j=i+1}^\infty X_j$ we define

$$\mathscr{R}^* = \left\{S: S = \prod_i \beta_i \text{ for some } \beta\ s.t.\ \beta_i \in \mathscr{R}_i \text{ for each } i\right\}$$

$$\mathscr{R}^{**} = \{\beta: \text{for some } i,\ \beta = \alpha \times X_i^* \text{ where } \alpha \subset X^i \text{ and } \mu_i(\alpha) = 0\}\ ,$$

$$\mathscr{R} = \mathscr{R}^* \cup \mathscr{R}^{**}\ ,$$

g to be the function on $\mathscr{R}$ which is zero on $\mathscr{R}^{**}$ and given by

$$g(\beta) = \lim_i \mu_i(\beta^i)$$

on $\mathscr{R}^*$.

For $\beta \in \mathscr{R}^*$ and $x \in X$, let

$$\rho_i(x, \beta) = \nu_0(\beta_1) \prod_{j=1}^{i-1} \nu_j(x^j, \beta_{j+1}) ,$$

and

$$\rho(x, \beta) = \lim_i \rho_i(x, \beta) .$$

Let $\mathscr{R}^{*\prime} = \{\beta \in \mathscr{R}^*: g(\beta) < \infty, \rho_i(x, \beta)$ is uniformly bounded on β, and $\rho(x, \beta)$ exists for all $x \in \beta\}$ and $\mathscr{R}' = \mathscr{R}^{*\prime} \cup \mathscr{R}^{**}$ and use $\mathscr{R}'$ and g to generate a measure φ on X.

Our first objectives are to prove that φ and g agree on the covering family $\mathscr{R}'$ and that members of $\mathscr{R}$ are φ measurable. To do this we need and state a generalization of Tulcea's extension theorem for regular conditional probabilities. The final objective is to show that the product topology on X is inner regular and almost Lindelöf [1] whenever the component spaces are provided the spaces are of finite measure and the conditional measures are continuous [1]. The proof of this parallels that given for general product measures [2].

1. A generalization of Tulcea's extension theorem. Let a regular conditional measure system ν_i' be given as above and assume that $\nu_0'(X_1) = 1$ and $\nu_i'(x^i, X_{i+1}) = 1$ for each i and $x^i \in X^i$, i.e., ν_i' is a system of regular conditional probabilities. Define the measures μ_i' as above with $\mu_1' = \nu_0'$ and $\mu_{i+1}' = (\mu_i' \circ \nu_i')$ and let $\mathscr{K}$ be the family of subsets of X which are cylinders in X over sets which are μ_i' measurable for some i.

Now let Ψ be the measure on X generated by the covering family $\mathscr{K}$ and the content h defined by

$$h(\beta) = \mu_i'(\alpha)$$

where $\alpha \subset X^i$ and $\beta = \alpha \times X_i^* \in \mathscr{K}$.

The measure Ψ differs from the conventional Tulcea extension of the conditional probabilities ν_i' in that in going from μ_i' to μ_{i+1}' the sets $\alpha \subset X^{i+1}$ for which

$$\iint I_\alpha(x^{i+1}) \nu_i'(x^i, \cdot) dx_{i+1} \mu_i' dx^i = 0$$

are assigned measure zero whereas, in the conventional extension they may not even be measurable. The conventional method of proof [3] for Tulcea's extension theorem, however, may be carried through for this new extension with essentially no changes. Therefore we give without proof the following.

THEOREM 1.1. *The members of $\mathscr{K}$ are Ψ measurable and $\Psi(\beta) = h(\beta)$ for each $\beta \in \mathscr{K}$.*

2. Agreement and measurability. Consider a member S of $\mathscr{R}^{*\prime}$ and let $\nu_0'(\cdot) = \nu_0(\cdot)/\nu_0(S_1)$ to get a normalized measure on S_1, and let $\nu_i'(x^i, \cdot) = \nu(x^i, \cdot)/\nu_i(x^i, S_{i+1})$ to get a regular conditional probability on S_{i+1}. Extending this system of probabilities as in §1 yields a (probability) measure Ψ_S on the space S. Let the measures μ_i' on S^i be associated with the ν_i' as in §1.

If $\beta \in \mathscr{R}^{*\prime}$ and $\beta \subset S$ then $\mu_i(\beta^i)$ is given by an i fold integral in

$$\begin{aligned}\mu_i(\beta^i) &= \int (i) \int I_{\beta^i}(x^i)\nu_{i-1}(x^{i-1}, \cdot)dx_i \cdots \nu_1(x^1, \cdot)dx_2\nu_0 dx_1 \\ &= \int (i) \int I_{\beta^i}(x^i)\rho_i(x, S)\nu_{i-1}'(x^{i-1}, \cdot)dx_i \cdots \nu_1'(x^1, \cdot)dx_2\nu_0' dx_1 \\ &= \int I_{\beta^i}(x^i)\rho_i(x, S)\mu_i' dx^i \\ &= \int I_{\beta^i}(x^i)\rho_i(x, S)\Psi_S dx\ .\end{aligned}$$

Thus, employing Lebesgue's theorem, we have

$$\begin{aligned}g(\beta) &= \lim_i \mu_i(\beta^i) \\ &= \lim_i \int I_{\beta^i}(x^i)\rho_i(x, S)\Psi_S dx \\ &= \int \lim_i I_{\beta^i}(x^i)\rho_i(x, S)\Psi_S dx \\ &= \int I_\beta(x)\rho(x, S)\Psi_S dx\ .\end{aligned}$$

Suppose now that $\mathscr{G} \subset \mathscr{R}'$, $\mathscr{G}$ is countable, and $S = \cup \mathscr{G}$, then the members of $\mathscr{G}$ are Ψ_S measurable since the members of $\mathscr{G}_1 = \mathscr{G} \cap \mathscr{R}^{*\prime}$ are countable intersections of members of $\mathscr{K}$ (i.e., cylinders over μ_i' measurable sets for some i) and members of $\mathscr{G}_2 = \mathscr{G} \cap \mathscr{R}^{**}$ have Ψ_S measure zero. Hence,

$$I_S(x) \leqq \sum_{\beta \in \mathscr{G}} I_\beta(x)$$

and

$$I_S(x) \leqq \sum_{\beta \in \mathscr{G}_1} I_\beta(x) \qquad \text{a.e. } \Psi_S\ .$$

Consequently,

$$\int I_S(x)\rho(x, S)\Psi_S dx \leqq \sum_{\beta \in \mathscr{G}_1} \int I_\beta(x)\rho(x, S)\Psi_S dx + 0$$

and

$$\begin{aligned} g(S) &\leqq \sum_{\beta \in \mathscr{G}_1} g(\beta) + 0 \\ &\leqq \sum_{\beta \in \mathscr{G}} g(\beta) , \end{aligned}$$

and we conclude $g(S) = \varphi(S)$ proving the

THEOREM 2.1. *If $S \in \mathscr{R}'$ then $\varphi(S) = g(S)$. Let*

$$\mathscr{M} = \{A\colon A = X^{i-1} \times \beta_i \times X_i^* \text{ for some } i \text{ and } \beta_i \in \mathscr{R}_i\}$$

and note that if $A \in \mathscr{M}$ and $S \in \mathscr{R}'$ then

$$S \cap A \in \mathscr{R}' \text{ and } S - A \in \mathscr{R}'$$

and

$$\varphi(S) = \varphi(S \cap A) + \varphi(S - A) .$$

We consequently learn that members of $\mathscr{M}$ are φ measurable since $\mathscr{R}'$ is the covering family for φ. Countable intersections of members of $\mathscr{M}$ are hence measurable proving the next

THEOREM 2.2. *If $\beta \in \mathscr{R}$ then β is φ measurable.*

For $x^i \in X^i$ let $\xi_0(\cdot) = \nu_i(x^i, \cdot)$, write $x^i y^j$ for the point $(x_1, \cdots, x_i, y_1, \cdots, y_j)$ and let $\xi_j(y^j, \cdot) = \nu_i(x^i y^j, \cdot), j = 1, 2, \cdots$. The regular conditional measure system ξ_j then determines a measure $\lambda_i(x^i, \cdot)$ on X_i^*. For $\beta \subset X$ let us agree that $\beta_x i = \{y\colon (x_1, \cdots, x_i, y_1, y_2, \cdots) \in \beta\}$. Then we may state the

THEOREM 2.3. *If β is φ measurable then*

$$\varphi(\beta) = \int \lambda_i(x^i, \beta_x i)\mu_i dx^i = \iint I_\beta(x^i y)\lambda_i(x^i, \cdot)dy\mu_i dx^i$$

and λ_i is a regular conditional measure associated with μ_i.

From [1, 1.6] we obtain the Fubini-like

THEOREM 2.4. *If f is φ integrable*[1] *then*

[1] $-\infty \leqq \int f(z)\varphi dz \leqq \infty$ and $\{z\colon f(z) \neq 0\}$ is σ-finite under φ.

$$\int f(z)\varphi\, dz = \iint f(x^i, y)\lambda_i(x^i, \cdot)dy\, \mu_i dx^i .$$

3. **Topological measures.** To review the topological notions in [1] let us suppose that T is a topological space with $\mathscr{T}$ being its family of open sets, and let θ be a measure on T for which the open sets are measurable. Then $\mathscr{T}$ is almost Lindelöf (a.L.) provided each covering of T by open sets contains a countable subfamily which covers almost all of T, and $\mathscr{T}$ is inner regular (i.r.) provided each open set can be approximated in measure by closed subsets of finite measure, i.e., for each $\beta \in \mathscr{T}$,

$$\theta(\beta) = \operatorname{Sup}_{\substack{\gamma \subset \beta,\ \theta(\gamma) < \infty \\ \gamma \text{ closed}}} .$$

Now let us assume that each of the spaces X_i is endowed with a topology $\mathscr{T}_i$ and that $\mathscr{T}^i$ is the product of the topologies $\mathscr{T}_j$, $1 \leqq j \leqq i$. Then the sequence $\mathscr{T}_i$ will be called a.L. and i.r. provided $\mathscr{T}_1$ is a.L. and i.r. relative to ν_0 and $\mathscr{T}_i$ is a.L. and i.r. relative to $\nu_{i-1}(x^{i-1}, \cdot)$ for each $x^{i-1} \in X^{i-1}$, and the sequence ν_i will be called continuous provided that for each $i = 1, 2, \cdots$, the function $\nu_i(\cdot, \beta)$ is finite and $\mathscr{T}^i$ continuous for each set β which is measurable under all measures $\nu_i(x^i, \cdot)$ where $x^i \in X^i$.

From [1, 2.3] and mathematical induction we obtain the

THEOREM 3.1. *If $\mathscr{T}_i$ is* a.L. *and* i.r., *ν_i is continuous and $\mu_i(X^i) < \infty$ for each i, then $\mathscr{T}^i$ is* a.L. *and* i.r. *relative to μ_i for each i.*

Let $\mathscr{T}$ be the product topology on X obtained from the $\mathscr{T}_i$. Then we have the

THEOREM 3.2. *If $\mathscr{T}_i$ is* a.L. *and* i.r., *ν_i is continuous, $\mu_i(X^i) < \infty$ for each i, and $\varphi(X) < \infty$ then $\mathscr{T}$ is* a.L. *and* i.r. *relative to φ.*

Proof. Suppose $A \in \mathscr{T}$, then for some countable family $\mathscr{G}$ such that each $\alpha \in \mathscr{G}$ is a cylinder $\alpha' \times \alpha''$ where $\alpha' \in \mathscr{T}^{i(\alpha)}$ and $\alpha'' = X^*_{i(\alpha)}$ we have $A = \cup \mathscr{G}$. Since α' above is $\mu_{i(\alpha)}$ measurable, α is φ measurable and consequently A is φ measurable. Since $\varphi(X) < \infty$ and each set α' can be $\mu_{i(\alpha)}$ approximated by a closed subset as closely as desired, it follows that each $\alpha \in \mathscr{G}$ can be φ approximated as closely as desired by the (closed) cylinders over those closed subsets. Since $\varphi(A) < \infty$ a finite subfamily $\mathscr{G}'$ of $\mathscr{G}$ can be chosen so that $\varphi(\cup \mathscr{G}')$ is as close to $\varphi(A)$ as desired. Hence A may be φ approximated as closely as desired by closed subsets (which are the union of the closed cylinders associated with $\mathscr{G}'$). Thus $\mathscr{T}$ is i.r. relative to φ.

To see that $\mathscr{T}$ is a.L., let $\mathscr{H}$ be an open covering of X and let $\mathscr{H}_i$ be the family of open sets in X^i such that each cylinder in X over one of these open sets is a subset of some member of $\mathscr{H}$. Thus, letting

$$\mathscr{C}_i = \{\beta\colon \beta = \alpha \times X_i^* \quad \text{for some} \quad \alpha \in \mathscr{H}_i\}$$

we see that members of $\mathscr{C}_i$ belong to the base for the topology $\mathscr{T}$ and that $X = \bigcup \mathscr{H} = \bigcup_i \bigcup \mathscr{C}_i$. Using the fact that $\mathscr{T}^i$ is both i.r. and a.L. we can select a countable subfamily $\mathscr{H}_i'$ of $\mathscr{H}_i$ for which $\mu_i(\bigcup \mathscr{H}_i - \bigcup \mathscr{H}_i') = 0$. Now, letting

$$\mathscr{C}_i' = \{\beta\colon \beta = \alpha \times X_i^* \quad \text{for some} \quad \alpha \in \mathscr{H}_i'\}$$

we have $\phi(\bigcup \mathscr{C}_i - \bigcup \mathscr{C}_i') = 0$ and taking $\mathscr{B}_i$ to be such a countable subfamily of $\mathscr{H}$ that each member of $\mathscr{C}_i'$ is a subset of some member of $\mathscr{B}_i$, we obtain further that $\phi(\bigcup \mathscr{C}_i - \bigcup \mathscr{B}_i) = 0$.

Finally, let $\mathscr{B} = \bigcup_i \mathscr{B}_i$ and conclude,

$$\begin{aligned} X - \bigcup \mathscr{B} &= \bigcup_i \bigcup \mathscr{C}_i - \bigcup_i \bigcup \mathscr{B}_i \\ &\subset \bigcup_i (\bigcup \mathscr{C}_i - \bigcup \mathscr{B}_i) \end{aligned}$$

and

$$\phi(X - \bigcup \mathscr{B}) \leqq \sum_i \phi(\bigcup \mathscr{C}_i - \bigcup \mathscr{B}_i) = 0\ .$$

Noting that $\mathscr{B}$ is a countable subfamily of $\mathscr{H}$ completes the proof.

References

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