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THE HIGHER ORDER DIFFERENTIABILITY OF SOLUTIONS OF ABSTRACT EVOLUTION EQUATIONS

P. SURYANARAYANA

In this paper, the regularity of the solution of the initial value problem for the abstract evolution equation

$$\frac{du}{dt} + A(t)u = f(t)\,, \quad u(0) \in X\,, \qquad 0 \leqq t \leqq T \tag{0.1}$$

and the associated homogeneous equation

$$\frac{du}{dt} + A(t)u = 0\,, \quad u(0) \in X\,, \qquad 0 \leqq t \leqq T \tag{0.2}$$

in a Banach space X is considered. Here $u = u(t)$ and $f(t)$ are functions from $[0, T]$ to X and $A(t)$ is a function on $[0, T]$ to the set of (in general) unbounded linear operators acting in X.

DEFINITION. $u(t)$ is called a strict solution of (0.1) or (0.2) in $(s, T]$ if

(i) $u(t)$ is strongly continuous in the closed interval $[s, T]$ and is strongly continuously differentiable in the semiclosed interval $(s, T]$,

(ii) $u(t) \in D(A(t))$, the domain of $A(t)$, for each $t \in (s, T]$,

(iii) $u(t)$ satisfies (0.1) resp. (0.2) in $(s, T]$, $u(s)$ coinciding with the given initial value at $t = s$.

It is assumed that $A(t)$ for each $t \in [0, T]$ satisfies the following conditions.

(i) $-A(t)$ generates a semigroup $\exp(-sA(t))$ of operators analytic in the sector $|\arg s| < \theta, s \neq 0, 0 < \theta < \pi/2$,

(ii) For any complex number λ satisfying $|\arg \lambda| < \pi/2 + \theta$, $0 < \theta < \pi/2$, $(\partial/\partial t)(\lambda + A(t))^{-1}$ exists in the operator topology and that there exist constants N and ρ independent of t and λ with $N > 0, 0 \leqq \rho < 1$ such that

$$\left\|\frac{\partial}{\partial t}(\lambda + A(t))^{-1}\right\| \leqq N\,|\lambda|^{\rho-1}\,.$$

The main result proved in the paper can be stated as follows. If, in addition to the above assumptions, $A(t)^{-1} \in C^{n+\alpha}[0, T]$ in the uniform operator topology, $B(t)$, a bounded operator for each $t \in [0, T]$ is of class $C^{n-1+\beta}[0, T]$, and $f(t) \in C^{n-1+\gamma}[0, T]$ in the strong topology, then the unique strict solution $u(t)$ of

$$\frac{du}{dt} + (A(t) + B(t))u = f(t)\,, \quad u(0) \in X\,. \qquad 0 \leqq t \leqq T$$

belongs to the class $C^{n+\delta}[s_0, T]$, $s_0 > 0$ arbitrary, $\delta > 0$ depending on α, β, γ and ρ. In this no assumption regarding the constancy of the domain $D(A(t))$ is made.

From the above it is clear that if further $A(t)^{-1} \in C^{\infty}[0, T]$, $B(t) \in C^{\infty}[0, T]$ and $f(t) \in C^{\infty}[0, T]$, then $u(t) \in C^{\infty}(0, T]$. It is shown by an example that the solution $u(t)$ need not be real analytic even though $A(t)^{-1}$ is real analytic and satisfies all other requirements.

The existence and uniqueness of strict solutions are established under varying hypotheses in a number of papers, Kato [3, 4, 5, 7], Tanabe [10, 11, 12], Kato and Tanabe [8] and Fisher [1] based on the theory of semigroups of operators. A survey of work done on the abstract evolution equation (0.1) is given in Kato [5]. Kato and Tanabe [8] established the existence and uniqueness of strict solutions without any assumptions on the constancy of the domain of the operators $A(t)$. They also proved that the solution $u(t)$ is analytic when $(-A(t))$ is a generator of an analytic semigroup for complex values of t in a convex neighbourhood of $[0, T]$ provided that the inhomogeneous term $f(t)$ is also analytic. On the other hand, when $D(A(t))$ is constant, Tanabe [12] proved that the solution of (0.2) is twice differentiable if $A(t)A(s)^{-1}$ is Holder continuously differentiable. P. E. Sobolevskii [9] showed that if

$$A(t)A(s)^{-1} \in C^{n+\varepsilon}[0, T]\,, \qquad f(t) \in C^{n}[0, T]\,,$$

then $u(t) \in C^{n+1}[0, T]$ and that $u(t)$ is real analytic if $A(t)A(0)^{-1}$ is real analytic.

The following notations are used throught the paper. X denotes a fixed Banach space. Σ denotes the closed sector in the complex plane consisting of the complex numbers λ satisfying

$$|\arg \lambda| \leqq \pi/2 + \theta\,, \qquad 0 < \theta < \pi/2\,.$$

E.1. For each $t \in [0, T]$, $A(t)$ is a densely defined closed linear operator acting in X. The resolvent set $\rho(-A(t))$ of $-A(t)$ contains Σ. The resolvent of $(-A(t))$ satisfies

$$\|(\lambda + A(t))^{-1}\| \leqq \frac{M}{|\lambda|} \qquad \text{for any } \lambda \in \Sigma,\ t \in [0, T]\,, \tag{1.1}$$

M being a constant independent of t and λ. (This implies that for each t, $-A(t)$ generates a semigroup $\exp(-sA(t))$ analytic in the sector $|\arg s| \leqq \theta$, $s \neq 0$. Hille-Phillips [2], Yosida [13]).

E.2.n. $A(t)^{-1}$ as a bounded operator for each $t \in [0, T]$ belongs to the class $C^{n+\alpha}[0, T]$ in the uniform operator topology (i.e. $d^n A(t)^{-1}/dt^n$ exists in the uniform operator topology and is Hölder continuous in the same topology with a Hölder exponent $\alpha > 0$).

E.3. ($K-T$-*condition*) For any $\lambda \in \Sigma$, $t \in [0, T]$, there exist constants N and ρ independent of t and λ with $N > 0$, $0 \leqq \rho < 1$ such that

$$\left\| \frac{\partial}{\partial t}(\lambda + A(t))^{-1} \right\| \leqq \frac{N}{|\lambda|^{1-\rho}} . \tag{1.2}$$

E.4.n. The inhomogeneous term $f(t)$ is of class $C^{n-1+\gamma}[0, T]$ in the strong topology for X, $0 < \gamma < 1$.

E.5.n. $B(t)$ for each $t \in [0, T]$ is a bounded operator and belongs to the class $C^{n-1+\beta}[0, T]$ in the uniform operator topology ($0 < \beta < 1$).

We first observe that if $(d/dt)A(t)^{-1}$ exists and $\lambda \in \rho(-A(t))$, then $(d/dt)(\lambda + A(t))^{-1}$ exists and

$$\begin{aligned} &\frac{d}{dt}(\lambda + A(t))^{-1} \\ &\quad = [1 - \lambda(\lambda + A(t))^{-1}]\frac{d}{dt}A(t)^{-1}[1 - \lambda(\lambda + A(t))^{-1}] . \end{aligned} \tag{1.3}$$

So $K - T$ condition always makes sense if $A(t)$ satisfies at least E.2.1 and we will always be taking $K - T$ condition in conjunction with E.2.1 at least.

We are now in a position to state our main results.

THEOREM 1. *Let* $A(t)$ *satisfy* E1, E.2.1, *and* E3, $B(t)$ *satisfy* E.5.1 *and* $f(t)$ *satisfy* E.4.1. *Then the unique strict solution* $u(t)$ *of*

$$\frac{du}{dt} + (A(t) + B(t))u = f(t) , \quad u(0) \in X , \qquad 0 \leqq t \leqq T \tag{1.4}$$

belongs to the class $C^{1+\delta}[s_0, T]$, $s_0 > 0$ *arbitrary*, $\delta > 0$ *depending on* α, β, γ *and* ρ.

THEOREM 2. *Let* $A(t)$ *satisfy* E.1, E.2.n, *and* E.3, $B(t)$ *satisfy* E.5.n *and* $f(t)$ *satisfy* E.4.n. *Then the unique strict solution* $u(t)$ *of* (1.4) *is of class* $C^{n+\delta}[s_0, T]$, $s_0 > 0$ *arbitrary*, $\delta > 0$ *depending on* α, β, γ *and* ρ.

COROLLARY 1. *If* $A(t)$ *satisfies* E.1, E.3 *and further*

(a) $$A(t)^{-1} \in C^{\infty}[0, T] ,$$

(b) $$B(t) \in C^{\infty}[0, T]$$

in the uniform operator topology and

(c) $$f(t) \in C^{\infty}[0, T]$$

in the strong topology of X, then the unique strict solution $u(t)$ of (1.4) *is of class $C^{\infty}(0, T]$ in the strong topology of X.*

COROLLARY 2. *Let us assume that $D(A(t))$ is independent of t, $A(t)$ satisfies* E.1 *and* E.2.n *and $f(t)$ satisfies* E.4.n. *Further let the bounded operator $A(t)A(s)^{-1}$ be once continuously differentiable in the uniform operator topology in $t \in [0, T]$ for any fixed $s \in [0, T]$. Then the unique strict solution $u(t)$ of* (0.1) *is of class $C^{n+\delta}[s_0, T]$ in the strong topology of X, $s_0 > 0$ arbitrary.*

2. Preliminaries and known results. We collect below some results from Kato and Tanabe [8] which will be used here very frequently.

THEOREM A (*Kato and Tanabe*). *Let $A(t)$ satisfy* E.1, E.2.1 *and* E.3 *and $f(t)$ satisfy* E.4.1. *Then the equation* (0.1) *has a unique strict solution $u(t)$ given by*

$$u(t) = U(t, 0)u(0) + \int_0^t U(t, \sigma)f(\sigma)d\sigma \; . \tag{2.1}$$

Here $U(t, s)$ is a bounded operator and is called evolution operator, Green's operator, propogator or fundamental solution. It is constructed as

$$\begin{aligned} U(t, s) = \exp(-(t-s)A(t)) \\ + \int_0^t \exp(-(t-\tau)A(t))R(\tau, s)d\tau \; , \end{aligned} \tag{2.2}$$

$R(t, s)$ being determined as the solution of the integral equation

$$R(t, s) - \int_s^t R_1(t, \tau)R(\tau, s)d\tau = R_1(t, s)$$

where

$$R_1(t, s) = -\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) \exp(-(t-s)A(s)) \; .$$

This $U(t, s)$ has the properties

(i) *$U(s, s) = I$ (The identity operator) for any $s \in [0, T]$*

(ii) *$U(t, r)U(r, s) = U(t, s)$, $0 \leqq s \leqq r \leqq t \leqq T$*

(iii) *The range of $U(t, s)$ is contained in $D(A(t))$ and*

$$
\begin{aligned}
\frac{\partial}{\partial t} U(t, s) &= -A(t)U(t, s) \\
&= A(t) \exp(-(t-s)A(t)) - R(t, s) \\
&\quad + \int_s^t A(t) \exp(-(t-\tau)A(t))(R(\tau, s) - R(t, s))d\tau \\
&\quad + \exp(-(t-s)A(t))R(t, s) \, .
\end{aligned} \tag{2.3}
$$

LEMMA 1. *Under the same assumptions as above, the following are true.*

(2.4) (a) $$\left\| \frac{\partial}{\partial t} \exp(-(t-s)A(t)) \right\| \leqq C(t-s)^{-1} \, .$$

(2.5) (b) $$\left\| \frac{\partial}{\partial s} \exp(-(t-s)A(t)) \right\| \leqq C(t-s)^{-1} \, .$$

(2.6) (c) $$\| R_1(t, s) \| \leqq C(t-s)^{-\rho} \, .$$

(2.7) (d) $$\| R(t, s) \| \leqq C(t-s)^{-\rho} \, .$$

(e) *For* $0 \leqq s < \tau < t \leqq T$,

$$
\begin{aligned}
&\| R(t, s) - R(\tau, s) \| \\
&\leqq C\left\{ \frac{t-\tau}{(t-s)(\tau-s)^\rho} + \frac{(t-\tau)^\alpha}{t-s} \right. \\
&\qquad \left. + \frac{(t-\tau)^{1-\rho}}{(t-s)^\rho} + \frac{(t-\tau)^\alpha}{(t-s)^\rho} \log \frac{t-s}{t-\tau} \right\} .
\end{aligned} \tag{2.8}
$$

(f) *For* $0 \leqq s < \tau < t \leqq T$,

$$\| R_1(t, s) - R_1(\tau, s) \| \leqq C\left\{ \frac{t-\tau}{(t-s)(\tau-s)^\rho} + \frac{(t-\tau)^\alpha}{t-s} \right\} . \tag{2.9}$$

(g) *Let*

$$W(t, s) = \int_s^t \exp(-(t-\tau)A(t))R(\tau, s)d\tau \, . \tag{2.10}$$

Then we have

$$
\begin{aligned}
\frac{\partial}{\partial t} W(t, s) &= \int_s^t \frac{\partial}{\partial t} \exp(-(t-\tau)A(t))(R(\tau, s) - R(t, s))d\tau \\
&\quad - \int_s^t R_1(t, \tau)d\tau R(t, s) \\
&\quad + \exp(-(t-s)A(t))R(t, s) \, .
\end{aligned} \tag{2.11}
$$

$$(2.12)\quad (\text{h})\qquad \left\|\frac{\partial}{\partial t}W(t, s)\right\| \leqq C\{(t-s)^{-\rho}+(t-s)^{\alpha-1}\}\,.$$

$$(2.13)\quad (\text{i})\qquad \begin{aligned}&\frac{\partial}{\partial t}\int_s^t \exp(-(t-\sigma)A(t))f(\sigma)d\sigma\\ &=\int_s^t \frac{\partial}{\partial t}\exp(-(t-\sigma)A(t))(f(\sigma)-f(t))d\sigma\\ &\quad -\int_s^t R_1(t, \sigma)f(t)d\sigma+\exp(-(t-s)A(t))f(t)\,.\end{aligned}$$

Note. Throughout this section and the following, C denotes a positive constant depending only on the fundamental constants $M, N, \theta, \rho, \alpha$ and those which appear in the assumptions of Theorem 1. The constant C is not necessarily the same at every occurence. We use C_ε to denote a constant depending on $\varepsilon > 0$ in addition to the constants mentioned above.

We also require a slightly weaker form of Theorem 6.1 [Kato and Tanabe [8]. We present it as

THEOREM B (*Kato and Tanabe*). *Let* $A(t)$ *satisfy* E.1, E.2.1, *and* E.3, $B(t)$ *satisfy* E.5.1 *and* $f(t)$ *satisfy* E.4.1. *Then the equation* (1.4) *has a unique strict solution given by*

$$u(t) = U_1(t, 0)u(0) + \int_0^t U_1(t, \sigma)f(\sigma)d\sigma$$

where

$$U_1(t, s) = U(t, s) - \int_s^t U(t, \sigma)B(\sigma)U_1(\sigma, s)d\sigma$$

$U(t, s)$ *being the evolution operator corresponding to* (0.1).

We now proceed to give the proofs of theorems stated in § 1. Section 3 will be devoted for the proof of Theorem 1 and § 4 for Theorem 2.

For the proof of Theorem 2, we need the following Theorem C from Kato [6], which is the same as Lemma 13.7.1 in Hille-Phillips [2].

DEFINITION 1. $H(\omega, 0)$ is the set of all densely defined closed linear operators T in a Banach space X satisfying

(i) the resolvent set $\rho(-T)$ contains a sector

$$|\arg \xi| \leqq \frac{\pi}{2}+\omega\,, \qquad\qquad 0<\omega<\frac{\pi}{2}$$

and

(ii) for any $\varepsilon > 0$,

$$\|(T+\xi)^{-1}\| \leqq \frac{M_\varepsilon}{|\xi|} \qquad \text{for } |\arg \xi| \leqq \frac{\pi}{2} + \omega - \varepsilon$$

with M_ε independent of ξ.

DEFINITION 2. $H(\omega, \beta)$, β real, is the set of operators T of the form $T = T_0 - \beta$ with $T_0 \in H(\omega, 0)$.

THEOREM C. *Let $T \in H(\omega, \beta)$ and let A be relatively bounded with respect to T so that*

$$\|Au\| \leqq a\|u\| + b\|Tu\|, \qquad u \in D(T) \subset D(A).$$

For any $\varepsilon > 0$, there exists a $\beta' > 0$, $\delta > 0$ depending on T, ω and ε only, such that $T + A \in H(\omega - \varepsilon, \beta')$ whenever $a < \delta$, $b < \delta$. If in particular $\beta = 0$ and $a = 0$, then $T + A \in H(\omega - \varepsilon, 0)$.

3. Proof of Theorem 1. In view of Theorem B, we have only to prove the Hölder continuity of du/dt. We do this in several steps.

Step I. We consider the solution $u(t)$ of the homogeneous equation (0.2) with the same assumptions on $A(t)$ as in Theorem 1.

Let $0 \leqq s < r < t \leqq T$.

As $(\partial/\partial t)U(t, s) = -A(t)U(t, s)$, $(t > s)$ it is enough to estimate

$$\|A(t)U(t, s) - A(r)U(r, s)\|.$$

From (2.3) we have

$$\begin{aligned} A(t)U(t, s) &- A(r)U(r, s) \\ &= [-R(t, s) + R(r, s)] \\ &\quad + [A(t)\exp(-(t-s)A(t)) - A(r)\exp(-(r-s)A(r))] \\ &\quad + [\exp(-(t-s)A(t))R(t, s) - \exp(-(r-s)A(r))R(r, s)] \\ &\quad + \left[\int_s^t A(t)\exp(-(t-\tau)A(t))(R(\tau, s) - R(t, s))d\tau \right. \\ &\qquad \left. - \int_s^r A(r)\exp(-(r-\tau)A(r))(R(\tau, s) - R(r, s))d\tau\right] \\ &= \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)} \qquad \text{(say)} \end{aligned}$$

$\|\text{(i)}\|$ is estimated by (2.8).

$$(\text{ii}) = \frac{\partial}{\partial s}\{\exp(-(t-s)A(t)) - \exp(-(r-s)A(r))\}$$
$$= \frac{1}{2\pi i}\int_\Gamma \lambda\{e^{(t-s)\lambda}(\lambda + A(t))^{-1} - e^{(r-s)\lambda}(\lambda + A(r))^{-1}\}d\lambda$$

where Γ is a smooth contour running in Σ from $\infty e^{-i((\pi/2)+\theta)}$ to $\infty e^{i((\pi/2)+\theta)}$.

$$= \frac{1}{2\pi i}\int_\Gamma e^{(t-s)\lambda}\lambda\int_r^t \frac{\partial}{\partial\sigma}(\lambda + A(\sigma))^{-1}d\sigma d\lambda$$
$$+ \frac{1}{2\pi i}\int_\Gamma\int_{r-s}^{t-s}\frac{\partial}{\partial\sigma}(e^{\sigma\lambda})d\sigma\lambda(\lambda + A(r))^{-1}d\lambda\ .$$

Using (1.1) and (1.2) we have

$$\|(\text{ii})\| \leqq C\left\{\frac{t-r}{(t-s)^{1+\rho}} + \frac{t-r}{(t-s)(r-s)}\right\}. \tag{3.1}$$

$$(\text{iii}) = \{\exp(-(t-s)A(t)) - \exp(-(r-s)A(r))\}R(t,s)$$
$$+ \exp(-(r-s)A(r))(R(t,s) - R(r,s))$$
$$= (\text{iii a}) + (\text{iii b}) \qquad (\text{say}).$$

$\|(\text{iii})a\| \leqq C(t-r)/(t-s)^\rho(r-s)$ using (2.7) and (2.4).
$\|(\text{iii b})\|$ is estimated by (2.8) since $\|\exp(-(r-s)A(r))\| \leqq M$.

$$(\text{iv}) = \int_s^r\{A(t)\exp(-(t-\tau)A(t)) - A(r)\exp(-(r-\tau)A(r))\}$$
$$\times\{R(\tau,s) - R(r,s)\}d\tau$$
$$+ \int_s^r A(t)\exp(-(t-\tau)A(t))(R(r,s) - R(t,s))d\tau$$
$$+ \int_r^t A(t)\exp(-(t-\tau)A(t))(R(\tau,s) - R(t,s))d\tau$$
$$= (\text{iv a}) + (\text{iv b}) + (\text{iv c})\ .$$

Using (3.1) and (2.8) we have

$$\|\text{iv(a)}\| < C\int_s^r(t-r)\left\{\frac{1}{(t-\tau)^{\rho+1}} + \frac{1}{(r-\tau)(t-\tau)}\right\}$$
$$\times\left\{\frac{r-\tau}{(r-s)(\tau-s)^\rho} + \frac{(r-\tau)^\alpha}{r-s} + \frac{(r-\tau)^{1-\rho}}{r-s} + \frac{(r-\tau)^\alpha}{(r-s)^\rho}\log\frac{r-s}{r-\tau}\right\}d\tau\ .$$

Estimating the various integrals on the right, with $\varepsilon > 0$ arbitrarily chosen, we can prove

$$\|(\text{iv a})\| < C_\varepsilon\left\{\frac{(t-r)^{1-\rho}}{(r-s)^{\rho-\alpha}} + \frac{(t-r)^{1-\varepsilon}}{(r-s)^{1+\rho-\varepsilon}} + \frac{(t-r)^{\alpha-\varepsilon}}{(r-s)^{\rho-\varepsilon}} + \frac{(t-r)^{1-\rho-\varepsilon}}{(r-s)^{\rho-\varepsilon}}\right\}.$$

$$\|(\text{iv b})\| \leqq \|R(t, s) - R(r, s)\| \int_s^r \frac{d\tau}{t-\tau}$$
$$= \|R(t, s) - R(r, s)\| \log \frac{t-s}{t-r}$$

and this can be estimated by (2.8).

$$\|(\text{iv c})\| \leqq C\int_r^t \frac{d\tau}{t-\tau} \|R(\tau, s) - R(t, s)\| d\tau \qquad \text{using (2.5)}$$
$$\leqq C\Big\{\frac{t-r}{(t-s)(r-s)^\rho} + \frac{(t-r)^\alpha}{t-s} + \frac{(t-r)^{1-\rho}}{(t-s)^\rho} + \frac{(t-r)^\alpha}{(t-s)^\rho}\Big(\frac{1}{\alpha^2} + \frac{1}{\alpha}\log\frac{t-s}{t-r}\Big)\Big\}$$

using (2.8) and estimating the respective integrals. Combining all these estimates, we note for $0 \leqq s < s_0 \leqq r < t \leqq T$,

$$\|A(t)U(t, s) - A(r)U(r, s)\| \leqq C(t-r)^\eta ,$$

$\eta = \min(1-\rho-\varepsilon, \alpha-\varepsilon)$ and C depends on $s_0 > s$, s, ε and T. ε can be chosen to make $\eta > 0$.

This establishes the Hölder continuity of the derivative of the solution of the equation (0.2) in every interval of the form $[s_0, T]$, $0 < s_0 < T$.

Step II. We now consider the solution $u(t)$ of the equation (0.1) with the same assumptions on $A(t)$ and $f(t)$ as in Theorem 1.

The solution of (0.1) is given by:

$$U(t) = U(t, s)u(s) + \int_s^t U(t, \sigma)f(\sigma)d\sigma , \qquad 0 \leqq s < t \leqq T ,$$

$u(s)$ being the initial value at $t = s$ and $U(t, s)$ the corresponding evolution operator.

In view of the result proved in Step I, it is enough to consider the case $u(s) = 0$.

Let $0 \leqq s < r < t \leqq T$.

From the defining equations of $U(t, s)$ and $W(t, s)$ on using (2.11), (2.12) and (2.13) we obtain

$$\frac{du}{dt} = \int_s^t \frac{\partial}{\partial t}\exp\{-(t-\sigma)A(t)\}(f(\sigma) - f(t))d\sigma$$
$$- \int_s^t R_1(t, \sigma)f(t)d\sigma + \exp(-(t-s)A(t))f(t)$$
$$+ \int_s^t \frac{\partial}{\partial t}W(t, \sigma)f(\sigma)d\sigma .$$

Using the Hölder continuity of $f(t)$, the estimates (2.4) through (2.13) and the estimates obtained in Step I, we can prove after some tedious computations that for $0 \leqq s < s_0 \leqq r < t \leqq T$,

$$\left\| \frac{du(t)}{dt} - \frac{du(r)}{dr} \right\| < K(t-r)^{\eta}$$

where $\eta = \min\{1-\rho-\varepsilon, \alpha-\varepsilon, \gamma-\varepsilon\}$, $\varepsilon > 0$ being arbitrarily chosen to make $\eta > 0$. K is a constant depending on s_0, s, ε and T but not on t and r. This establishes the Hölder continuity of the derivative of the solution of (0.1) in every interval of the form $[s_0, T]$, $0 < s_0 < T$.

Step III. The existence and uniqueness of the solution of the equation (1.4) is established in Theorem B. We have only to establish the Hölder continuity of the derivative du/dt of this solution. Because $u(t) \in C^1(0, T]$ and $B(t)$ is Hölder continuous, we have that $B(t)u(t) \in C^{\beta}(0, T]$. We can treat $u(t)$ as the solution of the equation

$$\frac{du}{dt} + A(t)u(t) = f(t) - B(t)u(t) , \qquad u_0 \in X \text{ given} .$$

As $A(t)$ satisfies the conditions used in Step II and $f(t) - B(t)u(t)$ is strongly Hölder continuous with Hölder exponent $\min(\gamma, \beta)$, by appealing to the result in Step II, we have $u(t) \in C^{1+\eta}[s_0, T]$,

$$\eta = \min(1-\rho-\varepsilon, \alpha-\varepsilon, \gamma-\varepsilon, \beta-\varepsilon) , \qquad s_0 > 0 ,$$

$\varepsilon > 0$ chosen arbitrarily to make $\eta > 0$.

This completes the proof of Theorem 1.

4. Proof of Theorem 2. The proof of Theorem 2 will be given after a few preparatory lemmas.

Let us first remark that if $u(t)$ is a strict solution of

$$\text{(1.4)} \qquad \frac{du}{dt} + (A(t) + B(t))u = f(t) , \quad u(0) \in X , \qquad 0 \leqq t \leqq T ,$$

then $e^{-Kt}u(t)$ is the strict solution of

$$\text{(4.1)} \qquad \frac{du}{dt} + (A(t) + B(t) + K)u = e^{-Kt}f(t) , \quad u(0) \in X , \quad 0 \leqq t \leqq T$$

and conversely. So we may, if necessary, consider the equation (4.1) instead of (1.4) with a suitable choice of K.

LEMMA 2. *If $A(t)$ satisfies* E.1, *$(d/dt)A(t)^{-1}$ exists as a bounded* *operator and* E.3 *holds, then for a suitable* $K > 0$,

$$A(t) + K + \left\{\frac{d}{dt}(A(t) + K)^{-1}\right\}(A(t) + K) \qquad \{= A(t; K) \text{ for short}\}$$

satisfies E.1 *with a possibly different constant* M.

Proof. We can regard $-A(t; K)$ as a perturbation of the analytic semi-group generator $-(A(t) + K)$. If $u \in D(A(t))$, we have

$$\left\|\frac{d}{dt}(A(t) + K)^{-1}(A(t) + K)u\right\| \leqq \left\|\frac{d}{dt}(A(t) + K)^{-1}\right\| \, \|(A(t) + K)u\| \, .$$

So $(d/dt)(A(t) + K)^{-1}(A(t) + K)$ is relatively bounded with respect to $(A(t) + K)$ with a relative bound

$$\leqq \left\|\frac{d}{dt}(A(t) + K)^{-1}\right\| .$$

According to Theorem C of § 2, $-A(t; K)$ generates an analytic semi-group if we can make

$$\left\|\frac{d}{dt}(A(t) + K)^{-1}\right\| < \frac{1}{1 + M} ,$$

M being the constant appearing in E.1. In view of $K - T$ condition we have

$$\left\|\frac{d}{dt}(A(t) + K)^{-1}\right\| \leqq \frac{N}{|K|^{1-\rho}}$$

uniformly for all $t \in [0, T]$. So if we choose $K > 0$ such that

$$NK^{\rho-1} < (1 + M \operatorname{Sec} \theta)^{-1} \tag{4.2}$$

(the term $\operatorname{Sec} \theta$ is introduced for convenience in work later on), we have for each $t \in [0, T]$, $-A(t; K)$ generates an analytic semigroup. Further, the resolvent set of each of these operators contains the sector Σ and

$$\|(\lambda + A(t; K))^{-1}\| \leqq \frac{M^1}{|\lambda|} \qquad \text{for } \lambda \in \Sigma \, , \tag{4.3}$$

M^1 being a constant independent of t and λ. This completes the proof of Lemma 2.

LEMMA 3. *If* $A(t)$ *satisfies* E.3 *and* E.2.n, *then for a suitable* $K > 0$, $A(t; K)$ *satisfies* E.2.n-1.

Proof. $A(t; K) = \{1 + (d/dt)(A(t) + K)^{-1}\}(A(t) + K)$. In view of E.3, we can choose $K > 0$ such that

$$\left\| \frac{d}{dt}(A(t) + K)^{-1} \right\| < 1$$

so that $\{1 + (d/dt)(A(t) + K)^{-1}\}^{-1}$ exists as a bounded operator. Then for such a choice of K,

$$A(t; K)^{-1} = (A(t) + K)^{-1}\left\{1 + \frac{d}{dt}(A(t) + K)^{-1}\right\}^{-1}.$$

Also in view of (1.3) and $A(t)^{-1} \in C^{n+\alpha}[0, T]$, it follows that

$$A(t; K)^{-1} \in C^{n-1+\alpha}[0, T].$$

Hence the Lemma is proved.

LEMMA 4. *If $A(t)$ satisfies* E.1, E.2.2 *and* E.3, *then for a suitable $K > 0$, $A(t; K)$ satisfies* E.3 *with a possibly different N but with the same ρ $(0 \leqq \rho < 1)$.*

Proof. By Lemma 1, $A(t; K)$ satisfies E.1 if $K > 0$ is chosen according to (4.2). Let $\lambda \in \Sigma$. From the second resolvent equation, we have

(4.4)

$$\begin{aligned}(\lambda + A(t; K))^{-1} &- (\lambda + A(t) + K)^{-1} \\ &= -(\lambda + A(t; K))^{-1}\frac{d}{dt}(A(t) + K)^{-1}(A(t) + K)(\lambda + A(t) + K)^{-1}.\end{aligned}$$

Since $A(t)$ satisfies E.2.2, $(\partial/\partial t)(A(t; K) + \lambda)^{-1}$ exists for $\lambda \in \Sigma$ noting (1.3). From (4.4) we have

$$\begin{aligned}&\frac{\partial}{\partial t}(A(t; K) + \lambda)^{-1} \\ &\quad = \frac{\partial}{\partial t}(\lambda + K + A(t))^{-1} \\ &\qquad - (\lambda + A(t; K))^{-1}\frac{d^2}{dt^2}(A(t) + K)^{-1}(A(t) + K)(A(t) + \lambda + K)^{-1} \\ &\qquad - (\lambda + A(t; K))^{-1}\frac{d}{dt}(A(t) + K)^{-1}\frac{\partial}{\partial t}\{(A(t) + K)(\lambda + K + A(t))^{-1}\} \\ &\qquad - \frac{\partial}{\partial t}(A(t; K) + \lambda)^{-1}\frac{d}{dt}(A(t) + K)^{-1}(A(t) + K)(A(t) + \lambda + K)^{-1} \\ &\quad = (1) + (2) + (3) + (4) \qquad \text{say}.\end{aligned}$$

Therefore

$$\left\| \frac{\partial}{\partial t}(A(t; K) + \lambda)^{-1} \right\| \leqq \|(1)\| + \|(2)\| + \|(3)\| + \|(4)\| .$$

Now

$$\|(1)\| \leqq \frac{N}{|\lambda + K|^{1-\rho}} \qquad \text{in view of E.3 .}$$

$$\begin{aligned} \|(2)\| &\leqq \|A(t; K) + \lambda)^{-1}\| \\ &\quad \times \left\| \frac{d^2}{dt^2}(A(t) + K)^{-1} \right\| \|(A(t) + K)(A(t) + \lambda + K)^{-1}\| \\ &\leqq \|A(t; K) + \lambda)^{-1}\| C_K\left(1 + \frac{M|\lambda|}{|\lambda + K|}\right) \qquad \text{using E.1} \end{aligned}$$

where

$$C_K = \sup_{0 \leqq t \leqq T} \left\| \frac{d^2}{dt^2}(A(t) + K)^{-1} \right\|$$

which is finite in view of E.2.2. Further from (4.4) we have

$$\begin{aligned} &\|(A(t; K) + \lambda)^{-1}\| \\ &\quad \leqq \|(A(t) + K + \lambda)^{-1}\| \\ &\qquad + \|(A(t; K) + \lambda)^{-1}\| \left\| \frac{d}{dt}(A(t) + K)^{-1} \right\| \|1 - \lambda(A(t) + \lambda + K)^{-1}\| \\ &\quad \leqq \frac{M}{|\lambda + K|} + \|(A(t; K) + \lambda)^{-1}\| \frac{N}{K^{1-\rho}}\left(1 + M\frac{|\lambda|}{|\lambda + K|}\right) \\ &\quad \leqq \frac{M}{|\lambda + K|} + \|(A(t; K) + \lambda)^{-1}\| \frac{N}{K^{1-\rho}}(1 + M \operatorname{Sec} \theta) \\ &\quad \leqq \frac{M}{|\lambda + K|} + \|(A(t; K) + \lambda)^{-1}\| \mu, \quad \mu < 1 \qquad \text{(because of (4.2)) .} \end{aligned}$$

Therefore

$$\|(A(t; K) + \lambda)^{-1}\| < \frac{M}{|\lambda + K|(1 - \mu)} .$$

Hence

$$\|(2)\| < \frac{MC_K}{|\lambda + K|} \frac{(1 + M \operatorname{Sec} \theta)}{1 - \mu} .$$

$$\begin{aligned} \|(3)\| &\leqq \|(A(t; K) + \lambda)^{-1}\| \\ &\quad \times \left\| \frac{d}{dt}(A(t) + K)^{-1} \right\| \left\| \frac{\partial}{\partial t}(1 - \lambda(A(t) + K + \lambda)^{-1}) \right\| \\ &< \frac{M_1 N^2 \operatorname{Sec} \theta}{K^{1-\rho}|\lambda + K|^{1-\rho}} . \end{aligned}$$

$$\| (4) \| \leqq \left\| \frac{\partial}{\partial t}(A(t; K) + \lambda)^{-1} \right\|$$

$$\times \left\| \frac{d}{dt}(A(t) + K)^{-1} \right\| \| 1 - \lambda(A(t) + K + \lambda)^{-1} \|$$

$$\leqq \left\| \frac{\partial}{\partial t}(A(t; K) + \lambda)^{-1} \right\| \mu ,$$

$$\mu = \frac{N(1 + M \operatorname{Sec} \theta)}{K^{1-\rho}} < 1 .$$

Combining all these estimates we have

$$\left\| \frac{\partial}{\partial t}(A(t; K) + \lambda)^{-1} \right\| \leqq \frac{N^1}{|\lambda + K|^{1-\rho}} \leqq \frac{N_1}{|\lambda|^{1-\rho}}$$

N^1, N_1 being constants which do not depend on t or λ. Thus $A(t; K)$ satisfies E.3 with the same ρ and this completes the proof of the Lemma.

Proof of Theorem 2. We wish to prove this theorem by induction. First the case $n = 1$ is Theorem B of Kato and Tanabe.

So let us now assume the theorem true for $n = m$ and make the induction hypothesis that $A(t)$ satisfy E.1, E.2. $m + 1$ and E.3, $B(t)$ satisfy E.5. $m + 1$ and $f(t)$ satisfy E.4. $m + 1$. Let $K > 0$ be so chosen to satisfy (4.2) and to allow

$$\left\{1 + \frac{d}{dt}(A(t) + K)^{-1} + B(t)(A(t) + K)^{-1}\right\}^{-1} \qquad (= B(t; K)^{-1})$$

exist as a bounded operator for each $t \in [0, T]$. This is possible because $\operatorname{Sup}_{0 \leqq t \leqq T} \| B(t) \|$ is finite and $A(t)$ satisfies E.1 and E.2.

As remarked earlier, we will consider the equation

$$\frac{du}{dt} + (A(t) + B(t) + K)u = e^{-Kt} f(t) , \tag{4.1}$$
$$u(0) = U_0 \in X , \qquad 0 \leqq t \leqq T$$

with K chosen above.

In view of Theorem B, equation (4.1) has a unique strict solution under our present hypotheses on $A(t)$, $B(t)$ and $f(t)$.

Let $F(t) = e^{-Kt} f(t)$,

$$g(t) = \left(B(t; K) \frac{d}{dt} B(t; K)^{-1}\right) F(t) . \tag{4.5}$$

In view of Lemmas 2, 3, and 4 and our induction hypothesis, we note that $A(t; K)$ satisfies E.1, E.2.m and E.3. Also because $A(t)^{-1} \in C^{m+1+\alpha}[0, T]$, $B(t) \in C^{m+\beta}[0, T]$, $f(t) \in C^{m+\gamma}[0, T]$, if follows that

(i) $B(t) + B(t; K)(d/dt)B(t; K)^{-1}$ is a bounded operator for each $t \in [0, T]$,

(ii) $B(t; K) \in C^{m+\nu}[0, T]$, $\nu = \min(\alpha, \beta)$

(iii) $B(t) + B(t; K)(d/dt)B(t; K)^{-1} \in C^{m-1+\nu}[0, T]$,

(iv) $(dF/dt) + g(t) \in C^{m-1+\eta}[0, T]$, $\eta = \min(\nu, \gamma)$.

So $A(t; K)$, $B(t) + B(t; K)(d/dt)B(t; K)^{-1}$, $g(t)$ satisfy respectively the conditions for $A(t)$, $B(t)$, $f(t)$ of the theorem with $n = m$.

Let $t_0 \in (0, T)$ be arbitrarily chosen. Then consider the equation

$$\frac{dv}{dt} + \left(A(t; K) + B(t) + B(t; K)\frac{d}{dt}B(t; K)^{-1}\right)v = -\left(\frac{dF}{dt} + g(t)\right) \tag{4.6}$$

$0 < t_0 \leqq t \leqq T$ with initial value at t_0,

$$v(t_0) = -F(t_0) + B(t_0; K)(A(t_0) + K)u(t_0)$$

where $u(t_0)$ is the value of the strict solution of (4.1) at $t = t_0$.

Because the equation (4.6) satisfies the conditions of theorem with $n = m$, we have that the unique strict solution $v(t)$ of (4.6) is of class $C^{m+\delta}[t_1, T]$, $t_1 > t_0$ arbitrary.

Let $w(t) = F(t) + v(t)$.

Clearly $w(t) \in C^{m+\delta}[t_1, T]$. Then

$$\begin{aligned}
&(A(t) + K)^{-1}B(t; K)^{-1}\frac{dw}{dt} + \left\{1 + (A(t) + K)^{-1}\frac{d}{dt}B(t; K)^{-1}\right\}w \\
&= (A(t) + K)^{-1}B(t; K)^{-1}\frac{dF}{dt} + \left\{1 + (A(t) + K)^{-1}\frac{d}{dt}B(t; K)^{-1}\right\}F \\
&\quad + (A(t) + K)^{-1}B(t; K)^{-1} \\
&\qquad \times \left[\frac{dv}{dt} + B(t; K)(A(t) + K)\left(1 + (A(t) + K)^{-1}\frac{d}{dt}B(t; K)^{-1}\right)v\right] \\
&= (A(t) + K)^{-1}B(t; K)^{-1}\frac{dF}{dt} + \left\{1 + (A(t) + K)^{-1}\frac{d}{dt}B(t; K)^{-1}\right\}F \\
&\quad + (A(t) + K)^{-1}B(t; K)^{-1}\left(-\frac{dF}{dt} - g(t)\right)
\end{aligned}$$

in view of (4.6) and noting that, $B(t; K)(A(t) + K) = A(t; K) + B(t) = F(t)$ by our choice of $g(t)$ (see 4.5).

Thus $w(t)$ satisfies

$$(A(t) + K)^{-1}B(t; K)^{-1}\frac{dw}{dt} + \left(1 + (A(t) + K)^{-1}\frac{d}{dt}B(t; K)^{-1}\right)w = F(t)\,, \qquad t_0 \leqq t \leqq T\,,$$

$$w(t_0) = F(t_0) + v(t_0) = B(t_0; K)(A(t_0) + K)u(t_0)\,.$$

Therefore

$$(A(t) + K)^{-1}\frac{d}{dt}(B(t; K)^{-1}w) + w = F(t) .$$

Writing $\xi(t) = B(t; K)^{-1}w$, we have

$$(A(t) + K)^{-1}\frac{d\xi}{dt} + \left\{1 + \frac{d}{dt}(A(t) + K)^{-1} + B(t)(A(t) + K)^{-1}\right\}\xi = F(t) ,$$
$$\xi(t_0) = B(t_0; K)^{-1}w(t_0) = (A(t_0) + K)u(t_0) , \qquad t_0 \leqq t \leqq T .$$

So

$$\frac{d}{dt}((A(t) + K)^{-1}\xi) + (A(t) + K + B(t))(A(t) + K)^{-1}\xi = F(t) .$$

Writing $(A(t) + K)^{-1}\xi = \zeta$, we have

$$\text{(4.7)} \qquad \begin{aligned} &\frac{d\zeta}{dt} + (A(t) + B(t) + K)\zeta = F(t) , \\ &\zeta(t_0) = u(t_0) , \qquad t_0 \leqq t \leqq T . \end{aligned}$$

Since (4.1) has a unique strict solution $u(t)$, $0 < t \leqq T$, and since the unique strict solution of (4.7) coincides with that of (4.7) at $t = t_0$, we conclude that

$$\zeta(t) = u(t) , \qquad \text{for } t \in [t_0, T] .$$

Now

$$u(t) = (A(t) + K)^{-1}\xi(t) = (A(t) + K)^{-1}B(t; K)^{-1}w(t) .$$

Because $w(t) \in C^{m+\delta}[t_1, T]$, $B(t; K)^{-1} \in C^{m+\nu}[0, T]$, and

$$(A(t) + K)^{-1} \in C^{m+1+\alpha}[0, T] ,$$

we have $u(t) \in C^{m+\delta}[t_1, T]$ and

$$(A(t) + K)u(t) = B(t; K)^{-1}w(t) \in C^{m+\delta}[t_1, T] .$$

Therefore

$$\frac{du}{dt} = -(A(t) + K)u(t) - B(t)u(t) \in C^{m+\delta}[t_1, T] .$$

Hence $u \in C^{m+1+\delta}[t_1, T]$.

Because $t_0 > 0$, and $t_1 > t_0$ are arbitrary, we have

$$u(t) \in C^{m+1+\delta}[t_1, T] , \qquad t_1 > 0$$

arbitrary. This completes the proof of Theorem 2.

Corollary 1 follows immediately.

Example of an operator $A(t)$ which satisfies E.1 *and* E.3 *with $A(t)^{-1}$ real analytic the corresponding evolution operator $U(t, s)$ of which is not real analytic*:

This example is the same as given in Kato [5].

Let $X = L^2[a, b]$, $0 < a < b < T$. $A(t)$ be a family of multiplication operators in X defined by

$$A(t)u(x) = (t - x)^{-2}u(x) , \qquad u(x) \in D(A(t)) .$$

These are positive and self adjoint operators and so E.1 is clearly satisfied. Because $(\partial/\partial t)(A(t) + \lambda)^{-1}$ is a multiplication operator defined by

$$\begin{aligned} \frac{\partial}{\partial t}(A(t) + \lambda)^{-1}u(x) &= \frac{2(t - x)^{-3}}{(\lambda + (t - x)^{-2})^2}u(x) , \\ \left\|\frac{\partial}{\partial t}(A(t) + \lambda)^{-1}\right\| &\leqq \operatorname*{Sup}_{t,x}\left|\frac{2(t - x)^{-3}}{(\lambda + (t - x)^{-2})^2}\right| \\ &\leqq 2 \operatorname*{Sup}_{t,x}\left|\left\{\frac{(t - x)^{-2}}{\lambda + (t - x)^{-2}}\right\}^{3/2}\frac{1}{\{\lambda + (t - x)^{-2}\}^{1/2}}\right| \\ &\leqq \frac{C}{|\lambda|^{1/2}} \qquad \text{for } \lambda \in \Sigma . \end{aligned}$$

Thus $K - T$ condition is satisfied with $\rho = 1/2$. The evolution operator to this $A(t)$ is given by

$$\begin{aligned} U(t, s)u(x) &= \exp\{(t - x)^{-1} - (s - x)^{-1}\}u(x) \qquad &&\text{if } x > t \text{ or } x < s \\ &= 0 &&\text{if } s \leqq x \leqq t . \end{aligned}$$

Let

$$\begin{aligned} \eta(x) &= e^{-(1/x)} \qquad &&\text{if } x > 0 \\ &= 0 &&\text{if } x \leqq 0 \\ U(t, s)u(x) &= \eta(x - t)\eta(s - x)u(x) . \end{aligned}$$

As $|\eta(x)| \leqq 1$, $\|U(t, s)\| \leqq 1$.

It is also clear that $U(t, t) = I$ and

$$U(t, r)U(r, s) = U(t, s) \qquad s \leqq r \leqq t .$$

It can easily be shown that

$$\operatorname*{Sup}_{t,x}\left|\frac{\eta(x - t - h)\eta(s - x) - \eta(x - t)\eta(s - x)}{h} - \eta^1(x - t)\eta(s - x)\right| < \varepsilon$$

for sufficiently small h so that

$$\frac{\partial}{\partial t} U(t, s) = -(x - t)^{-2} U(t, s)$$
$$= -A(t) U(t, s) .$$

This implies that $U(t, s)$ is the evolution operator. Now $A(t)^{-1}$, being multiplication by $(t - x)^2$, is analytic in t.

Also $U(t, s) = 0$ if $s \leqq a$ and $t \geqq b$ and $U(t, s) \neq 0$ otherwise. Hence $U(t, s)$ is not analytic in t.

We note that this is due to the fact that even though $A(t)^{-1}$ has an analytic extension for t complex, $-A(t)$ is not the generator of an analytic semigroup.

Proof of Corollary 2. In view of Theorem 2, it is enough to show that $A(t)$ satisfies $K - T$ condition under the assumption that the bounded operator $A(t)A(s)^{-1}$ is continuously differentiable in the uniform operator topology in $t \in [0, T]$ for any s in $[0, T]$.

Because $(A(t)A(s)^{-1})^{-1} = A(s)A(t)^{-1}$, we have

$$\operatorname*{Lim}_{r \to t} A(s) \frac{A(r)^{-1} - A(t)^{-1}}{r - t}$$

exists in the uniform operator topology. $A(s)$ being closed, we have that $A(s)(dA(t)^{-1}/dt)$ is a bounded operator for any $s \in [0, T]$. In particular $A(t)(dA(t)^{-1}/dt)$ is a bounded operator. Because of the continuous differentiability of $A(s)A(t)^{-1}$, we can find a constant C such that

$$\left\| A(t) \frac{dA(t)^{-1}}{dt} \right\| < C \qquad \text{for } t \in [0, T] .$$

$$\frac{d}{dt}(A(t) + \lambda)^{-1} = A(t)(A(t) + \lambda)^{-1} \frac{d}{dt} A(t)^{-1} A(t)(A(t) + \lambda)^{-1}$$
$$= (A(t) + \lambda)^{-1} A(t) \frac{d}{dt} A(t)^{-1} A(t)(A(t) + \lambda)^{-1}$$

therefore

$$\left\| \frac{d}{dt}(A(t) + \lambda)^{-1} \right\| < \frac{M}{|\lambda|} C(1 + M) .$$

Thus $K - T$ condition is verified with $\rho = 0$.

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