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FANO BUNDLES AND SPLITTING THEOREMS ON PROJECTIVE SPACES AND QUADRICS

VINCENZO ANCONA, THOMAS PETERNELL AND JAROSLAW A. WIŚNIEWSKI

The aim of this paper is to describe the structure of Fano bundles in dimension ≥ 4.

Introduction. In this paper rank 2 vector bundles E on projective spaces $\mathbb{P}_n$ and quadrics Q_n are investigated which enjoy the additional property that their projectized bundles $\mathbb{P}(E)$ are Fano manifolds, i.e. have negative canonical bundles. Such bundles are shortly called Fano bundles. Up to dimension 3 Fano bundles are completely classified by [SW], [SW′], [SW″], [SSW]. The aim of this paper is to describe the structure of Fano bundles in dimension ≥ 4. Namely we prove the following

MAIN THEOREM. *Let E be a rank 2 Fano bundle on $\mathbb{P}_n$ or Q_n, $n \geq 4$. Then up to some explicit exceptions on Q_4 and Q_5 (see ex. (2.1), (2.2), (2.3)), E splits into a direct sum of line bundles.*

A rank 2 bundle E on $\mathbb{P}_n$ is Fano if and only if the "$\mathbb{Q}$-vector bundle" $E \otimes (\det E^*)/2 \otimes \mathscr{O}(\frac{n+1}{2})$ is ample, i.e.

$$\mathscr{O}_{\mathbb{P}(E)}(2) \otimes \pi^* \left(\det E^* \otimes \mathscr{O}\left(\frac{n+1}{2}\right)\right) \quad \text{is ample}.$$

If we normalize E in the following sense: $E_0 = E \otimes (\det E^*)/2$, so that $c_1(E_0) = 0$; then E is Fano iff $E_0(\frac{n+1}{2})$ is ample. Similarly on quadrics. In other words, we show that bundles with $E_0(\frac{n+1}{2})$ ample must split (on $\mathbb{P}_n$, $n \geq 4$). In other words: ample bundles with $c_1(E) \leq n+1$ split.

We prove even more:

THEOREM (9.1). *Let F be an ample 2-bundle on $\mathbb{P}_n$. Then F splits if one of the following assumptions hold*:

(1) $n = 4$, $c_1(F) \leq 6$,
(2) $n = 5$, $c_1(F) \leq 8$,

(3) $n = 6$ *or* 7, $c_1(F) \leq \frac{4n+2}{3}$,
(4) $n \geq 8$, $c_1(F) \leq \frac{5n-1}{3}$.

For testing the well-known conjecture of Hartshorne, that every 2-bundle on $\mathbb{P}_n$ ($n \geq 5$, or 6, or 7) should split, it would certainly be interesting to prove better bounds than in (9.1).

It is equally interesting to prove splitting theorems assuming only information of E on the lines in $\mathbb{P}_n$. The archaeopteryx of these theorems is the uniform splitting theorem. In the last section we prove among other things:

THEOREM (10.11). *Let E be a 2-bundle on $\mathbb{P}_n$. Assume for every line $L \subset \mathbb{P}_n$:*

$$E|L = \mathcal{O}(a_1(L)) \oplus \mathcal{O}(a_2(L))$$

with $|a_1(L) - a_2(L)| < \frac{n}{2} - 1$.

Then E splits.

1. Preliminaries. In this section we fix notations, give basic definitions and some elementary propositions which will be frequently used in the later sections.

(1.1) We will consider vector bundles only on the projective space $\mathbb{P}_n$ and on the n-dimensional quadric Q_n. If E is a vector bundle, we let $\mathbb{P}(E)$ be its associated projective bundle—taking hyperplanes in the fibers of E. We always let

$$\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) \quad \text{and} \quad \eta = \pi^*(c_1(\mathcal{O}_X(1))),$$

where $\pi\colon \mathbb{P}(E) \to X$ is the projection and $X = \mathbb{P}_n$ or Q_n. If E is a 2-bundle on X, we denote by $c_i(E)$ its Chern classes, $i = 1, 2$ and consider them as numbers. Since we work only in dimension at least 4, we have

$$H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$$

with the possible exception of Q_4; in this case

$$H^4(Q_4, \mathbb{Z}) \simeq \mathbb{Z}^2,$$

and we fix generators H_1, H_2 and identify $c_2(E) = aH_1 + bH_2$ with the pair (a, b).

DEFINITION 1.2. Let E be a vector bundle on a projective manifold X.

(1) E is said to be a Fano bundle if $\mathbb{P}(E)$ is a Fano manifold, i.e. $-K_{\mathbb{P}(E)}$ is ample.

(2) E is said to be nef ("numerically effective") if $\mathscr{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle, i.e.:

$$c_1(\mathscr{O}_{\mathbb{P}(E)}(1)) \cdot C \geq 0$$

for all curves $C \subset \mathbb{P}(E)$.

REMARK 1.3. Let $r = \operatorname{rk} E$. Since

$$-K_{\mathbb{P}(E)} = \mathscr{O}_{\mathbb{P}(E)}(r) \otimes \pi^*(-K_X \otimes \det E^*),$$

E is Fano iff the "$\mathbb{Q}$-vector bundle"

$$E \otimes \frac{\det E^*}{r} \otimes \frac{-K_X}{r}$$

is ample, i.e. the $\mathbb{Q}$-Cartier divisor

$$O_{\mathbb{P}\left(E \otimes \frac{\det E^*}{r} \otimes \frac{-K_X}{r}\right)}(1)$$

is ample. We will often abbreviate $E \otimes (\det E^*)/r$ by E_0; we have $c_1(E_0) = 0$.

PROPOSITION 1.4. (A) *Let F be a nef 2-bundle on an n-dimensional projective manifold X with $b_2(X) = 1$, $b_4(X) = 1$ where the square of a generator of $H^2(X, \mathbb{Z})$ generates $H^4(X, \mathbb{Z})$. Let $c_i = c_i(F)$ (as numbers). Then:*

(1) $c_2 \geq 0$,
(2) $c_1^2 \geq 2c_2$ *if* $n \geq 3$,
(3) $c_1^2 \geq 3c_2$ *if* $n \geq 5$,
(4) $c_1^2 \geq (2 + \sqrt{2})c_2$ *if* $n \geq 7$,
(5) $c_1^2 \geq (\frac{5}{2} + \frac{\sqrt{5}}{2})c_2$ *if* $n \geq 9$,
(6) $c_1^2 \geq (2 + \sqrt{3})c_2$ *if* $n \geq 11$.

If F is ample, all inequalities are strict.

(B) *Let F be a nef 2-bundle on Q_4. Write $c_1 = c_1(F)$, $c_2 = c_2(F) = (a, b)$. Then:*

(1) $a \geq 0$, $b \geq 0$,
(2) $c_1^2 \geq a$, $c_1^2 \geq b$,
(3) $a^2 + b^2 - 3c_1^2(a + b) + 2c_1^4 \geq 0$.

Again the inequalities are strict for F ample.

Proof. (1) in (A) or (B) is well known. The other inequalities follow from positivity of the Segre classes for ample or nef bundles (**[FL]**, [**Fu**]) and the following computations for the Segre classes $s_i = s_i(F)$:

$$\begin{aligned}
s_1 &= c_1\,, \\
s_2 &= c_1^2 - c_2\,, \\
s_5 &= c_1(c_1^2 - c_2)(c_1^2 - 3c_2)\,, \\
s_7 &= c_1(c_1^2 - 2c_2)(c_1^2 - (2 - \sqrt{2}c_2))(c_1^2 - (2 + \sqrt{2})c_2)\,, \\
s_9 &= c_1 \left(c_1^2 - \left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right) c_2\right) \left(c_1^2 - \left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right) c_2\right) \\
&\quad \times \left(c_1^2 - \left(\frac{5}{2} + \frac{\sqrt{5}}{2}\right) c_2\right) \left(c_1^2 - \left(\frac{5}{2} - \frac{\sqrt{5}}{2}\right) c_2\right)\,, \\
s_{11} &= c_1(c_1^2 - c_2)(c_1^2 - 2c_2)(c_1^2 - 3c_2)(c_1^2 - (2 + \sqrt{3}c_2)) \\
&\quad \times (c_1^2 - (2 - \sqrt{3})c_2)\,.
\end{aligned}$$

(3) is just the semi-positivity of s_4.

Later we will also use s_6 on Q_6:

$$s_6 = 2c_1^6 - 10c_1^4c_2 + 6c_1^2c_2^2 - c_2^3\,.$$

An important tool will be le Potier's vanishing theorem [SS]:

THEOREM 1.5. *Let X be a projective manifold, E an ample vector bundle of rank r. Then:*

$$H^i(X, E \otimes K_X) = 0 \quad \textit{for } i \geq r\,.$$

We will also use

PROPOSITION 1.6. *Let F be a 2-bundle on $X = \mathbb{P}_n$ or on $X = Q_n$ with $n \geq 5$.*

Assume

$$H^0(X, F) \neq 0\,, \qquad H^0(X, F(-1)) = 0\,,$$

and that $c_2(F) \leq 1$. Then F splits.

The same holds for $X = Q_4$, provided $a < 0$ or $b < 0$ or $a = b = 0$ where $c_2(F) = (a, b)$.

Proof. Take $s \in H^0(F)$, $s \neq 0$, and let $Z = \{s = 0\}$. If $Z = \varnothing$, clearly F splits. If $Z \neq \varnothing$, then Z is of pure codimension 2, and

$$\deg Z = c_2(F)\,.$$

By our assumption on $c_2(F)$, we obtain a contradiction.

(1.7) A rank 2-bundle E on $\mathbb{P}_n$ or Q_n $(n \geq 3)$ is called (semi-) stable if for every line bundle $\mathscr{L} \subset E$:

$$c_1(\mathscr{L}) < \frac{c_1(E)}{2} \left(c_1(\mathscr{L}) \leq \frac{c_1(E)}{2} \right).$$

If E is stable on $\mathbb{P}_n$, it is well known that

$$(*) \qquad c_1^2(E) < 4c_2(E)$$

([**Ba**]). This is also true for quadrics Q_n; observe that for $n \neq 4$, $(*)$ is just an inequality of numbers, for $n = 4$ $(*)$ means

$$\int_{Q_4} c_1^2(E) \wedge \omega^2 < 4 \int_{Q_4} c_1^2(E) \wedge \omega^2$$

for every Kähler form ω on Q_4.

In order to see $(*)$ for quadrics, one can proceed as follows. If E is semistable then E carries an "approximate" Hermite-Einstein connection and hence

$$c_1^2(E) \leq 4c_2(E);$$

see [**Ko**].

Now assume $c_1^2 = 4c_2$. Since we may also assume $c_1(E) = 0$, we have $c_2(E) = 0$. But it is obvious that such an E cannot be stable. Thus a stable bundle satisfies $c_1^2 < 4c_2$.

(1.8) Some further notations: $h^i(X, \mathscr{F})$ will always be the dimension of $H^i(X, \mathscr{F})$; K_X will denote canonical line bundle of the complex manifold X; $[x]$ denotes the integral part of x.

2. Statement of the main result. Before stating our main result we shortly review some facts on special rank 2 vector bundles on quadrics.

EXAMPLE 2.1. We denote by S' and S'' the two "spinor bundles" on the 4-dimensional quadric Q_4. These are bundles of rank 2 with Chern classes $c_1(S') = c_1(S'') = -1$ and $c_2(S') = (1, 0)$, $c_2(S'') = (0, 1)$. Since $S'(1)$ and $S''(1)$ are globally generated, they are Fano bundles, i.e. $\mathbb{P}(S')$ and $\mathbb{P}(S'')$ are Fano manifolds (see [**Ot1**]). We will need in the sequel the following fact due to Ottaviani ([**Ot1**, Remark 3.4]): Every stable 2-bundle on Q_4 with Chern classes $c_1 = -1$ and $c_2 = (1, 0)$ (resp. $(0, 1)$) is isomorphic to S' (resp. S'').

EXAMPLE 2.2. Applying the Serre correspondence (see e.g. [**OSS**]) to the union of two disjoint planes in Q_4 we can construct a family of stable rank 2-bundles F with

$$c_1(F) = -1, \quad c_2(F) = (1, 1).$$

Their moduli space can be identified with $\mathbb{P}_7 \backslash Q_6$ (**[Ot2**, Remark 3.4]). By **[OT2]**, $F(2)$ is generated by global sections and thus F is a Fano bundle. Moreover: every stable 2-bundle on Q_4 with Chern classes $c_1 = -1$ and $c_2 = (1, 1)$ is isomorphic to some F described above.

EXAMPLE 2.3. On Q_5 there is a family of stable 2-bundles C with $c_1(C) = -1$, $c_2(C) = 1$. These were introduced in **[Ot2]**, where they are called Cayley bundles. Again $C(2)$ is globally generated; hence Cayley bundles are Fano. Moreover we have by **[Ot2**, main theorem and Theorem 3.2]: Every stable rank 2-bundle on Q_5 with Chern classes $c_1 = -1$, $c_2 = 1$ is isomorphic to a Cayley bundle. No Cayley bundle extends to Q_6.

We are now able to state the main result of this paper.

MAIN THEOREM 2.4. (1) *Let E be a Fano bundle of rank 2 on $\mathbb{P}_n$, $n \geq 4$. Then E splits as a direct sum of two line bundles.*

(2) *Let E be a Fano bundle of rank 2 on Q_n, $n \geq 4$. Then either E splits or:*

(a) *$n = 4$ and E is—up to a twist—a spinor bundle or one of the bundles described in* (2.2);

(b) *$n = 5$ and E is—up to a twist—a Cayley bundle* (*Example* (2.3)).

Fano 2-bundles on $\mathbb{P}_n$ or Q_n with $n \leq 3$ are classified in **[SW]** and **[SSW]**. Let E be a 2-bundle on $X = \mathbb{P}_n$ or Q_n. Since E is Fano if and only if $E \otimes (\det E^*)/2 \otimes -K_X/2$ is ample, we can restate Theorem 2.4 as follows.

COROLLARY 2.5. (1) *Let E be a normalized 2-bundle on $\mathbb{P}_n$, $n \geq 4$. If $c_1(E) = 0$ assume that $E(\frac{n+1}{2})$ is ample. If $c_1(E) = -1$, assume that $E(\frac{n+2}{2})$ is ample. Then E splits.*

(2) *Let E be a normalized 2-bundle on Q_n, $n \geq 4$. If $c_1(E) = 0$, assume that $E(\frac{n}{2})$ is ample. If $c_1(E) = -1$, assume that $E(\frac{n+1}{2})$ is ample. Then either E splits or E is as in* 2.4 (2)(a), (b).

The rest of this section is devoted to the proof of the following important technical result.

PROPOSITION 2.6. *Let E be a normalized Fano bundle of rank 2 on $\mathbb{P}_n$, $n \geq 4$. Then:*

(1) *If $c_1(E) = -1$ and n is odd, then $E([\frac{n}{2}] + 3)$ is generated by global sections and $E([\frac{n}{2}] + 2)$ is ample.*

(2) *In the other cases,* $E([\frac{n}{2}]+2)$ *is generated by global sections and* $E([\frac{n}{2}]+1)$ *is ample.*

In particular, in all cases $E(n)$ *is generated by global sections and ample.*

Proof. The ampleness statements are just translations of (1.3).

(1) The le Potier vanishing Theorem (1.5) gives

$$H^i(\mathbb{P}_n, E(t)) = 0$$

for $i \geq 2$ and $t \geq k+2-(n+1) = -k$ with $k = [\frac{n}{2}]$.

In particular:

$$H^i(\mathbb{P}_n, E(k+3-i)) = 0 \quad \text{for } i \geq 2.$$

Now we claim that this holds also for $i = 1$.

Consider on $\mathbb{P}(E)$ the divisor

$$D = 3\xi + (3k+5)\eta.$$

D is clearly ample; hence by Kodaira vanishing

$$H^1(\mathbb{P}(E), D + K_{\mathbb{P}(E)}) = 0,$$

i.e. $0 = H^1(\mathbb{P}(E), \xi + (k+2)\eta) \cong H^1(\mathbb{P}_n, E(k+2))$, whence our claim.

Now $E(k+3)$ is globally generated by the Castelnuovo-Mumford lemma.

(2) We treat shortly the case $n = 2k$ and $c_1(E) = 0$ leaving the remaining cases to the reader.

The le Potier vanishing theorem gives now

$$H^i(\mathbb{P}_n, E(k+2-i)) = 0, \qquad i \geq 2,$$

while the Kodaira vanishing theorem applied to the ample divisor $3\xi + (3k+2)\eta$ yields

$$H^1(\mathbb{P}_n, E(k+1)) = 0.$$

Thus $E(k+2)$ is globally generated.

The corresponding result for Q_n reads

PROPOSITION 2.7. *Let* E *be a normalized Fano bundle of rank* 2 *on* Q_n, $n \geq 4$. *Then:*

(1) *If* $c_1(E) = 0$ *and* n *is even,* $E(\frac{n}{2})$ *is ample and* $E(\frac{n}{2}+1)$ *is globally generated.*

(2) *In the other cases,* $E([\frac{n}{2}]+1)$ *is ample and* $E([\frac{n}{2}]+2)$ *is globally generated. In particular,* $E(n-1)$ *is ample and generated in all cases.*

The proof of (2.7) is just an adaptation of (2.6) and will be omitted. The proof of Main Theorem 2.4 will be given in the subsequent sections; several cases have to be treated separately.

3. The case $\mathbb{P}_n$, $c_1^2(E) \geq 4c_2(E)$. In this section we shall prove

PROPOSITION 3.1. *Let* E *be a Fano 2-bundle on* $\mathbb{P}_n$, $n \geq 4$. *Assume* $c_1^2(E) \geq 4c_2(E)$. *Then* E *splits.*

The proof rests on the following result due to Holme and Schneider [**HS**, Theorem 4.2].

PROPOSITION 3.2. *Let* F *be a 2-bundle on* $\mathbb{P}_n$ *admitting a section whose zero locus is of pure codimension* 2. *If* F *is not stable and if moreover*

(3.2.1) $$c_2(F) < (n-1)(c_1(F) - n + 2),$$

then F *splits.*

COROLLARY 3.3. *Let* F *be a globally generated 2-bundle on* $\mathbb{P}_n$. *If* F *is not stable and if* (3.2.1) *holds, then* F *splits.*

Proof. Let $s \in H^0(F)$ be a general section. Then $Z = \{s = 0\}$ is either empty (hence F splits) or Z is smooth of codimension 2. In this second case now apply (3.2).

Proof of (3.1). We may assume E to be normalized. E is unstable by [**Ba**], because of the inequality $c_1^2(E) \geq 4c_2(E)$ (which is invariant under twists). Put $F = E(n)$. Then by (2.6) F is globally generated. Since $c_2(E) \leq 0$ and $c_1(F) = c_1(E) + 2n$, we have

$$c_2(F) = c_2(E) + c_1(E)n + n^2 \leq c_1(E)n + n^2;$$

hence (3.2.1) holds as is easily verified. Thus F—as well as E—splits by (3.3).

4. The case Q_n, $n \geq 5$, **and** $c_1^2(E) \geq 4c_2(E)$. We now treat the analogous case to §3 for quadrics Q_n, $n \geq 5$. The case Q_4 will be done later.

PROPOSITION 4.1. *Let* E *be a Fano 2-bundle on* Q_n, $n \geq 5$. *Assume* $c_1^2(E) \geq 4c_2(E)$. *Then* E *splits. In order to prove* (4.1) *we need the following analogy to* (3.2):

PROPOSITION 4.2. *Let* F *be a 2-bundle on* Q_n, $n \geq 5$, *admitting a section whose zero locus is of pure codimension* 2. *If* F *is unstable and if moreover*

(4.2.1) $$c_2(F) \leq (n-2)(c_1(F) - n + 2) + n - 3,$$

then F *splits.*

Postponing the proof of (4.2) for a moment we have as in §3 the immediate

COROLLARY 4.3. *Let* F *be a globally generated 2-bundle on* Q_n, $n \geq 5$. *If* F *is unstable and if* (4.2.1) *holds, then* F *splits.*

Proof of 4.1. Let E be normalized. Since $c_1^2(E) \geq 4c_2(E)$, E is unstable (1.7). By (2.7), $F = E(n-1)$ is generated by global sections. Now

$$c_2(F) = c_2(E) - c_1(E)(n-1) + (n-1)^2 \leq c_1(E)(n-1) + (n-1)^2,$$

so (4.2.1) holds. Hence F (and E) splits by (4.3).

Proof of 4.2. The proof of (4.2) follows the same lines as that one of (3.2), so we give only a sketch, following **[Ra]** and **[HS]**. We may assume that our section vanishes in codimension 2, so we have a sequence

$$0 \to \mathcal{O}_{Q_n} \to F \to J_X(c_1) \to 0,$$

where $c_1 = c_1(F)$ and $X = \{s = 0\}$ is a locally complete intersection of codimension 2 in Q_n and of degree $d = c_2 = c_2(F)$. For $t \in \mathbb{Z}$ let

$$e(t) = c_2 - c_1 t + t^2 = c_2(F(-t)).$$

For a fixed point $p \in Q_n$ let S_p be the set of lines $l \subset Q_n$ with $p \in l$, and let

$$\Sigma_k = \Sigma_{k,p} = \{l \in S_p \mid \text{length}(l \cap X) \geq k\}$$

be the set of k-secant lines through p contained in Q_n.

Then we have (compare **[Ra]**).

PROPOSITION 4.3.1. *Assume* $k \leq n-3$ *and* $e(0)\cdot e(1)\cdot \cdots \cdot e(k) \neq 0$. *Then* $\dim \Sigma_{k+1} \geq n-k-2$. *In particular,* $\Sigma_{k+1} \neq \varnothing$.

Proof of 4.3.1. Since $\Sigma_0 = S_p \simeq Q_{n-2}$ it suffices by induction on k to show the following. If $C \subset \Sigma_k$ is an irreducible curve with $C \cap \Sigma_{k+1} = \varnothing$ and with $\min\{\text{length}(l \cap X) | l \in C\} = k$, then $e(k) = 0$. But this is proved by easily adapting the proof of the proposition in **[Ra]** to our situation.

Arguing as in **[Ra]** we obtain

LEMMA 4.3.2. *If* $c_1(F) \geq c_2(F)/(n-3) + n - 3$ *or if* $c_2(F) \leq n-3$, *then* F *splits.*

Finally, the proof of Theorem (4.2) in **[HS]** can be copied almost word for word to give a proof of (4.2) (note that the inequality (4.2.1) is equivalent to $e(n-2) \leq n-3$).

5. The case $\mathbb{P}_n$, $n \geq 6$, and $c_1^2(E) < 4c_2(E)$.

PROPOSITION 5.1. *There is no Fano 2-bundle* E *on* $\mathbb{P}_n$, $n \geq 6$, *with* $c_1^2(E) < 4c_2(E)$.

The proof of (5.1) will be based on the following result of **[HS]** (Corollary 3.4 and Proposition 6.1).

PROPOSITION 5.2. *Let* F *be a 2-bundle on* $\mathbb{P}_n$ *admitting a section whose zero locus is smooth and of pure codimension* 2. *Assume* $c_1^2(F) < 4c_2(F)$. *Then:*
(1) $c_1(F) \geq 2n+3$ *for* $n \geq 6$,
(2) $c_1(F) \geq 3n$ *for* $n \geq 8$.

Actually only (1) is used at this place but (2) will be needed later.

Proof of 5.1. Assume E to be a normalized Fano bundle of rank 2 on $\mathbb{P}_n$, $n \geq 6$, with $c_1^2(E) < 4c_2(E)$. By (2.6) $E(n)$ is globally generated. Now take a general section of $E(n)$ which vanishes along a smooth 2-codimension subvariety (of course the zero locus is non-empty). Hence $c_1(F) \geq 2n+3$ by 5.2(i); hence $c_1(E) \geq 3$, contradicting the fact that E is normalized.

6. The case: Q_n, $n \geq 12$, and $c_1^2(E) < 4c_2(E)$.

PROPOSITION 6.1. *There are no Fano 2-bundles on Q_n, $n \geq 12$, with $c_1^2(E) < 4c_2(E)$.*

In order to prove (6.1) we must have a substitute of (5.2) which is given by

PROPOSITION 6.2. *Let F be a 2-bundle on Q_n, $n \geq 12$, admitting a section whose zero locus is smooth and of pure codimension 2. Assume $c_1^2(F) < 4c_2(F)$. Then*

$$c_2(F) > \frac{71}{4}\left(\sin\frac{\pi}{n-1}\right)^{-2}.$$

First we show how (6.1) is derived from (6.2).

Proof of 6.1. Suppose again E to be normalized and let $F = E(\frac{n}{2}+1)$ if $c_1(E) = 0$ and n even, $F = E([\frac{n}{2}] + 2)$ otherwise. In any case $c_1(F) \leq n + 3$, E being normalized. By (1.4), we have $c_1^2(F) > 3c_2(F)$; thus

$$c_2(F) \leq \frac{1}{3}c_1(F)^2 \leq \frac{1}{3}(n+3)^2,$$

F being globally generated (2.7), (6.2) applies to F. Hence (6.2.1) leads to a contradiction, since for $n \geq 12$ we have an inequality

$$\frac{1}{3}(n+3)^2 < \frac{71}{4}\left(\sin\left(\frac{\pi}{n-1}\right)\right)^{-2},$$

for $n \geq 12$.

Proof of 6.2. We mimic step by step the proof of the corresponding Theorem 2.2 of **[Sch]** on $\mathbb{P}_n$. Note that the Segre class $s_k(E)$ can be written as

$$s_k(E) = \tilde{s}_k(E)h^k$$

with $\tilde{s}_k(E) \in \mathbb{Z}$ and h the class of a hyperplane section of Q_n. According to the fact that the normal bundle of a submanifold of Q_n is always globally generated, we find as in **[Sch**, Corollary 1.2] that

$$\tilde{s}_k(E) \geq 0 \quad \text{for } k \leq n-2.$$

Now write

$$c_1(E) = \delta + \overline{\delta}, \quad c_2(E) = |\delta|^2$$

with $\delta = re^{i\varphi}$, $r \geq 0$, $-\pi \leq \varphi < \pi$.

Then $\tilde{s}_k(E) = \sum_{\nu=0}^{k} \delta^{k-\nu}\overline{\delta}^{-\nu}$.

Repeating the proof of Proposition 2.1 of **[Sch]** we get

LEMMA 6.2.1.

$$|\varphi| < \frac{\pi}{n-1}.$$

Now put $F(Q_n) = \min\{m \in \mathbb{N} | m = 4c_2(G) - c_1^2(G)$, with G a topological 2-bundle on $Q_n\}$, analogously $F(\mathbb{P}_n)$.

As in **[Sch]** we obtain from (6.2.2)

$$c_2(F) \geq \frac{1}{4} F(Q_n) \sin^2\left(\frac{\pi}{n-1}\right).$$

Since $n \geq 12$, we find a linearly embedded $\mathbb{P}_6$ in Q_n; hence

$$F(Q_n) \geq F(\mathbb{P}_6).$$

By **[Sch]**: $F(\mathbb{P}_6) \geq 71$, hence

$$F(Q_n) \geq 71,$$

finishing the proof of (6.2).

7. The case $\mathbb{P}_n$, $n = 4, 5$, and $c_1^2(E) < 4c_2(E)$. This is the last case to finish the main theorem for projective spaces.

PROPOSITION 7.1. *There are no Fano 2-bundles on $\mathbb{P}_n$, $n = 4, 5$ with $c_1^2(E) < 4c_2(E)$.*

Proof. Assume E is such a Fano 2-bundle. We may assume E to be normalized. Let $c_i = c_i(E)$ and introduce the $\mathbb{Q}$-vector bundle

$$E_0 = E\left(-\frac{c_1}{2}\right).$$

The fact that E is Fano can be expressed as $E_0(\frac{5}{2})$ (if $n = 4$) resp. $E_0(3)$ (if $n = 5$) to be ample.

As usual let ξ be the class of $\mathcal{O}_{\mathbb{P}(E)}(1)$, η the class of the pull-back of the hyperplane divisor.

(1) $n = 4$. Applying (1.4) to $E_0(\frac{5}{2})$ gives (by $c_1^2 > 2c_2$)

$$c_2(E_0) < \frac{25}{4};$$

hence $c_2 = c_2(E) \leq 6$ (and of course we also have $c_2 > 0$). Moreover

$$0 < s_4\left(E_0\left(\frac{5}{2}\right)\right) = (c_1^4 - 3c_1^2c_2 + c_2^2)(E_0).$$

Both inequalities easily imply

$$c_2(E) \leq 3.$$

By the Schwarzenberger conditions

$$c_2(c_2+1) \equiv 0\ (12) \quad (\text{if } c_1 = 0),$$
$$c_2(c_2+2) \equiv 0\ (12) \quad (\text{if } c_1 = -1),$$

we conclude that only the case $c_1(E) = 0$, $c_2(E) = 3$ is left.

By Riemann-Roch we obtain

$$\chi(\mathbb{P}_4, E(-2)) = 0.$$

The le Potier vanishing theorem applied to $E(3)$ yields

$$H^i(\mathbb{P}_4, E(-2)) = 0, \qquad i \geq 2.$$

Hence

$$H^0(\mathbb{P}_4, E(-2)) \neq 0,$$

and consequently $\zeta - 2\eta$ is an effective divisor on $\mathbb{P}(E)$. Thus

$$(\zeta - 2\eta)(2\zeta + 5\eta)^4 \geq 0.$$

On the other hand one computes easily

$$(\zeta - 2\eta)(2\zeta + 5\eta)^4 < 0,$$

a contradiction. Thus also the case $c_1 = 0$, $c_2 = 3$ is excluded.

(2) $n = 5$. This case is even simpler. The ampleness of $E_0(3)$ gives by $c_1^2 > 3c_2$ (1.4):

$$c_2(E) < 3 \quad \text{if } c_1 = 0,$$
$$c_2(E) \leq 3 \quad \text{if } c_1 = -1.$$

The claim follows again by using the Schwarzenberger conditions on $\mathbb{P}_4$.

8. The case: Q_n, $4 \leq n \leq 11$. We now treat the final case of low-dimensional quadrics in order to finish the proof of the main theorem.

PROPOSITION 8.1. *Let E be a Fano 2-bundle on Q_4. Then either E splits or is—up to a twist—a spinor bundle or one of the bundles of Example* (2.2).

Proof. As usual we assume E to be normalized and let $c_i = c_i(E)$, $E_0 = E(-\frac{c_1}{2})$, $\zeta = \mathcal{O}_{\mathbb{P}(E)}(1)$. Moreover let $c = c_2(E_0)$, $\zeta_0 = \zeta - c_1\eta/2$.

We write $c_2 = (a, b)$, $c = (a_0, b_0)$, so that $(a, b) = (a_0 + c_1^2/4, b_0 + c_1^2/4)$. We will use Riemann-Roch on Q_4:

$$\chi(E(-2)) = \frac{1}{12}(a(a+1) + b(b+1)) \quad \text{if } c_1(E) = 0;$$
$$\chi(E(-1)) = \frac{1}{12}(a(a-1) + b(b-1)) \quad \text{if } c_1(E) = -1.$$

The Fano condition says that $E_0(2)$ is ample. Hence by (1.4):

$$c_2(E_0(2)) > 0, \quad c_1^2(E_0(2)) > 2c_2(E_0(2)),$$

and consequently we obtain the bounds

$$-3 \le a \le 3, \quad -3 \le b \le 3 \quad (\text{if } c_1 = 0),$$
$$-3 \le a \le 4, \quad -3 \le b \le 4 \quad (\text{if } c_1 = -1).$$

LEMMA 8.1.1. *Suppose* $\chi(Q_4; E) > 0$ *and moreover:*
(1) *if* $c_1 = 0$: $a + 1 < 0$ *or* $b + 1 < 0$ *or* $a = b = -1$
(2) *if* $c_1 = -1$: $a + 2 < 0$ *or* $b + 2 < 0$ *or* $a = b = -2$.
Then E *splits.*

Proof. (1) Assume $c_1 = 0$. So $E(2)$ is ample. By le Potier vanishing:

$$H^i(Q_4, E(t-2)) = 0 \quad \text{for } i \ge 2, \ t \ge 0.$$

Hence

$$\chi(Q_4, E) = h^0(E) - h^1(E).$$

Since $\chi(E) > 0$, we conclude $h^0(E) \ne 0$.

By duality: $H^0(E(-2)) \simeq H^4(E(-2)) = 0$.

If now $H^0(E(-1)) \ne 0$, then by (1.6) E splits, since $c_2(E(-1)) = (a+1, b+1)$. If $H^0(E(-1)) = 0$, use (1.6) for E instead of $E(-1)$.

(2) The case $c_1 = -1$ is done in the same way starting with the ample bundle $E(3)$. We omit the details.

Since the condition $\chi(E) > 0$ is always satisfied if $a + b \le 0$ (by Riemann-Roch), the following cases are settled by Lemma 8.1.1:

$c_1 = 0$: $a < -1$, $b \le 2$ and $a \le 2$, $b < -1$ and $(a, b) = (-1, -1)$.

$c_1 = -1$: $a < -2$, $b \le 3$, and $a \le 3$, $b < -2$, and $(a, b) = (-2, -2)$.

(a) Suppose now $c_1 = 0$.

Riemann-Roch on Q_3 and Q_4 for the bundle E gives the following congruences:

$$a + b \equiv 0\ (2), \qquad -23(a+b) + a^2 + b^1 \equiv 0\ (12).$$

Hence (a, b) must be one of the following: $(-1, 3)$, $(3, -1)$, $(0, 0)$, $(2, 2)$, $(3, 3)$.

(a_1) In the first two cases we have

$$\zeta(\zeta_0 + 2\eta)^4 = -6 < 0;$$

hence ζ cannot be effective, i.e.

$$H^0(E) = 0.$$

In particular E is stable. This is a contradiction because for $(a, b) = (-1, 3)$ or $(3, -1)$ the discriminant $c_1^2 - 4c_2(E) \geq 0$.

(a_2) Let now $(a, b) = (0, 0)$. Then $\chi(E) = 2$. By le Potier vanishing we get $h^0(E) \neq 0$. On the other hand $h^0(E(-1)) = 0$, since

$$(\zeta - \eta)(\zeta_0 + 4\eta)^4 = -32 < 0.$$

Now apply (1.6) to obtain the splitting of E.

(a_3) If $(a, b) = (2, 2)$ then $\chi(E(-2)) = 1$ by Riemann-Roch; hence le Potier vanishing gives $H^0(E(-2)) \neq 0$, contradicting

$$(\zeta - 2\eta)(\zeta_0 + 2\eta)^4 = -120 < 0.$$

(a_4) If $(a, b) = (3, 3)$, we have

$$(\zeta_0 + 2\eta)^5 = a_0^2 + b_0^2 - 40(a_0 + b_0) + 160 < 0$$

(observe $(a, b) = (a_0, b_0)$), which is in contradiction to the ampleness of $E(2)$.

(b) We consider now the case $c_1 = -1$. We have a congruence

$$-13(a+b) + a^2 + b^2 \equiv 0\ (12).$$

Thus the only possible values for (a, b) are: $(4, -3)$, $(-3, 4)$, $(3, -2)$, $(-2, 3)$, $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, $(0, 4)$, $(4, 0)$, $(1, 4)$, $(4, 1)$, $(3, 3)$, $(4, 4)$. The first two cases are settled by (8.1.1) since then $\chi(E) = 2$. In the next three ones, we have $\chi(E) > 0$, hence $h^0(E)$ by le Potier vanishing; moreover $h^0(E(-1)) = 0$ since

$$(\zeta - \eta)(\zeta_0 + 2\eta)^4 < 0.$$

So E splits by (1.6).

In the case $(0,0)$, $(1,0)$ or $(0,1)$, E must be stable because $H^0(E)=0$ by

$$\zeta(\zeta_0+2\eta)^4<0.$$

So E is as in (2.1) or (2.2).

The case $(0,4)$, $(4,0)$ are ruled out as in a_1. The remaining cases finally contradict

$$(\zeta_0+2\eta)^5>0.$$

PROPOSITION 8.2. *Let E be a Fano 2-bundle on Q_n, $5\le n\le 11$, with $c_1^2(E)<4c_2(E)$. Then $n=5$ and E is—up to a twist—a Cayley bundle* (2.3).

Proof. Again suppose E to be normalized. Let $E_0=E(-c_1/2)$. E being Fano, $E_0(\frac{n}{2})$ is an ample $\mathbb{Q}$-vector bundle. By positivity of the Segre classes of $E_0(\frac{n}{2})$ we obtain

$$c_1^2\left(E_0\left(\frac{n}{2}\right)\right)>\alpha c_2\left(E_0\left(\frac{n}{2}\right)\right)$$

with

$$\alpha=\begin{cases}3 & \text{for } n=5,6,\\ 2+\sqrt{2} & n=7,8,\\ \frac{5}{2}+\frac{\sqrt{5}}{2} & n=9,10,\\ 2+\sqrt{3} & n=11\end{cases}$$

(see **[FL]**, **[FU]**).

Hence: $n^2>\alpha(c_2(E_0)+n^2/4)$.

Since $c_2(E_0)>0$, we easily obtain

$$c_2(E)\le 2,$$

with the exception $n=6$, $c_1(E)=-1$.

Let us first consider this exceptional case. Then we have only $c_2(E)\le 3$. Assume $c_2=3$. Then we compute

$$s_6\left(E\left(\frac{7}{2}\right)\right)=(2c_1^6-10c_1^4c_2+6c_1^2c_2^2-c_2^3)\left(E\left(\frac{7}{2}\right)\right)<0,$$

contradicting ampleness of $E(\frac{7}{2})$.

So we may assume $c_2(E)\le 2$ in all cases. The case $c_2=2$ is ruled out as follows: if $c_1(E)=0$ (resp. $c_1(E)=-1$) take a $Q_5\subset Q_n$ (resp. $Q_4\subset Q_n$) and Riemann-Roch gives $\chi(Q_5, E|Q_5)\notin\mathbb{Z}$ (resp. $\chi(Q_4, E|Q_4)\notin\mathbb{Z}$).

By observing $\chi(Q_4, E|Q_4)\notin\mathbb{Z}$, also the case $c_1(E)=0$, $c_2(E)=1$ is impossible. It remains to consider the case $c_1(E)=-1$, $c_2(E)=1$.

If E is unstable, apply (4.2) to the bundle $F = E(n-1)$ which is globally generated by (2.7) (the condition (4.2.1) is immediately verified). So E splits.

If E is stable, the restriction $E|Q_5$ to a generic linear $Q_5 \subset Q_n$ is stable again with $c_1 = -1$, $c_2 = 1$. Hence by (2.3), $E|Q_5$ is a Cayley bundle.

Since no Cayley extends to Q_6 (Ottaviani, see 2.3), we must have $n = 5$. The proof of (8.2) is now complete.

Combining all results of §§3–8 gives a proof of the Main Theorem.

9. Generalizations. The Main Theorem for projective spaces can be improved considerably (we will not consider the case of quadrics here):

THEOREM 9.1. *Let F be an ample 2-bundle on $\mathbb{P}_n$. Then F splits under one of the following assumptions.*

(1) $n = 4$, $c_1(F) \leq 6$,
(2) $n = 5$, $c_1(F) \leq 8$,
(3) $n = 6$ *or* 7, $c_1(F) \leq \frac{4n+2}{3}$,
(4) $n \geq 8$, $c_1(F) \leq \frac{5n-1}{3}$.

REMARK. (9.1) can be reformulated as follows. Assume that F is a $\mathbb{Q}$-vector bundle with $c_1(F) = 0$. Then e.g. (1) says that in case $n = 4$, F splits if $F(3)$ is ample. We should also mention the Horrocks-Mumford bundle H in this context. It has $c_1(H) = -1$ and $c_2(H) = 4$; moreover $H(4)$ is generated by global sections. So the statement (1) or (9.1) is almost sharp, see also (9.2) below.

Part (1) of Theorem 9.1 will follow from the more general statement:

PROPOSITION 9.2. *Let E be a 2-bundle on $\mathbb{P}_4$. Let $E_0 = E \otimes (\det E^*/2)$. If $E_0(3)$ is nef, then E splits.*

For the proof of (9.2) we will need

LEMMA 9.3. *Let E be a normalized 2-bundle on $\mathbb{P}_n$ such that $E(m)$ is nef for some $m \in \mathbb{Q}$. Let $r \in \mathbb{Z}$ be the maximal number such that*

$$H^0(\mathbb{P}_n, E(-r)) \neq 0.$$

Then either E splits or

(a) $r \leq m - 2$ (*if* $c_1(E) = 0$) *or*
(b) $r \leq m - 3$ (*if* $c_1(E) = -1$).

Proof. We treat only the case $c_1(E) = 0$, the other case being similar. Let $s \in H^0(E(-r))$, $s \neq 0$, and let $Z = \{s = 0\}$. If $Z = \varnothing$, E splits. So assume $Z \neq \varnothing$. By our assumptions, Z is locally a complete intersection of codimension 2. If $\deg Z = 1$, Z is a complete intersection and E splits. So let $\deg Z \geq 2$. Then take a 2-secant line L of Z with $L \neq Z$.

Then $E(-r)|L$ has a section with at least two zeros; hence

$$E(-r)|L = \mathcal{O}_L(2+k) \oplus \mathcal{O}_L(-2r-2-k)$$

for some $k \geq 0$. Hence

$$E(m)|L = \mathcal{O}_L(2+k+r+m) \oplus \mathcal{O}_L(m-r-2-k)$$

and by nefness of $E(m)|L$ we conclude.

Proof of 9.2. We may assume E to be normalized.

(a) First let $c_2(E) \leq 0$. So E is unstable. Let r be the biggest positive integer such that

$$H^0(\mathbb{P}_4, E(-r)) \neq 0.$$

Assume that E does not split. Then we deduce from (9.3): $r \leq l$ in case $c_1(E) = 0$; $r \leq \frac{1}{2}$ if $c_1(E) = -1$. In the second case $r \leq 0$; we must have $r = 1$, and thus $E(-1)$ has a section whose zero locus Z is either empty or of codimension 2 with $\deg Z = c_2(E) + 1 \leq 1$. But then clearly E splits.

(b) Now we consider the case $c_2(E) > 0$. Let $c = c_2(E_0)$. By nefness of $E_0(3)$ we obtain

$$0 \leq c_1(E_0(3))^2 - 2c_2(E_0(3)) = 36 - 2(c+9);$$

hence $c \leq 9$. On the other hand, the highest Segre class $s_4(E_0(3)) \geq 0$; hence $c^2 - 90c + 405 \geq 0$, which together with $c \geq 9$, proves $c \leq 45 - \sqrt{1620} < 5$.

Hence $c_2(E) \leq 4$ if $c_1(E) = 0$ and $c_2(E) \leq 5$ if $c_1(E) = -1$. By the Schwarzenberger conditions we find:

$$c_1(E) = 0, \quad c_2(E) = 3 \quad \text{or} \quad c_1(E) = -1, \quad c_2(E) = 4.$$

In both cases a short computation shows

$$\zeta(\zeta_0 + 3\eta)^4 < 0;$$

hence $H^0(E) = 0$ and E is thus stable.

By **[BE]** there is no stable 2-bundle on $\mathbb{P}_4$ with $c_1 = 0$, $c_2 = 3$; by **[DS]** the only stable 2-bundle on $\mathbb{P}_4$ with $c_1 = -1$, $c_2 = 4$ is the Horrocks-Mumford bundle for which it is easy to see that $E_0(3)$ is not nef (restrict to jumping lines). This completes the proof of (9.2).

REMARK. (1) In the proof of (9.2) one shows also the following stronger statement: let E be an unstable 2-bundle on $\mathbb{P}_4$, assume $E_0(3)$ to be nef on every line $L \subset \mathbb{P}_4$. Then E splits.

(2) It would be interesting to do the next step in (9.2): assume only $E_0(4)$ to be nef. This leads to some interesting problems. Let e.g. E be a (semi-stable) 2-bundle on $\mathbb{P}_4$ with $c_1 = -1$, $c_2 = 6$ and assume $E_0(4)$ even to be generated by global sections. Take a general section with smooth zero locus X. Then

$$K_X = \mathcal{O}_X(2),$$

i.e. X is a "half-canonical" surface in $\mathbb{P}_4$ with $\deg X = 18$. Half-canonical surfaces are investigated in **[DPPS]** and it is shown that they cannot exist (or are complete intersections) with possible exceptions in degree 18 and 22 (and some other restrictions). "Of course" one expects that half-canonical surfaces are complete intersections in these degrees, too.

Part (2) of (9.1) will be a consequence of

PROPOSITION 9.4. *Let E be a 2-bundle on $\mathbb{P}_5$ such that $E_0(4)$ is nef. Then E splits.*

Proof. As usual we suppose E normalized.

(a) Assume $c_2(E) \leq 0$; so E is not stable. Let r be the maximal positive integer such that

$$H^0(E(-r)) \neq 0.$$

By (9.3): $r \leq 2$ if $c_1(E) = 0$; $r \leq \frac{3}{2}$ if $c_1(E) = -1$. If $c_1(E) = -1$ we have $r = 1$, so by $c_2(E(-1)) = c_2(E) \leq 0$, E splits (1.6). This argument settles also $c_1(E) = 0$ and $r = 1$. Finally let $c_1(E) = 0$ and $r = 2$. Then (1.6) settles the case $c_2(E) \leq -3$. Take a linear $\mathbb{P}_4 \subset \mathbb{P}_5$ and use the Schwarzenberger condition for $E|\mathbb{P}_4$ to obtain $c_2(E) = 0$ or -1. But in both cases:

$$(\zeta - 2\eta)(\zeta + 4\eta)^5 < 0,$$

contradicting the nefness of $\zeta + 4\eta$.

(b) Assume now $c_2(E) > 0$. From $c_1^2(E_0(4)) > 3c_2(E_0(4))$ we deduce $c_2(E) \leq 5$. Now the Schwarzenberger condition for $E|\mathbb{P}_4$ implies:

$$c_1(E) = 0, \quad c_2(E) = 3, \quad \text{or} \quad c_1(E) = -1, \quad c_2(E) = 4.$$

In both cases:

$$\zeta(\zeta_0 + 4\eta)^5 < 0;$$

hence $H^0(E) = 0$, and consequently E is stable.

Now for a general linear $\mathbb{P}_4 \subset \mathbb{P}_5$, $E|\mathbb{P}_4$ is again stable with the same Chern classes, so $c_1(E) = 0$, $c_2(E) = 3$ is ruled out by **[BE]** and the other case by **[DS]**, since the Horrocks-Mumford bundle does not extend to $\mathbb{P}_5$.

Proof of 9.1, *parts* (3) *and* (4). Let $c_i = c_i(F)$. If $c_1 \leq n+1$, F is a Fano bundle and hence F splits by the Main Theorem. So assume now $c_1 \geq n+2$.

First let us show that

(1) $F(c_1 - n)$ is generated by global sections. In fact, $H^i(\mathbb{P}_n F(t)) = 0$ for $i \geq 2$, $t \geq -n-1$ by le Potier vanishing; moreover by Kodaira vanishing for the divisor ζ_F:

$$0 = H^1(\mathbb{P}(F), 3\zeta_F + K_{\mathbb{P}(F)}) \simeq H^1(\mathbb{P}_n, F(c_1 - n - 1)).$$

So (1) follows from the Castelnuovo-Mumford lemma. As a consequence we obtain

(2) $c_1^2(F) \geq 4c_2(F)$.

In fact, if $c_1^2(F) < 4c_2(F)$ we can apply—using (1)—Proposition 5.2 for $F(c_1 - n)$ conflicting our assumptions.

We suppose c_1 to be even, the odd case being treated similarly. Let $E = F(-c_1/2)$. Let r be the maximal integer such that

$$H^0(E(-r)) \neq 0.$$

Since E is unstable by (2), r must be positive.

Since $H^n(\mathbb{P}_n, F(-n-1)) = 0$ we have by duality $H^0(\mathbb{P}_n, E(-c_1/2)) = 0$; hence $r < c_1/2$.

Since $c_2(E) \leq 0$, we have moreover $c_2(E(-r)) \leq r^2 \leq c_1^2/4$, so our assumption yields

$$c_2(E(-r)) < (n-1)(n+5).$$

Since any section of $E(-r)$ vanishes nowhere or in codimension 2, E splits by [HS, 4.7].

10. Numerical splitting of rank 2-bundles on $\mathbb{P}_n$. In the previous sections we considered Fano bundles E on $\mathbb{P}_n$, i.e. $E_0(\frac{n+1}{2})$ is ample, E_0 denoting the normalization $E(c_1(E^*)/2)$. Here we want only to make an assumption on the behaviour of E on the lines and try to get some information.

Let E always denote a vector bundle of rank 2 on $\mathbb{P}_n$.

10.1. DEFINITION. (1) For a line $L \subset \mathbb{P}_n$ put

$$\delta_L(E) = \delta_L = a_2 - a_1,$$

if $E|L = \mathscr{O}(a_1) \oplus \mathscr{O}(a_2)$ with $a_1 \leq a_2$.

(2) For $x \in \mathbb{P}_n$ define

$$\delta_x^{\max} = \max\{\delta_L | L \text{ a line through } x\} \text{ and}$$
$$\delta_x^{\min} = \min\{\delta_L | L \text{ a line through } x\}.$$

10.2. DEFINITION. For $x \in \mathbb{P}_n$ let $\mathbb{P}_x$ be the variety of lines through x. Write $\delta_x^{\min} = \delta_0 < \delta_1 < \cdots < \delta_r = \delta_x^{\max}$, where δ_i are the "splitting types" realized by E on some line passing through x.

Define $V_{\delta_i} = \{L \in \mathbb{P}_x | \delta_L = \delta_i\}$.

10.3. REMARK. We have clearly:

(a) $\overline{V}_{\delta_0} = \mathbb{P}_x$,

(b) $\overline{V}_{\delta_i} = V_{\delta_i} \cup \bigcup_{j>i}(\overline{V}_{\delta_i} \cap V_{\delta_j})$.

10.4. DEFINITION. If $\delta_L > 0$, the ruled surface $\mathbb{P}(E|L)$ has a unique exceptional section C_L (i.e. $C_L^2 < 0$). We define a map (for fixed $x \in \mathbb{P}_n$)

$$\Phi_{\delta_i}: V_{\delta_i} \to \mathbb{P}(E_x) \simeq \mathbb{P}_1 \qquad (i > 0)$$

by setting

$$\Phi_{\delta_i}(L) = C_L \cap \mathbb{P}(E_x).$$

It is easy to check that Φ_{δ_i} is holomorphic.

The key to this section is

10.5. THEOREM. *Assume that for some δ_i the map ϕ_{δ_i} has a fiber containing a compact curve. Then E splits numerically:*

$$c_1(E) = a + b, \quad c_2(E) = ab, \quad \textit{where } E|L_{\delta_i} = \mathscr{O}(a) \oplus \mathscr{O}(b).$$

In other words E has the same Chern classes as a decomposable bundle.

REMARK. The assumption means that there is a "compact" family $(L_t)_{t\in T}$ of lines through x with T compact, such that $C_{L_t} \cap \mathbb{P}(E_x)$ does not depend on t.

Proof. After normalizing T we obtain a compact curve C and a geometrically ruled surface $p\colon S \to C$ with a map $\psi\colon S \to \mathbb{P}(E)$ such that $\psi(p^{-1}(c)) = C_{L_t}$ where c is a point in C over t. By our assumption the ruled surface contains a section, say C_0, such that $\psi(C_0) = \phi_{\delta_t}(t)$ for any $t \in T$.

Now consider the relative Euler sequence

$$0 \to \omega_{\mathbb{P}(E)/\mathbb{P}_n}(1) \to \pi^*(E) \to \mathcal{O}_{\mathbb{P}(E)}(1) \to 0$$

where $\mathbb{P}(E)$ is the projective bundle taking hyperplanes and $\omega_{\mathbb{P}(E)/\mathbb{P}_n}$ is the relative dualizing sheaf.

Since $\omega_{\mathbb{P}(E)/\mathbb{P}_n}(1) \simeq \mathcal{O}_{\mathbb{P}(E)}(-1) \otimes \pi^*(\mathcal{O}(a_1 + a_2))$ we obtain by tensoring with $\pi^*(\mathcal{O}(-a_1))$:

$$\begin{aligned} 0 &\to \mathcal{O}_{\mathbb{P}(E)}(-1) \otimes \pi^*(\mathcal{O}(a_2)) \to (\pi^* E)(-a_1) \\ &\to \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*(\mathcal{O}(-a_1)) \to 0. \end{aligned}$$

Now we have

$$(*) \qquad \psi^*((\mathcal{O}_{\mathbb{P}(E)}(1)) \otimes \pi^*(\mathcal{O}(-a_1))) \simeq \mathcal{O}_s:$$

this has only to be checked on C_0 (obvious!) and on a fiber $p^{-1}(c)$. But for this it is sufficient to see

$$\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*(\mathcal{O}(-a_1))|C_L \simeq \mathcal{O}$$

which is clear since

$$\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*(\mathcal{O}(-a_1))|C_L = \mathcal{O}_{\mathbb{P}(\mathcal{O}_1(a_2-a_1)\otimes\mathcal{O}_L)}(1)|C_L$$

and since C_L is the exceptional section (see [**Ha**, Chap. V.2]).

Since $\pi \circ \psi$ is generically finite, $(*)$ implies

$$c_2(E(-a_1)) = c_2(\psi^*(\pi^*(E(-a_1)))) = 0.$$

Hence $c_2(E) = a_1 a_2$.

An obvious consequence of 6.5 is

10.6. COROLLARY. *Assume that* E *does not split numerically. Then*
(1) $\dim V_{\delta_x^{\max}} \leq 1$ *and*
(2) $\dim \overline{V}_{\delta_i} - \dim(\overline{V}_{\delta_i} \cap \bigcup_{j>i} V_{\delta_j}) \leq 2$, *for* $\delta_i < \delta_x^{\max}$.

10.7. THEOREM. *Assume that there is some* $x \in \mathbb{P}_n$ *such that*

$$\delta_x^{\max} - \delta_x^{\min} < n - 2.$$

Then E *splits numerically.*

Proof. First observe that in general

$$\#\{V_{\delta_i}\} \leq \frac{1}{2}(\delta_x^{\max} - \delta_x^{\min}) + 1 < \frac{n}{2}$$

by our assumption.

On the other hand (10.6) implies: $2 \cdot \#\{V_{\delta_i}\} \geq n$, if E does not split numerically. Both inequalities being incompatible, E has to split numerically.

For $n = 3$ Theorem 10.7 says that every uniform (w.r.t. lines through x) 2-bundle E numerically splits. Of course it is well known that E in fact splits. But already for $n = 5$, the assumption of 10.7 is less restrictive than uniformity.

Another immediate consequence of 10.5 is

10.8. COROLLARY. *If there is some* $x \in \mathbb{P}_n$ *and some* i *such that* V_{δ_i} *contains a compact surface, then* E *splits numerically.*

10.9. COROLLARY. *Assume that* E *is a semi-stable 2-bundle on* $\mathbb{P}_n$, $n \geq 4$, *with* $c_1(E) = 0$. *Assume that there is some* $x \in \mathbb{P}_n$ *and some* $a > 0$ *such that for all* $L \in \mathbb{P}_x$ *either* $E|L = \mathcal{O} \oplus \mathcal{O}$ *or* $E|L = \mathcal{O}(a) \oplus \mathcal{O}(-a)$.
Then E *splits numerically.*

Proof. Of course we may assume that $E|L = \mathcal{O}(a) \oplus \mathcal{O}(-a)$ for some line. Since $c_1(E) = 0$, the jumping lines of E form a divisor D in $G(1, n)$ (=lines in $\mathbb{P}_n$, see e.g. **[OSS]**). Hence $D \cap \mathbb{P}_x$—which is the set of jumping lines through x—is a divisor in $\mathbb{P}_x$. Therefore we obtain a compact surface in $V_{\delta_1} = \overline{V}_{\delta_1} = \overline{V}_{2a}$, since $n \geq 4$.

In order to prove splitting criteria rather than merely criteria for numerically splitting we prove

10.10. PROPOSITION. *Let E be a 2-bundle on $\mathbb{P}_n$. Choose $x \in \mathbb{P}_n$ generic such that $\delta_x^{\max}$ is minimal. If* $\dim V_{\delta_x^{\max}} > \frac{n}{2}$, *then there is* $s \in H^0(E((-c_1 + \delta_x^{\max})/2))$ *with* $s(x) \neq 0$.

Proof. (a) We claim that $\phi := \phi_{\delta_x^{\max}}: V := V_{\delta_x^{\max}} \to \mathbb{P}(E_x)$ is constant. In fact, otherwise by our assumption ϕ would have fibers of dimension $> \frac{n}{2} - 1$. On the other hand we have $V \subset \mathbb{P}_x \simeq \mathbb{P}_{n-1}$; so algebraic sets in V of dimension $> \frac{n}{2} - 1$ must meet, contradiction.

(b) Let $D = \bigcup\{C_L | X \in \mathbb{P}_n,\ L \in \mathbb{P}_x \text{ and } \delta_L = \delta_x^{\max}\} \subset \mathbb{P}(E)$. If x is general, then $D \cap \mathbb{P}(E_x)$ consists of one point by (a). Hence there is an irreducible component $D_0 \subset D$ and some $d \in \mathbb{Z}$ such that

$$D_0 \in |\mathscr{O}_{\mathbb{P}(E)}(1) \otimes \pi^*(\mathscr{O}(d))|.$$

Taking a line L through our general x and observing

$$D_0 \cap \pi^{-1}(L) = C_L,$$

we conclude

$$d = \frac{-c(E) + \delta_x^{\max}}{2}.$$

10.11. THEOREM. *Let E be a 2-bundle on $\mathbb{P}_n$. Assume $\delta_L < \frac{n}{2} - 1$ for every line $L \subset \mathbb{P}_n$. Then E splits.*

Proof. This follows from (the proof of) Proposition 10.10 since our condition implies

$$\dim V_{\delta_x^{\max}} > \frac{n}{2}$$

for every x; hence the section constructed in the proof of 10.10 does not vanish at any point and consequently E splits.

Again 10.11 can be viewed as a generalization of the fact that uniform 2-bundles on $\mathbb{P}_n$, $n \geq 3$, split.

10.12. REMARK. Most of the above can be applied to manifolds containing "plenty of lines" if we only can control their cohomology. For example, if X is a Fano manifold of index $r > \frac{1}{2}\dim X + 1$ (recall that the index r is the largest integer dividing $-K_X$ in $\mathrm{Pic}(X)$ and that for $r > \frac{1}{2}\dim X + 1$, we have $\mathrm{Pic}(X) = \mathbb{Z}$ by **[Wi]**) then through every point of X there passes a line (i.e. a rational curve whose intersection with the ample generator H of $\mathrm{Pic}(X)$ is 1). For a 2-bundle E on X we can define (via normalization) the splitting type of E on any such line. Similarly we can define $\delta_x^{\max}$, $\delta_x^{\min}$. Then we obtain an equivalent of 10.7.

10.13. THEOREM. *Let X and E be as above, let moreover $b_4(X) = 1$. If $\delta_x^{\max} - \delta_x^{\min} < r - 3$ for some $x \in X$, then E splits numerically, i.e. $c_1(E) = (a + b)c_1(H)$, $c_2(E) = (ab)c_1(H)^2$ for some $a, b \in \mathbb{Z}$.*

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"U. Dini"
Univ. di Firenze
Viale Moragagni 67A
50139 Firenze, Italy

Univ. Bayreuth
95440 Bayreuth, Germany

and

Uniwersytet Warszawski
ul. Banacha 2
02-097 Warszawski, Poland