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ON REAL EIGENVALUES OF COMPLEX MATRICES

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This paper contains many inter-related results dealing with the general question of determination of real eigenvalues of complex matrices. We first discuss the relationship between the number of elementary divisors associated with real eigenvalues of a matrix A and the signature of a Hermitian matrix H when AH is also Hermitian. We then obtain sets of equivalent conditions for a matrix to be similar to a real matrix; for a matrix to be symmetrizable; and for a matrix to be similar to a real diagonal matrix. As corollaries we obtain results on the eigenvalues and elementary divisors of products of two Hermitian matrices. Some of the results are not new; these are included to give a more complete survey of what is known in these particular areas.

Recently a theorem on the stability of complex matrices, due to Lyapunov, has been generalized by Taussky [15, 16], and independently, by Ostrowski and Schneider [12]. Their result may be stated as follows: Given a complex matrix A, there exists a Hermitian H for which $AH + HA^* > 0$ (positive definite) if and only if A has no imaginary eigenvalues. Further, if $AH + HA^* > 0$, the numbers of eigenvalues of A with positive and negative real parts equal respectively the numbers of eigenvalues of H which are positive and negative.

Further generalizations of these results have been obtained by Schneider and this author [4, 6], under the condition that $AH + HA^* \geqq 0$ (positive semi-definite). This paper will use these results and methods to prove the theorems mentioned in the synopsis above.

I wish to acknowledge with thanks the contribution of Professor Emilie Haynsworth, who pointed out to me the connection between [7] and my results, and thus sparked this investigation. I also wish to thank the referee for many helpful comments, and for references to several related papers, especially [13], [14], and [17], with which I had not been familiar.

2. **Definitions.** We define the inertia of a complex matrix A to be $InA = (\pi(A), \nu(A), \delta(A))$, where $\pi(A)$, $\nu(A)$, $\delta(A)$ are respectively the number of eigenvalues of A with positive, negative, and zero real parts. We shall always let G, H and K represent Hermitian matrices; we denote the signature of H by $\sigma(H) = \pi(H) - \nu(H)$. We shall define, as in [12], $R(AH) = \frac{1}{2}(AH + HA^*)$.

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We shall use several simple propositions throughout the paper. The first two assume a matrix of the form (we let $\cdot$ denote a zero matrix)

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \cdot & A_{22} \end{bmatrix} \left(\text{or } A = \begin{bmatrix} A_{11} & \cdot \\ A_{21} & A_{22} \end{bmatrix}\right).$$

We let λ be an eigenvalue of A, and let

$$a_1 \geqq a_2 \geqq \cdots \geqq a_r,\ b_1 \geqq b_2 \geqq \cdots \geqq b_s,\ c_1 \geqq c_2 \cdots \geqq c_t$$

be the degrees of the elementary divisors associated with λ in respectively A, A_{11}, and A_{22} (let $a_i = 0$ for $i > r$, etc.).

PROPOSITION 1. *$a_i \leqq b_1 + c_1$ for all i.*

PROPOSITION 2. *$b_i \leqq a_i$ for all i, and $s \leqq r$; if $c_1 = c_2 = \cdots = 0$, then $b_i = a_i$ for all i.*

Proofs. Proposition 2 follows immediately from the more precise inequality

$$a_{t+1} \leqq b_i \leqq a_i \text{ for all } i$$

proved in Appendix A of [5].

To prove Proposition 1, we first assume $\lambda = 0$ (if $\lambda \neq 0$, we consider $A - \lambda I$ in place of A). Let $\{x_j, j = 1, \cdots, a_i\}$ be a lower Jordan chain (see [8], p. 201) associated with $\lambda = 0$ and define $x_0 = 0$; we have $Ax_j = x_{j-1}$, $j = 1, \cdots, a_i$. Let

$$x_j = \begin{bmatrix} y_j \\ z_j \end{bmatrix};\ \text{then } Ax_j = \begin{bmatrix} A_{11}y_j + A_{12}z_j \\ A_{22}z_j \end{bmatrix} = \begin{bmatrix} y_{j-1} \\ z_{j-1} \end{bmatrix}.$$

Suppose k is the minimal j for which $z_j \neq 0$; then it follows from the above calculation that $A_{11}y_j = y_{j-1}$, $j = 1, \cdots, k-1$, and $A_{22}z_j = z_{j-1}$, $j = k+1, \cdots, a_i$ and $A_{22}z_k = 0$. Thus $\{y_j, j = 1, \cdots, k-1\}$ and $\{z_j, j = k, \cdots, a_i\}$ form parts of Jordan chains for A_{11} and A_{22} respectively. As b_1 and c_1 are respectively the maximal lengths of such chains, $k - 1 \leqq b_1$ and $a_i - k + 1 \leqq c_1$, so that $a_i \leqq b_1 + c_1$.

Our third proposition is quite different. If $AH = K$, and S is nonsingular, let $B = SAS^{-1}$, $H_0 = SHS^*$, and $K_0 = SKS^*$. Then $BH_0 = K_0$ and B is similar to A; $InH_0 = InH$ and $InK_0 = InK$ (by Sylvester's Law of Inertia). Thus we have

PROPOSITION 3. *If $AH = K$, we may replace A by a matrix similar to it and either H or K by a matrix complex-congruent to it, leaving invariant the eigenvalues and elementary divisors of A*

and the inertias of H and K. In particular, we could assume A to be in some variant of Jordan canonical form or $H = H_{11} \oplus 0$, where H_{11} is nonsingular. We shall also always assume that A, H, and K are partitioned conformably into submatrices.

It will be convenient to make the following definitions. Let $\gamma(A)$ be the number of a_i which are associated with real eigenvalues λ. Let $\gamma_0(A)$ be the number of a_i which are odd and associated with real λ. Let $\gamma_1(A)$ be the number of a_i which are either odd and associated with real nonzero λ, or even and associated with $\lambda = 0$. Let $a(A)$ be the sum of all a_i associated with real λ.

One last comment: throughout the paper we shall discuss matrices of the form AH or H_1H_2, where (say) H_2 has some special property. All our results will remain true if we replace AH by HA, or assume H_1 has the desired special property instead of H_2.

3. Elementary divisors associated with real eigenvalues. Drazin and Haynsworth in [7] proved that a necessary and sufficient condition that A have (at least) m elementary divisors associated with real eigenvalues is that there exists an $H \geqq 0$, of rank m, for which AH is Hermitian. Our first theorem generalizes the conditions on H.

THEOREM 1. *Let A be a complex matrix. A necessary and sufficient condition that $\gamma(A) \geqq m$ is that there exists a Hermitian H for which*

$$|\sigma(H)| = m \tag{1}$$

and AH is Hermitian.

If AH is Hermitian and H is nonsingular, then

$$|\sigma(H)| \leqq \gamma_0(A) , \tag{2}$$

$$|\sigma(AH)| \leqq \gamma_1(A) . \tag{3}$$

Proof. The necessity of the first assertion is contained in the Drazin-Haynsworth theorem (If $H \geqq 0$, of rank m, then $|\sigma(H)| = m$). To prove sufficiency, we assume that AH is Hermitian (i.e., $AH - HA^* = 0$) for some H satisfying (1). Then if $A = iB$, $\gamma(A)$ is also the number of elementary divisors associated with imaginary eigenvalues of B. We have $R(BH) = \frac{1}{2}(BH + HB^*) = 0$. We may apply Theorem 3 of [4], which gives a set of bounds on the inertia of H when $R(BH) \geqq 0$, of rank r (obviously here $r = 0$). For $r = 0$, bound (14) of [4] becomes $|\sigma(H)| \leqq \gamma(A)$. As $|\sigma(H)| = m$ was assumed, the sufficiency is proved.

If H is nonsingular, then (2) is merely a restatement of the second display of § 9 of [4]. Also, (3) is a restatement of a theorem by

Loewy [10, p. 69] as given by Bromwich [3, p. 349]. This completes the proof.

4. Complex matrices similar to real matrices.

The next result contains a conjugate-transpose analogue for complex matrices of a theorem proved by Taussky and Zassenhaus [18]: That every matrix with elements in a field F may be taken into its transpose by similarity transformation by a matrix symmetric in F.

THEOREM 2. *Let A be a complex matrix. The following four conditions are equivalent*:

(i) *A is similar to a real matrix.*

(ii) *There exists a Hermitian H for which $H^{-1}AH = A^*$.*

(iii) *There exists a nonsingular Hermitian H for which AH is Hermitian.*

(iv) *There exist Hermitian matrices G and H, with H nonsingular, for which $A = GH$.*

Proof. Suppose there exists a matrix T so that $B = T^{-1}AT$, where B is real. By the Taussky-Zassenhaus theorem, there exists a real symmetric S so that $S^{-1}BS = B' = B^*$. Calculation shows that $(TST^*)^{-1} A(TST^*) = A^*$; clearly TST^* is Hermitian. Conversely, if $H^{-1}AH = A^*$, A is similar to A^*, and conjugate eigenvalues of A must have elementary divisors with identical degrees. Thus A must be similar to a real matrix.

The equivalence of (ii), (iii), and (iv) is obvious.

5. Products of Hermitian matrices.

As corollaries of the Drazin-Haynsworth theorem and our previous theorems, we obtain results on the eigenvalues and elementary divisors of products of two Hermitian matrices. Some are not new; some are easily proved independently. They are all presented, however, as, taken together they give a fairly complete description of the eigenvalues of a product of two Hermitian matrices.

Corollary 1 extends a result credited by MacDuffee [11, p. 65] to Klein [9].

COROLLARY 1. *If H_1 and H_2 are Hermitian, then*

$$\gamma(H_1H_2) \geqq |\sigma(H_2)| , \tag{4}$$

$$\alpha(H_1H_2) \geqq |\sigma(H_2)| + \delta(H_2) . \tag{5}$$

If H_2 is nonsingular, then

$$\gamma_0(H_1H_2) \geqq |\sigma(H_2)|, \ \gamma_1(H_1H_2) \geqq |\sigma(H_1)| . \tag{6}$$

Proof. We first suppose H_2 is nonsingular. Then for $A = H_1H_2$, and $H = H_2^{-1}$, $AH = H_1$ is Hermitian and (4) and (6) follow from Theorem 1 (as $InH_2 = InH_2^{-1}$). This completes the proof of (6).

If H_2 is singular, we may assume $H_2 = K_{11} \oplus 0$, where K_{11} is nonsingular (there exists a unitary U so that $U^*H_2U = K_{11} \oplus 0$; we write H_1 for UH_1U^*). Then

$$H_1H_2 = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} K_{11} & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} H_{11}K_{11} & \cdot \\ H_{21}K_{11} & \cdot \end{bmatrix}. \tag{7}$$

As H_{11} and K_{11} are Hermitian and K_{11} is nonsingular, we may apply (4) to H_{11} and K_{11} to obtain

$$\gamma(H_{11}K_{11}) \geqq |\sigma(K_{11})| . \tag{8}$$

As $\pi(K_{11}) = \pi(H_2)$ and $\nu(K_{11}) = \nu(H_2)$, we have

$$|\sigma(K_{11})| = |\sigma(H_2)| . \tag{9}$$

By Proposition 2 applied to all real eigenvalues,

$$\gamma(H_1H_2) \geqq \gamma(H_{11}K_{11}) . \tag{10}$$

Combining (8), (9), and (10) we have proved (4) for all H_1 and H_2.

It is clear from our definitions that

$$\alpha(A) \geqq \gamma(A) \tag{11}$$

for any A; hence (5) follows from (4) when H_2 is nonsingular (i.e. $\delta(H_2) = 0$). When H_2 is singular we again assume, as in (7), $H_1H_2 = H_{11}(K_{11} \oplus 0)$, where the zero matrix has order $\delta(H_2)$; clearly

$$\alpha(H_1H_2) = \alpha(H_{11}K_{11}) + \delta(H_2) . \tag{12}$$

From (8), (9), and (11) (with $A = H_{11}K_{11}$), we have

$$\alpha(H_{11}K_{11}) \geqq \gamma(H_{11}K_{11}) \geqq |\sigma(K_{11})| = |\sigma(H_2)| ,$$

and substituting this in (12) we obtain (5). We have proved Corollary 1.

COROLLARY 2. *If H_1 and H_2 are Hermitian, then H_1H_2 is similar to a real matrix.*

Proof. If H_2 is nonsingular, this is part of Theorem 2. If H_2 is singular, as in Corollary 1 we may assume $H_2 = K_{11} \oplus 0$, with K_{11} nonsingular. Now H_1H_2 is given by (7).

We shall use Proposition 2 for $A = H_1H_2$ and $A_{11} = H_{11}K_{11}$. To avoid confusion, we attach a superscript (λ) to each a_i and b_i associated with the eigenvalue λ. As $H_{11}K_{11}$ is similar to a real matrix by

Theorem 2, and $A_{22} = 0$, we have for all nonreal λ,

$$a_i^{(\lambda)} = b_i^{(\lambda)} = b_i^{(\overline{\lambda})} = a_i^{(\overline{\lambda})} .$$

This implies that H_1H_2 is similar to a real matrix.

REMARK 1. Corollary 2 implies that H_1H_2 is similar to H_1H_2 (we have that H_1H_2 is similar to $\overline{H_1H_2}$ and hence to $(H_1H_2)^* = H_2H_1$) for all Hermitian H_1 and H_2, a property not enjoyed by all pairs of matrices: for example take

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

REMARK 2. We note that when H_2 is nonsingular the above result is trivial ($(H_1H_2)^* = H_2H_1 = H_2(H_1H_2)H_2^{-1}$). However, the singular case cannot be handled in the obvious way by continuity arguments on $H_2 + \varepsilon I$, as the elementary divisors structure is not a continuous function of the elements of the matrix. The same comment applies to the two corollaries below.

COROLLARY 3. *If H_1 and H_2 are Hermitian and $H_2 > 0$, then H_1H_2 is diagonalizable, with all real eigenvalues, and In $H_1H_2 =$ In H_1.*

Proof. Let $A = H_1H_2$ and $H = H_2^{-1} > 0$; then $AH = H_1$. By the Drazin-Haynsworth theorem, for $m =$ order A, we have that H_1H_2 is diagonalizable, with all real eigenvalues. By Corollary 3 of [12], In $H_1H_2 =$ *In* H_2.

REMARK 3. Corollary 3 is well known; cf. [19, p. 108, problem 6].

Corollary 4 (below) has connections with previous work on symmetrizable operators; we shall discuss this further in § 6.

COROLLARY 4. *If H_1 is Hermitian and $H_2 \geqq 0$, then H_1H_2 has all real eigenvalues; nonzero eigenvalues have linear elementary divisors, zero eigenvalues have elementary divisors of degree less than or equal two. We have*

$$\pi(H_1H_2) \leqq \pi(H_1),\ \nu(H_1H_2) \leqq \nu(H_1) . \tag{13}$$

Proof. Let $H_2 = K_{11} \oplus 0$, where $K_{11} > 0$. Again H_1H_2 is given by (7). By Corollary 3, $H_{11}K_{11}$ is diagonalizable, with all real eigenvalues. As $(H_1H_2)_{22} = 0$, we have by Proposition 2

$$a_i^{(\lambda)} = b_i^{(\lambda)} = 1$$

for all $\lambda \neq 0$ and all nonzero $a_i^{(\lambda)}$. We also have by Proposition 1

$$a_1^{(0)} \leqq b_1^{(0)} + c_1^{(0)} \leqq 2$$

for all nonzero $a_i^{(0)}$.

All that remains to be proved is (13); but this follows from Corollary 4 of [12].

6. Symmetrizable matrices. In a recent paper [14] Silberstein discusses symmetrizable operators in unitary spaces. (Other results on symmetrizable operators may be found in Reid [13] and Zaanen [20]). We specialize his definition to our finite dimensional setting:

DEFINITION. The complex matrix A is symmetrizable if there exists a Hermitian H, $H \geqq 0$, for which

(i) $Hx = 0$ implies $Ax = 0$.

(ii) HA is Hermitian.

THEOREM 3. *Let A be a complex matrix. The following conditions are equivalent:*

(i) *A has all real eigenvalues; nonzero eigenvalues have linear elementary divisors, zero eigenvalues have elementary divisors of degree less than or equal two.*

(ii) *A is symmetrizable.*

(iii) *There exists a nonsingular H for which $AH \geqq 0$.*

(iv) *There exist Hermitian matrices H and K, with $K \geqq 0$, for which $A = HK$.*

(v) *There exist Hermitian matrices H and K, with H nonsingular and $K \geqq 0$, for which $A = HK$.*

Proof. That (i) $\Leftrightarrow$ (ii) is due to Silberstein (Theorems 3.1 and 3.2 of [14]). That (iii) $\Rightarrow$ (i) follows from our Corollary 4. We however, shall prove (i) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

Suppose A satisfies (*i*). Then for some P, $P^{-1}AP = J = D \oplus 0 \oplus (\sum_i \oplus J_i)$, where D is a real nonsingular diagonal and each

$$J_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Define $K = D \oplus I \oplus (\sum_i \oplus K_i)$, where each

$$K_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad J_i K_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then K is nonsingular, Hermitian, and $JK \geqq 0$. We define $H = PKP^*$, and then $AH = (PJP^{-1})(PKP^*) \geqq 0$.

Suppose (iii) holds; then $A = HK$, where $K = H^{-1}A = H^{-1}(AH)H^{-1} \geqq$

0. That (v) $\Rightarrow$ (iv) is obvious. Suppose (iv) holds, and $A = HK$, where $K \geqq 0$. Then $KA = KHK$ is Hermitian, and $Kx = 0 \Rightarrow Ax = HKx = 0$. The proof is complete.

As a corollary we give a slight generalization of a result due (in a more general Setting) to Reid [13].

COROLLARY 5. *If* $AH \geqq 0$ *for some* H, *rank* $H = r \neq 0$, *then* $\alpha(A) \geqq r$. *In particular, if* $H \neq 0$, A *has a real eigenvalue.*

Proof. We may assume by Proposition 3 that $H = H_{11} \oplus 0$, H_{11} nonsingular. Then

$$AH = \begin{bmatrix} A_{11} A_{12} \\ A_{21} A_{22} \end{bmatrix} \begin{bmatrix} H_{11} & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} A_{11}H_{11} & \cdot \\ A_{21}H_{11} & \cdot \end{bmatrix} . \tag{14}$$

As AH is Hermitian, $A_{21}H_{11} = 0$; as H_{11} nonsingular, $A_{21} = 0$. Now by Theorem 3, as $A_{11}H_{11} \geqq 0$, A_{11} has all real eigenvalues. As $A_{21} = 0$, obviously $\alpha(A) \geqq \alpha(A_{11}) = r$.

7. Elementary divisors associated with positive and negative eigenvalues. We give a corollary to Theorem 3 similar in nature to Theorem 1.

COROLLARY 6. *A necessary and sufficient condition for* A *to have at least* p *and* q *elementary divisors associated with, respectively, positive and negative eigenvalues is that there exist a Hermitian* H *for which* $\pi(H) = p$, $\nu(H) = q$, *and* $AH \geqq 0$, *of rank* $p + q$.

Proof. The proof of the necessity is modeled after the corresponding proof of the Drazin-Haynsworth theorem. Let $\beta_1, \cdots, \beta_{p+q}$ be real eigenvalues of A, of which p are positive and q are negative; let $V_1, \cdots, V_{p+q}$ be a set of linearly independent associated eigenvectors for $\beta_1, \cdots, \beta_{p+q}$. Let $D = \text{diag}\ (\beta_1, \cdots, \beta_{p+q})$ and let $V = (V_1, \cdots, V_{p+q})$. We have $AV = VD$ and

$$AVDV^* = VD^2V^* = VDV^*A^* .$$

We take $H = VDV^*$; clearly $\pi(H) = p$, $\nu(H) = q$. As $D^2 > 0$, of order $p + q$, $AH = HA^* = VD^2V^* \geqq 0$, of rank $p + q$.

To prove sufficiency, we assume by Proposition 3 that $H = H_{11} \oplus 0$, where H_{11} is nonsingular. As in Corollary 5, $A_{21} = 0$ and $AH = A_{11}H_{11} \oplus 0$. We have

$$\text{rank}\ H_{11} = \text{rank}\ H = p + q = \text{rank}\ AH = \text{rank}\ A_{11}H_{11}$$

and $A_{11}H_{11} \geqq 0$; therefore $A_{11}H_{11} > 0$.

By Theorem 1 of [12], $InA_{11} = InH_{11} = (p, q, 0)$. By our Theorem

3, as A_{11} and H_{11} are nonsingular, A_{11} has all real nonzero eigenvalues with linear elementary divisors. Thus A_{11} has p and q elementary divisors associated with respectively positive and negative eigenvalues. By Proposition 2, as $A_{21} = 0$, A has at least as many of each.

8. Matrices similar to real diagonal matrices.

We next give conditions for a complex matrix to be diagonalizable, with all real eigenvalues. For the case when A is real, some of these have been given by Taussky [17].

COROLLARY 7. *The matrix A is diagonalizable, with all real nonzero eigenvalues, if and only if there exists an H for which $AH > 0$. If $AH > 0$, then $InA = InH$.*

Proof. This is Corollary 6 for $p + q =$ order A.

COROLLARY 8. *Let A be a complex matrix. The following conditions are equivalent:*

(i) *A is diagonalizable, with all real eigenvalues,*

(ii) *There exists an H for which $AH \geqq 0$, with rank A = rank H = rank AH,*

(iii) *There exists an $H > 0$ for which AH is Hermitian,*

(iv) *There exists an $H \geqq 0$ for which AH is Hermitian and rank A = rank AH,*

(v) *There exist Hermitian matrices G and H, with $H > 0$, for which $A = GH$.*

Further, if (ii) *holds, then $InA = InH$. If either* (iii) *or* (iv) *holds, then $InA = InAH$. If* (v) *holds, then $InA = InG$.*

REMARK 4. If A is a real matrix, all of the Hermitian matrices G and H of Corollaries 7 and 8 may be chosen to be real symmetric. In fact, all constructions of G and H, given in proof or referred to, may be used to obtain such real G and H. The real case of Corollary 8, (i) $\Leftrightarrow$ (v), is known; cf. [17, p. 133].

REMARK 5. That (i) $\Leftrightarrow$ (iii) is contained in Theorem 3.3 of [14]. That (i) $\Leftrightarrow$ (v) has been noted by Taussky (Amer. Math. Monthly 66 (1959), p. 427, problem 4846; published solution by Parker, Amer. Math. Monthly 67 (1960) p. 192). That (v) $\Rightarrow$ (i) is our Corollary 3. Alternate proofs for Corollary 7 and Corollary 8, (i) $\Leftrightarrow$ (iii), may be obtained using the equivalence of (i) and (v).

Proof of Corollary 8. (i) $\Rightarrow$ (ii). Let $S^{-1}AS = B = \operatorname{diag}(\beta_1, \cdots, \beta_n)$, where the β_i are real. Obviously $B^2 \geqq 0$, with rank B = rank B^2; and if we let $H = SBS^*$, $AH \geqq 0$, with rank A = rank H = rank AH.

(ii) $\Rightarrow$ (i). Assume $H = H_{11} \oplus 0$, H_{11} nonsingular. As in the proof of Corollary 5, $A_{21} = 0$ and $AH = A_{11}H_{11} \oplus 0 \geqq 0$. As rank $H =$ rank AH, $A_{11}H_{11} > 0$ and by Corollary 7, A_{11} is diagonalizable, with all real nonzero eigenvalues. As rank $A =$ rank $H =$ rank $H_{11} = =$ rank A_{11}, and $A_{21} = 0$, we have $A_{22} = 0$. The proof of (i) now follows from Proposition 1. That $InA = InH$ follows from (i) and Corollary V.2 of [6].

(i) $\Leftrightarrow$ (iii). This is the Drazin-Haynsworth theorem for $m =$ order A. That $InA = InAH$ follows from Corollary 3 of [12].

(i) $\Rightarrow$ (iv). Assume $S^{-1}AS = B = \text{diag}\,(\beta_1, \cdots, \beta_n)$, where the β_i are real. Let $H = SB^2S^* \geqq 0$.

Clearly $AH = SB^3S^*$ is Hermitian, rank $A =$ rank AH.

(iv) $\Rightarrow$ (i). Suppose now AH is Hermitian, rank $A =$ rank AH, and $H = H_{11} \oplus 0$, where $H_{11} > 0$. By (iii), as $(AH)_{11} = A_{11}H_{11}$ is Hermitian, A_{11} is diagonalizable, with all real eigenvalues, and $InA_{11} = In(A_{11}H_{11})$. As before, $A_{21} = 0$, and $AH = A_{11}H_{11} \oplus 0$. As rank $A =$ rank $AH =$ rank $A_{11}H_{11} =$ rank A_{11}, we must have $A_{22} = 0$, so that all eigenvalues of A are real. Further, all nonzero eigenvalues have linear elementary divisors by Proposition 1.

We now prove that zero eigenvalues also have linear elementary divisors. Let S_{11} be a matrix for which $S_{11}^{-1}A_{11}S_{11} = D \oplus 0$, where D is a nonsingular diagonal. Let $S = S_{11} \oplus I$. Then

$$S^{-1}AS = B = \begin{bmatrix} D & \cdot & B_{13} \\ \cdot & \cdot & B_{23} \\ \cdot & \cdot & \cdot \end{bmatrix},$$

where A_{22} corresponds to the zero matrix in the lower-right corner of B. As rank $D =$ rank $A_{11} =$ rank A, obviously $B_{23} = 0$ and all zero eigenvalues have linear elementary divisors, (again by Proposition 1).

As $A_{22} = 0$, $\pi(A) = \pi(A_{11}) = \pi(A_{11}H_{11}) = \pi(AH)$, and similarly $\nu(A) = \nu(AH)$. The proof is complete.

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